Electronic Journal of Graph Theory and Applications **1** (2) (2013), 118–124

Electronic Journal of
Graph Theory and Applications

Sequence of maximal distance codes in graphs or other metric spaces

C. Delorme

University Paris Sud 91405 Orsay CEDEX - France.

Charles.Delorme@lri.fr

Abstract

Given a subset C in a metric space E, its successor is the subset $s(C)$ of points at maximum distance from C in E. We study some properties of the sequence obtained by iterating this operation. Graphs with their usual distance provide already typical examples.

Keywords: maximal distance code, graph, metric space.
Mathematics Subject Classification : 94B25.

1. Introduction

We consider a metric space E endowed with a distance d. The distance $d(x, C)$ from a point x of E to a subset C of E is as usual the infimum of the distances of x to the points of C, that is $d(x, C) = \inf_{y \in C} d(x, y)$. We consider then the supremum $r(C) = \sup_{x \in E} d(x, C)$ of the distances to C, and the subset $s(C)$ of elements of E such that $d(x, C) = r(C)$.

We already can give two common-sense properties:

$$\text{If } B \subset C, \text{ then } r(B) \geq r(C). \tag{1}$$

$$\text{If } B \subset C \text{ and } r(B) = r(C), \text{ then } s(C) \subset s(B). \tag{2}$$

We may start from any subset C_0 of E and examine the sequence of subsets of E such that $C_{i+1} = s(C_i)$ for $i \geq 0$.

Received: 2 July 2013, Revised: 9 September 2013, Accepted: 26 October 2013.

Note that $s(C)$ is always closed in E, maybe empty. Since the distance $d(x, C)$ is equal to the distance $d(x, \bar{C})$ of x to the closure of C, we may suppose without loss of generality that we deal only with closed subsets of E.

Let us get rid off of two special cases.

- If $C = E$ (or if C is dense in E), then $r(C) = 0$ and $s(C) = E$.
- If $C = \emptyset$, then $r(C) = \infty$ and $s(C) = E$.

Note that if E is compact and nonempty, then $s(C)$ also is nonempty.

We may observe the following behaviour.

Proposition 1. *For a succession $C, s(C), s^2(C), s^3(C)$ of subsets obtained by the process, and if $C, s(C)$ are nonempty we have*

$$r(s(C)) \geq r(C). \tag{3}$$

$$\textit{If } r(C) = r(s(C)), \textit{ then } C \subset s^2(C) \textit{ and } s^3(C) = s(C). \tag{4}$$

Proof. Consider a point x of $s(C)$, the distances of the points $y \in C$ to x are at least $r(C)$. Thus the distance $d(y, s(C))$ is at least $r(C)$. The supremum on E of distances to $s(C)$ is thus also at least $r(C)$. Hence the inequality.

If $r(C) = r(s(C))$, since the points y of C already satisfy $d(y, s(C)) = r(s(C))$, we have $C \subset s^2(C)$. Then $r(s^2(C)) \geq r(s(C))$.

Moreover we have already $s(C) \subset s^3(C)$ (owing to Eq. (3)), but since $C \subset s^2(C)$, the common-sense remark (Eq. (2)) gives $s^3(C) \subset s(C)$. Hence the equality. $\square$

If the metric space is finite, we clearly get a sequence of subsets of E that is ultimately periodic. If the full set occurs in the sequence, the period is 1. Otherwise, the period is 2.

We will show examples where the metric space E is a graph, with its usual metric. Its subsets will be called *codes*, and $r(C)$ is known under the name of *covering radius* of C. The *minimum distance* is the smallest distance between two different vertices of the code.

Let us recall that a *path of length* n (respectively a *one-directional ray*, respectively a *two-directional ray*) is isomorphic to the graph with vertex set $\{0, 1, 2, \ldots, n\}$ (respectively $\mathbb{N}$, respectively $\mathbb{Z}$) with edges connecting two numbers x, y if $|x - y| = 1$. These kinds of graphs will be used in sections 3 and 4.

2. Examples

2.1. A tree with 5 vertices

The graph is the tree with five vertices labeled from 1 to 5. The four edges are the pairs $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}$.

The successors of a code C_0 are shown in Table 1.

Then the codes are alternately the codes 3 and 4.

Table 1. A sequence of codes in a graph of order 5

	codes	covering radius
0	{2,3,5}	1
1	{1,4}	2
2	{5}	3
3	{1}	3
4	{4,5}	3

Table 2. A sequence of codes in a graph of order 7

	codes	covering radius
0	{2,4,5,6,7}	1
1	{1,3}	2
2	{5,7}	3
3	{4}	4
4	{7}	4
5	{4,5}	4

2.2. A graph with 7 vertices and 7 edges

The vertices are labeled from 1 to 7 and the seven edges are $\{1,2\}$, $\{2,3\}$, $\{1,3\}$, $\{1,4\}$, $\{2,5\}$, $\{3,6\}$, $\{6,7\}$. The code C_0 and its successors are given in Table 2.

3. Codes on paths

In this section, we will show that one can build sequences with an arbitrarily long nonperiodic part.

Table 3. Production rule for modified Fibonacci words

replace	by
a_1	b_2
a_2	b_1
b_1	b_2a_1
b_2	a_2b_1

Consider the sequence of *modified Fibonacci words* w_n and their symmetrics w'_n (Table 4). The word w_{i+1} (respectively w'_{i+1}) is obtained by replacing each letter of w_i (respectively w'_i) with the rule given in Table 3.

The length of word $w_i, i \geq 1$ is then the Fibonacci number F_i. The word w_i contains F_{i-1} letters b and F_{i-2} letters a and the indices are alternately 1 and 2. A letter with index 1 is followed

Table 4. Modified Fibonacci words

n	w_n	w'_n
1	a_2	a_1
2	b_1	b_2
3	b_2a_1	a_2b_1
4	$a_2b_1b_2$	$b_1b_2a_1$
5	$b_1b_2a_1a_2b_1$	$b_2a_1a_2b_1b_2$
6	$b_2a_1a_2b_1b_2b_1b_2a_1$	$a_2b_1b_2b_1b_2a_1a_2b_1$
7	$a_2b_1b_2b_1b_2a_1a_2b_1b_2a_1a_2b_1b_2$	$b_1b_2a_1a_2b_1b_2a_1a_2b_1b_2b_1b_2a_1$

with the same letter with index 2 unless it is the last letter of the word, and similarly a letter with index 2 follows the same letter with index 1 unless it is the first letter of the word.

The sequence of words has some weak resemblance to sequence A008351 of [1].

We may note $w_{n+3} = w_n w_{n+1} w'_{n+1}$ and $w'_{n+3} = w_{n+1} w'_{n+1} w'_n$.

We now choose two integers α and β with $0 \leq \alpha < \beta$, and we build the path P of length $\alpha F_{n-2} + \beta F_{n-1}$, by concatenating subpaths of length α for each letter a and β for each letter b. We then put a code C_1 in P by choosing each vertex just after the paths labeled 2 or just before the paths labeled 1, and C_0 is the complement of C_1. The distances between a vertex of the path and the closest vertex of the code C_1 is at most β, and this distance β occurs precisely for vertices preceding a subpath labeled b_2 or following a subpath labeled b_1. The code C_2 formed with these vertices is also the one created with the word w_{n-1} and the lengths $\alpha' = \beta$ and $\beta' = \alpha + \beta$. See

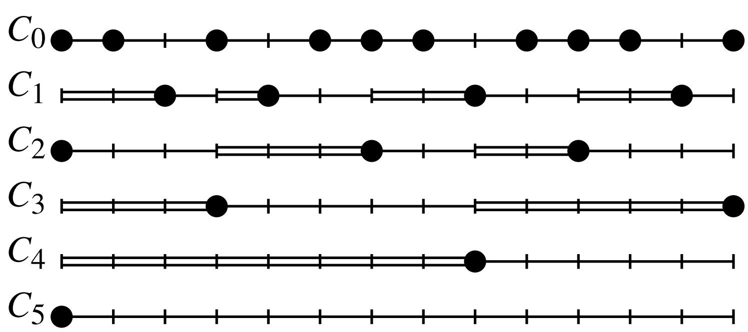

Figure 1. Successive codes on a path for $n = 6$, $\alpha = 1$ and $\beta = 2$.

Figure 1 of codes C_0 to C_5 with $n = 6$, $\alpha = 1$ and $\beta = 2$, where double lines show the subpaths labeled a_2 or b_2. Of course C_n and C_{n-1} are the ends of the path P.

We may note features for the sequence of codes build by this method.

- If $\alpha = 1$ and $\beta = 2$ the length of the path is F_{n+1}.
- $C_{k+3} \subset C_k$ for $0 \leq k \leq n - 3$.

4. Codes on rays

4.1. Nonperiodic sequences

Here we will build codes on infinite graphs such that the sequence of codes is itself infinite.

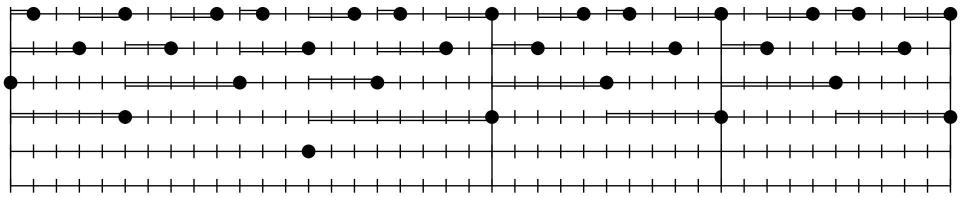

Figure 2. How to get a delayed occurrence of empty code on a ray

Noticing that w_n is always a prefix of w_{n+3}, we can define three infinite words that are the limits of the w_n's, namely $W_k = \lim_{n\to\infty} w_{3n+k}$, for $k = 0, 1, 2$.

We can build from each of these infinite words a one-directional ray and a code on it in the same way as before, and get an infinite sequence of codes $C_0, C_1, \ldots, C_n, \ldots$ with covering radiuses $\beta, \beta+\alpha, 2\beta+\alpha, \ldots, F_n\alpha + F_{n+1}\beta, \ldots$ and minimal distances $2\alpha, 2\beta, \ldots, 2(F_n\beta + F_{n-1}\alpha), \ldots$.

Similarly w'_n is always a suffix of w'_{n+3}, and we can define the three 'left-infinite words' W'_k, that are the symmetrics of the W_k's. Concatenating W'_n and W_n gives a 'word' infinite in both directions, that provides a two-directional ray with a sequence of codes having the same parameters as the codes on the one-directional ray.

4.2. Arrival of empty code

Let us take a ray. Let us consider the word consisting on w_{n+3} followed by an infinite sequence of concatenated $w'_n w_n$, with $n \geq 1$. We make a code C as formerly. Then the code $s(C)$ is obtained in the same way with w_{n+2} followed by concatenated $w'_{n-1}w_{n-1}$. When arriving at $w_4 w'_1 w_1 \ldots$ we observe that the code following this one has just a vertex and therefore is followed by the empty code (Figure 2).

4.3. Ultimately 2-periodic sequences

Codes whose sequence of successors is on rays is ultimately 2-periodic can be build as follows.

Concatenating infinitely many copies of $w_n w'_n$ and building the code C_0 like above, we get for C_{n-2} the code associated to $b_1b_2b_1b_2\ldots$, with vertices at positions $0, 2m, 4m, \ldots$, where $m = \alpha F_{n-2} + \beta F_{n-1}$, and then C_{n-1} (associated to $a_2a_1a_2a_1\ldots$) has its points at positions $m, 3m, 5m, \ldots$ and $C_n = C_{n-2}$.

5. Remarks and questions

5.1. Graphs and general metric spaces

If the graph is finite, the sequence of codes (starting from any code) is ultimately periodic of period 1 or 2.

> *What is the minimum order of a graph whose nonperiodic part has length k (i.e. $C_k \neq C_{k-2}$ and $C_{k+1} = C_{k-1}$)?*

Table 5. Some upper bounds for orders of graphs

k	bound for n
2	$3 = F_3 + 1$
3	$4 = F_4 + 1$
4	[Example 2.1] $5 < 6 = F_5 + 1$
5	[Example 2.2] $7 < 9 = F_6 + 1$
$k \geq 6$	$F_{k+1} + 1$

If the diameter of the graph is D, then this length is at most $D+1$. The given examples provide upper bounds (Table 5).

The examples with paths show that for a real segment it is possible to have a sequence of arbitrary finite length before the periodic part. The same conclusion holds for a space isometric to a sphere S^1.

However, that leaves open the question:

Is it possible for a compact *metric space to have an infinite sequence of codes?*

Of course, if any increasing sequence of distances in the set is stationary (for example the usual distance in rings of p-adic integers, see [2]), the sequence is ultimately periodic of period 1 or 2.

The Hausdorff distance between closed parts of a compact metric space endowed with distance d is defined by

$$\partial(X,Y) = \max(\max_{x\in X}\min_{y\in Y} d(x,y), \max_{y\in Y}\min_{x\in X} d(x,y)).$$

The set of closed parts of a compact metric space endowed with that distance constitutes a compact metric space (see [4, ch.7 §3 ex. 7, p. 279]), but the function successor is in general not continuous, as shown by an example: E is the real segment $[0,2]$ and $C(\varepsilon) = \{1+\varepsilon\}$. Then for $\varepsilon > 0$ the successor is $\{0\}$, but for $\varepsilon = 0$ the successor becomes $\{0,2\}$. This contributes to the difficulty of the question.

We may note however that if the sequences Γ_n and $s(\Gamma_n)$ of codes in a compact metric space E are convergent, then $\lim(s(\Gamma_n)) \subset s(\lim(\Gamma_n))$. In a sequence of $s^n(C)$, we can extract convergent subsequences Γ_n and $s(\Gamma_n)$ and then the distances $\partial(E, s^n(C))$ converge to $\partial(E, \lim(\Gamma_n)) = \partial(E, s(\lim(\Gamma_n))$ and $s^3(\lim(\Gamma_n)) = s(\lim(\Gamma_n))$.

5.2. *Other functions*

Clearly, the same behaviour occurs is the distance is replaced by a function satisfying $d(x,y) = d(y,x)$ and $d(x,x) < d(x,y)$ if $x \neq y$, like the *unilateral distance* in oriented graphs [3].

References

[1] On-Line Encyclopedia of Integer Sequences. `http://oeis.org/A008352`

[2] Y. Amice, Les nombres p-adiques. Presses Universitaires de France, sér. Le mathématicien 14 (1975)

[3] F. Comellas, C. Dalfó and M. A. Fiol, The Manhattan product of digraphs, *Electronic Journal of Graph Theory and Applications* **1** (1) (2013),11-27.

[4] J. R. Munkres, Topology, a first course, Prentice-Hall (1975).