



## On the spectrum of a class of distance-transitive graphs

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### Abstract

Let  $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$  be the Cayley graph on the cyclic additive group  $\mathbb{Z}_n$  ( $n \geq 4$ ), where  $S_1 = \{1, n-1\}, \dots, S_k = S_{k-1} \cup \{k, n-k\}$  are the inverse-closed subsets of  $\mathbb{Z}_n - \{0\}$  for any  $k \in \mathbb{N}$ ,  $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ . In this paper, we will show that  $\chi(\Gamma) = \omega(\Gamma) = k+1$  if and only if  $k+1|n$ . Also, we will show that if  $n$  is an even integer and  $k = \frac{n}{2} - 1$  then  $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 \text{wr}_I \text{Sym}(k+1)$  where  $I = \{1, \dots, k+1\}$  and in this case, we show that  $\Gamma$  is an integral graph.

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### 1. Introduction

In this paper, a graph  $\Gamma = (V, E)$  always means a simple connected graph with  $n$  vertices (without loops, multiple edges and isolated vertices), where  $V = V(\Gamma)$  is the vertex set and  $E = E(\Gamma)$  is the edge set. The size of the largest clique in the graph  $\Gamma$  is denoted by  $\omega(\Gamma)$  and the size of the largest independent sets of vertices by  $\alpha(\Gamma)$ . A graph  $\Gamma$  is called a vertex-transitive graph if for any  $x, y \in V$  there is some  $\pi$  in  $\text{Aut}(\Gamma)$ , the automorphism group of  $\Gamma$ , such that  $\pi(x) = y$ . Let  $\Gamma$  be a graph, the complement  $\bar{\Gamma}$  of  $\Gamma$  is the graph whose vertex set is  $V(\Gamma)$  and whose edges are the pairs of nonadjacent vertices of  $\Gamma$ . It is well known that for any graph  $\Gamma$ ,  $\text{Aut}(\Gamma) = \text{Aut}(\bar{\Gamma})$ .

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[8]. If  $\Gamma$  is a connected graph and  $\partial(u, v)$  denotes the distance in  $\Gamma$  between the vertices  $u$  and  $v$ , then for any automorphism  $\pi$  in  $Aut(\Gamma)$  we have  $\partial(u, v) = \partial(\pi(u), \pi(v))$ .

Let  $k$  be a positive integer, a  $k$ -colouring of a graph  $\Gamma$  is a mapping  $f: V(\Gamma) \rightarrow \{1, \dots, k\}$  such that  $f(x) \neq f(y)$  for any two adjacent vertices  $x$  and  $y$  in  $\Gamma$ , and if such a mapping exists we say that  $\Gamma$  is  $k$ -colorable. The chromatic number  $\chi(\Gamma)$  of  $\Gamma$  is the minimum number  $k$  such that  $\Gamma$  is  $k$ -colorable. Let  $\Gamma$  be a graph and  $\mathcal{I}(\Gamma)$  denote the set of all independent sets of the graph  $\Gamma$ . A fractional colouring of a graph  $\Gamma$  is a weight function  $\mu: \mathcal{I}(\Gamma) \rightarrow [0, 1]$  such that for any vertex  $x$  of  $\Gamma$ ,  $\sum_{x \in I \in \mathcal{I}(\Gamma)} \mu(I) \geq 1$ , and if such a weight function exists we say that  $\Gamma$  is fractional colouring. The fractional chromatic number of a graph  $\Gamma$  is denoted by  $\chi_f(\Gamma)$  and defined in [9, Page 134]. Also a fractional clique of a graph  $\Gamma$  is denoted by  $\psi_f(\Gamma)$  and defined in [9, Page 134].

Let  $\Upsilon = \{\gamma_1, \dots, \gamma_{k+1}\}$  be a set and  $K$  be a group then we write  $Fun(\Upsilon, K)$  to denote the set of all functions from  $\Upsilon$  into  $K$ , we can turn  $Fun(\Upsilon, K)$  into a group by defining a product:

$$(fg)(\gamma) = f(\gamma)g(\gamma) \quad \text{for all } f, g \in Fun(\Upsilon, K) \quad \text{and } \gamma \in \Upsilon,$$

where the product on the right is in  $K$ . Since  $\Upsilon$  is finite, the group  $Fun(\Upsilon, K)$  is isomorphic to  $K^{k+1}$  (a direct product of  $k + 1$  copies of  $K$ ) via the isomorphism  $f \rightarrow (f(\gamma_1), \dots, f(\gamma_{k+1}))$ . Let  $H$  and  $K$  be groups and suppose  $H$  acts on the nonempty set  $\Upsilon$ . Then the wreath product of  $K$  by  $H$  with respect to this action is defined to be the semidirect product  $Fun(\Upsilon, K) \rtimes H$  where  $H$  acts on the group  $Fun(\Upsilon, K)$  via

$$f^x(\gamma) = f(\gamma^{x^{-1}}) \quad \text{for all } f \in Fun(\Upsilon, K), \gamma \in \Upsilon \quad \text{and } x \in H.$$

We denote this group by  $Kwr_{\Upsilon}H$ . Consider the wreath product  $G = Kwr_{\Upsilon}H$ . If  $K$  acts on a set  $\Delta$  then we can define an action of  $G$  on  $\Delta \times \Upsilon$  by

$$(\delta, \gamma)^{(f,h)} = (\delta^{f(\gamma)}, \gamma^h) \quad \text{for all } (\delta, \gamma) \in \Delta \times \Upsilon,$$

where  $(f, h) \in Fun(\Upsilon, K) \rtimes H = Kwr_{\Upsilon}H$  [6].

Eigenvalues of an undirected graph  $\Gamma$  are the eigenvalues of an arbitrary adjacency matrix of  $\Gamma$ . Harary and Schwenk [10] defined  $\Gamma$  to be integral, if all of its eigenvalues are integers. For a survey of integral graphs see [3]. In [2] the number of integral graphs on  $n$  vertices is estimated. Known characterizations of integral graphs are restricted to certain graph classes, see [1].

Let  $G$  be a finite group and  $S$  a subset of  $G$  that is closed under taking inverses and does not contain the identity. A Cayley graph  $\Gamma = Cay(G, S)$  is a graph whose vertex-set and edge-set are defined as follows:

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x, y\} \mid x^{-1}y \in S\}.$$

It is well known that every Cayley graph is vertex-transitive.

For any graph  $\Gamma$ ,  $\omega(\Gamma) \leq \chi(\Gamma)$  [8]. Also it is well known that for bipartite graphs  $\omega(\Gamma) = \chi(\Gamma) = 2$ . Let  $\Gamma$  be the  $Cay(\mathbb{Z}_n, S_k)$  where  $\mathbb{Z}_n$  ( $n \geq 4$ ), is the cyclic additive group with identity  $\{0\}$ , and for any  $k \in \mathbb{N}$ ,  $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ ,  $S_1 = \{1, n-1\}, \dots, S_k = S_{k-1} \cup \{k, n-k\}$  are inverse-closed subsets of  $\mathbb{Z}_n - \{0\}$ . In this paper we will show that  $\chi(\Gamma) = \omega(\Gamma) = k + 1$  if and only if  $k+1|n$ , also we show that if  $n$  is an even integer and  $k = \frac{n}{2} - 1$  then  $Aut(\Gamma) \cong \mathbb{Z}_2wr_I Sym(k+1)$ , where  $I = \{1, \dots, k+1\}$ .

## 2. Definitions and Preliminaries

**Proposition 2.1.** [11] For any graph  $\Gamma$  we have

$$\omega(\Gamma) \leq \omega_f(\Gamma) \leq \chi_f(\Gamma) \leq \chi(\Gamma).$$

**Proposition 2.2.** [8] If  $\Gamma$  is vertex transitive graph, then we have

$$\omega_f(\Gamma) = \frac{|V(\Gamma)|}{\alpha(\Gamma)}$$

**Definition 1.** [4] Let  $\Gamma$  be a graph with automorphism group  $Aut(\Gamma)$ . We say that  $\Gamma$  is symmetric if, for all vertices  $u, v, x, y$  of  $\Gamma$  such that  $u$  and  $v$  are adjacent, also,  $x$  and  $y$  are adjacent, there is an automorphism  $\pi$  such that  $\pi(u) = x$  and  $\pi(v) = y$ . We say that  $\Gamma$  is distance-transitive if, for all vertices  $u, v, x, y$  of  $\Gamma$  such that  $\partial(u, v) = \partial(x, y)$ , there is an automorphism  $\pi$  such that  $\pi(u) = x$  and  $\pi(v) = y$ .

*Remark 2.1.* [4] Let  $\Gamma$  be a graph. It is clear that we have a hierarchy of conditions:

**distance-transitive  $\Rightarrow$  symmetric  $\Rightarrow$  vertex-transitive**

**Definition 2.** [4], [5] For any vertex  $v$  of a connected graph  $\Gamma$  we define

$$\Gamma_r(v) = \{u \in V(\Gamma) \mid \partial(u, v) = r\},$$

where  $r$  is a non-negative integer not exceeding  $d$ , the diameter of  $\Gamma$ . It is clear that  $\Gamma_0(v) = \{v\}$ , and  $V(\Gamma)$  is partitioned into the disjoint subsets  $\Gamma_0(v), \dots, \Gamma_d(v)$ , for each  $v$  in  $V(\Gamma)$ . The graph  $\Gamma$  is called distance-regular with diameter  $d$  and intersection array  $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$ , if it is regular of valency  $k$  and for any two vertices  $u$  and  $v$  in  $\Gamma$  at distance  $r$  we have  $|\Gamma_{r+1}(v) \cap \Gamma_1(u)| = b_r$ , and  $|\Gamma_{r-1}(v) \cap \Gamma_1(u)| = c_r$ , ( $0 \leq r \leq d$ ). The numbers  $c_r, b_r$  and  $a_r$ , where

$$a_r = k - b_r - c_r \quad (0 \leq r \leq d),$$

is the number of neighbours of  $u$  in  $\Gamma_r(v)$  for  $\partial(u, v) = r$ , are called the intersection numbers of  $\Gamma$ . Clearly  $b_0 = k, b_d = c_0 = 0$  and  $c_1 = 1$ .

*Remark 2.2.* [4] It is clear that if  $\Gamma$  is distance-transitive graph then  $\Gamma$  is distance-regular.

**Lemma 2.1.** [4] A connected graph  $\Gamma$  with diameter  $d$  and automorphism group  $G = Aut(\Gamma)$  is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer  $G_v$  is transitive on the set  $\Gamma_r(v)$ , for each  $r \in \{0, 1, \dots, d\}$ , and  $v \in V(\Gamma)$ .

**Theorem 2.1.** [5] Let  $\Gamma$  be a distance-regular graph which the valency of each vertex as  $k$ , with diameter  $d$ , adjacency matrix  $A$  and intersection array,

$$\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}.$$

Then the tridiagonal  $(d + 1) \times (d + 1)$  matrix



**Proposition 3.2.** Let  $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$  be the Cayley graph on the cyclic group  $\mathbb{Z}_n$  ( $n \geq 4$ ), where  $S_1 = \{1, n - 1\}, \dots, S_k = S_{k-1} \cup \{k, n - k\}$  are the inverse-closed subsets of  $\mathbb{Z}_n - \{0\}$  for any  $k \in \mathbb{N}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ . If  $n$  is an even integer and  $k = \frac{n}{2} - 1$  then  $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 \text{wr}_I \text{Sym}(k + 1)$ , where  $I = \{1, \dots, k + 1\}$ .

*Proof.* Let  $V(\Gamma) = \{1, \dots, n\}$  be the vertex set of  $\Gamma$ . By assumptions and Proposition 2.2, the size of the largest independent set of vertices in the  $\Gamma$  is 2, because  $\Gamma$  is a vertex transitive graph and the size of every clique in the graph  $\Gamma$  is  $k + 1$ . Thus, the size of the every independent set of vertices in the  $\Gamma$  is 2. Therefore for any  $x \in V(\Gamma)$ , there is exactly one  $y \in V(\Gamma)$  such that  $x^{-1}y = k + 1$ . Hence, if  $x^{-1}y = k + 1$  then two vertices  $x$  and  $y$  adjacent in the complement  $\bar{\Gamma}$  of  $\Gamma$ , so  $\bar{\Gamma}$  contains  $k + 1$  components  $\Gamma_1, \dots, \Gamma_{k+1}$  such that for any  $i \in I = \{1, \dots, k + 1\}, \Gamma_i \cong \Gamma_1 \cong K_2$ , where  $K_2$  is the complete graph of 2 vertices. Therefore  $\bar{\Gamma} \cong (k + 1)K_2$ , hence by Theorem 2.2,  $\text{Aut}(\bar{\Gamma}) \cong \text{Aut}(K_2) \text{wr}_I \text{Sym}(k + 1) = \mathbb{Z}_2 \text{wr}_I \text{Sym}(k + 1)$ , so  $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 \text{wr}_I \text{Sym}(k + 1)$ .  $\square$

**Example 2.** Let  $\Gamma = \text{Cay}(\mathbb{Z}_{12}, S_5)$  be the Cayley graph on the cyclic group  $\mathbb{Z}_{12}$ , then  $\chi(\Gamma) = \omega(\Gamma) = 6$ , and  $\text{Aut}(\Gamma) = \mathbb{Z}_2 \text{wr}_I \text{Sym}(6)$ , where  $I = \{1, \dots, 6\}$ .

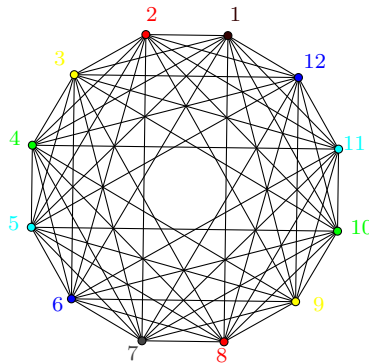


Figure 3:  $\chi(\Gamma) = \omega(\Gamma) = 6$

**Proposition 3.3.** Let  $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$  be the Cayley graph on the cyclic group  $\mathbb{Z}_n$  ( $n \geq 4$ ), where  $S_1 = \{1, n - 1\}, \dots, S_k = S_{k-1} \cup \{k, n - k\}$  are the inverse-closed subsets of  $\mathbb{Z}_n - \{0\}$  for any  $k \in \mathbb{N}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ . If  $n$  is an even integer and  $k = \frac{n}{2} - 1$  then  $\Gamma$  is a distance-transitive graph.

*Proof.* By Lemma 2.1, it is sufficient to show that vertex-stabilizer  $G_v$  is transitive on the set  $\Gamma_r(v)$  for every  $r \in \{0, 1, 2\}$  and  $v \in V(\Gamma)$ , because of  $\Gamma$  is a vertex-transitive graph. We know  $V(\Gamma) = \{1, 2, \dots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 1, \dots, n\}$  is the vertex set of  $\Gamma$ . Let  $G = \text{Aut}(\Gamma)$ . Consider the vertex  $v = 1$  in the  $V(\Gamma)$ , then  $\Gamma_0(v) = \{1\}, \Gamma_1(v) = \{2, \dots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 2, \dots, n\}$  and  $\Gamma_2(v) = \{\frac{n}{2} + 1\}$ . Let  $\rho = (2, 3, \dots, \frac{n}{2}, \frac{n}{2} + 2, \dots, n)$  be the cyclic permutation of the vertex set of  $\Gamma$ . It is an easy task to show that  $\rho$  is an automorphism of  $\Gamma$ . We can show that  $H = \langle (2, 3, \dots, \frac{n}{2}, \frac{n}{2} + 2, \dots, n) \rangle$  acts transitively on the set  $\Gamma_r(v)$  for each  $r \in \{0, 1, 2\}$ , because  $H$  is a cyclic group. Note that if  $1 \neq v \in V(\Gamma)$  then, we can show that vertex-stabilizer  $G_v$  is transitive on the set  $\Gamma_r(v)$  for each  $r \in \{0, 1, 2\}$ , because  $\Gamma$  is a vertex-transitive graph.  $\square$

**Proposition 3.4.** Let  $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$  be the Cayley graph on the cyclic group  $\mathbb{Z}_n$  ( $n \geq 4$ ), where  $S_1 = \{1, n-1\}, \dots, S_k = S_{k-1} \cup \{k, n-k\}$  are the inverse-closed subsets of  $\mathbb{Z}_n - \{0\}$  for any  $k \in \mathbb{N}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ . If  $n$  is an even integer and  $k = \frac{n}{2} - 1$  then  $\Gamma$  is an integral graph.

*Proof.* By Remark 2.2, it is clear that  $\Gamma$  is distance-regular, because  $\Gamma$  is a distance-transitive graph. Let  $V(\Gamma) = \{1, 2, \dots, n\}$  be the vertex set of  $\Gamma$ . Consider the vertex  $v = 1$  in the  $V(\Gamma)$ , then  $\Gamma_0(v) = \{1\}, \Gamma_1(v) = \{2, \dots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 2, \dots, n\}$  and  $\Gamma_2(v) = \{\frac{n}{2} + 1\}$ . Let be  $u$  in the  $V(\Gamma)$  such that  $\partial(u, v) = 0$  then  $u = v = 1$  and  $|\Gamma_1(v) \cap \Gamma_1(u)| = 2k$ , hence  $b_0 = 2k$  and by Definition 2,  $a_0 = 2k - b_0 = 0$ . Also, if  $u$  in the  $V(\Gamma)$  and  $\partial(u, v) = 1$  then two vertices  $u, v$  are adjacent in  $\Gamma$ , so  $|\Gamma_0(v) \cap \Gamma_1(u)| = 1$  and  $|\Gamma_2(v) \cap \Gamma_1(u)| = 1$ , hence  $c_1 = 1, b_1 = 1$  and  $a_1 = 2k - b_1 - c_1 = 2k - 2$ . Finally, if  $u$  in the  $V(\Gamma)$  and  $\partial(u, v) = 2$  then two vertices  $u, v$  are not adjacent in  $\Gamma$ , so  $|\Gamma_1(v) \cap \Gamma_1(u)| = 2k$ , hence  $c_2 = 2k$  and  $a_2 = 2k - c_2 = 0$ . So the intersection array of  $\Gamma$  is  $\{2k, 1; 1, 2k\}$ . Therefore by Theorem 2.1, the tridiagonal  $(3) \times (3)$  matrix,

$$\begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 2k & 0 \\ 1 & 2k - 2 & 1 \\ 0 & 2k & 0 \end{bmatrix},$$

determines all the eigenvalues of  $\Gamma$ . It is clear that all the eigenvalues of  $\Gamma$  are  $2k, 0, -2$  and their multiplicities are  $1, k+1, k$ , respectively. So  $\Gamma$  is an integral graph.  $\square$

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