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TECHNICAL PAPER

On the Oscillation of the Generalized Food-Limited Equations with Delay

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Abstract - The objective of the paper is to find conditions for the oscillation of the food-limited equation. We established conditions for the oscillation of all solutions of the generalized food-limited equation by transforming the equation to a non-linear delay differential equation and then to a scalar delay differential equation and using the property of the scalar delay differential equation to obtain our result. Similarly we establish conditions for the oscillation of all solutions of the food-limited equation with several delays by transforming the equation to a scalar differential equation to obtain the oscillatory property.

Keywords: Autonomous system, non-autonomous system, delay differential equations, Lébesque measurable functions, several delays, non-Oscillation

Introduction

Most of the differential equation models of population dynamics have been derived based on the foundation format

$$\frac{dN(t)}{dt} = \begin{Bmatrix} \text{an individual's contribution} \\ \text{to population change in unit time} \end{Bmatrix} N(t) \qquad (1.01)$$

where $N(t)$ denotes the density of a population of a single species at time *t*. If one assumes that an individual's contribution to the change in population in unit time is denoted by a function say $f(t, N)$ defined suitably for all $t > 0, N \geq 0$, then one obtains from equation (1.01) the so called Kolmogrov formulation in the form

$$\frac{dN(t)}{dt} = f\big(t, N(t)\big)N(t) \qquad (1.02)$$

Various choices of f together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in f lead to several well-known ordinary differential equations of population dynamics. For example, if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation

$$\frac{dN(t)}{dt} = rN(t) \qquad (1.03)$$

And if one assumes $f(t, N) \equiv r - (r/k)N$ for some positive constants *r* and *k*, one gets the logistic equation

$$\frac{dN(t)}{dt} = rN(t)\left\{1 - \frac{N(t)}{k}\right\} \qquad (1.04)$$

Since the logistic equation (1.04) implies monotonic approach as $t \to \infty$ of the population density to the steady state $N(t) \equiv k$ it has been desirable to look for modifications of equation (1.04) in order to have fluctuating (non-monotonic) solutions of model equations. If one assumes

$$f(t, N) \equiv r - (r/k)N(t - \tau) \qquad (1.05)$$

for some constant $\tau > 0$ then equation (1.02) leads to Hutchinson (1948) delay-logistic equation

$$\frac{dN(t)}{dt} = rN(t)\left\{1 - \frac{N(t-\tau)}{k}\right\} \qquad (1.06)$$

Smith (1963) proposed an alternative to the logistic equation (1.06) for a food-limited-population

$$N'(t) = rN(t)\frac{k - N(t)}{k + crN(t)}, \quad t \geq 0 \qquad (1.07)$$

Here *N*, *r*, *k* are the mass of the population, the rate of increase with unlimited food and value of *N* at saturation, respectively. The constant $1/c$ is the rate of replacement of mass in the population at saturation.

$$N'(t) = rN(t)\left\{1 - \frac{N(h(t))}{k}\right\}, h(t) \leq t \qquad (1.08)$$

The delay logistic equation is known as Hutchinson's equation, if *r* and *k* are positive constants and $h(t) = t - \tau$ for a positive constant τ which is a maturation delay. Hutchinson's equation has been investigated by many authors; see Jones (1962), Kakutani and Markus (1958). Delay logistic equation (1.08) was studied by Zang and Gopalsamy

(1988), Gopalsamy (1992), who gave sufficient conditions for the oscillation and non-oscillation of (1.08). Gopalsamy *et al.* (1988) and Grove *et al.* (1993) established the oscillation criteria for the autonomous delay food-limited equation

$$N'(t) = rN(t)\frac{k-N(t-\tau)}{k+crN(t-\tau)}, \quad t \geq 0 \qquad (1.09)$$

So and Yu (1995) investigated the stability properties of the non-linear differential equation with a constant delay

$$N'(t) = rN(t)\frac{k-N^{l}(t-\tau)}{k+crN^{l}(t-\tau)}, \quad t \geq 0 \qquad (1.10)$$

Equation (1.11) is a generalization of food-limited equations (1.07) and (1.09). Saker and Sheba (2003) discussed the oscillation and global attractivity of impulsive nonlinear delay periodic model of population dynamics and presented sufficient conditions for the oscillation of all positive solution and also established sufficient conditions for the global attractivity of the solution, see also Saker (2005), while Saker (2008) and Braverman and Saker (2007) studied the qualitative behavior of discrete nonlinear delay respiratory dynamics models. The variation of environment plays an important role in many biological and ecological dynamics. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in stable environment. Thus, the assumption of periodicity of the parameters in the system incorporates the periodicity of the environment, e.g., seasonal effects of weather, food supplies, mating habits, e.t.c. Berezansky and Braverman (2003) obtain oscillation properties of a non-autonomous food-limited equation with a non-constant delay

$$N'(t) = r(t)N(t)\frac{k-N(h(t))}{k+s(t)N(g(t))} \qquad (1.11)$$

This also generalizes equation (1.09). They compared oscillation properties of equation (1.11) and some linear delay differential equations. As a corollary, they obtained explicit oscillation condition for equation (1.11) and they proposed that their results can be extended to the generalized delay-food limited equation

$$N'(t) = r(t)N(t)\frac{k-N(h(t))|N(h(t))|^{l-1}}{k+s(t)N(g(t))|N(g(t))|^{l-1}} \qquad (1.12)$$

where $l > 0$, and the food-limited equation with several delays.

$$N(t)' = \sum_{k=1}^{\infty} r_k(t)N(t)\frac{k-N(h_k(t))}{k+s_k(t)N_k(g)} \qquad (1.13)$$

where k > 0.

In this paper, we shall obtain results for the oscillation of all solutions of the delay food-limited equations (1.12) and (1.13) which is an extension of the results obtained by Berezansky and Braverman (2003) for equation (1.12) using the same methods. The following results will be utilized in the process of establishing our results.

Methodology

We are going to consider the generalized Food-Limited equation (2.01) and present the following lemmas which will be used in the proof of the main results see Koplatadze and Chanturija (1982), Gopalsamy *et al* (1990) and Grammatikopoulos *et al* (2003). The first two lemmas will be used in proving Theorem 3.1 while the other two lemmas, that is, lemma 2.3 and lemma 2.4 will assist in establishing the conditions for the oscillation of the food-limited equation with delay (Theorem 3.2). Consider the generalized food-limited equation

$$N'(t) = r(t)N(t)\frac{k-N(h(t))|N(h(t))|^{l-1}}{k+s(t)N(g(t))|N(g(t))|^{l-1}} \qquad (2.01)$$

where $t \geq 0, h(t) \leq t, g(t) \leq t$.

let $l > 0$ and assume that conditions (A) and (B) below hold.

A. $r(t)$ and $s(t)$ are Lébesgue measurable locally essentially bounded functions, $r(t) \geq 0$ and $s(t) \geq 0$.

B. $h, g: [0,\infty) \to \mathbb{R}$ are Lébesgue measurable functions, $h(t) \leq t,\ g(t) \leq t$ $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} h(t) = \infty$.

Substituting $y(t) = N(t)|N(t)|^{l-1}$, equation (2.01) becomes

$$y'(t) = lr(t)y(t)\frac{k-y(h(t))}{k+s(t)y(g(t))} \qquad (2.02)$$

The substitution $x(t) = \frac{y(t)}{k} - 1$ transforms equation (2.02) to the equation

$$x'(t) = -lr(t)x(h(t))\frac{x(t)+1}{1+s(t)[x(g(t))+1]} \qquad (2.03)$$

Consider equation (2.03) under conditions A and B above.

Together with equation (2.03), we consider for each $t_0 \geq 0$ an initial value problem

$$x'(t) = -lr(t)x(h(t))\frac{x(t)+1}{1+s(t)[1+x(g(t))]}, \quad t \geq t_0 \qquad (2.04)$$

$$x(t) = \phi(t),\ t < t_0,\ \ x(t_0) = x_0 \qquad (2.05)$$

We also assume that the following holds:

C. $\phi: (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

Consider the linear delay differential equation

$$x'(t) + lr(t)x(h(t)) = 0, \ t \geq 0. \quad (2.06)$$

Lemma 2.1: Let A and B hold for equation (2.06) see Koplatadze and Chanturija (1982).
Then the following hypotheses are equivalent:

1. The differential inequality

$$x'(t) + lr(t)x(h(t)) \leq 0, \quad t \geq 0 \quad (2.07)$$

   has an eventually positive solution;

2. There exists $t_0 \geq 0$ such that the inequality

$$u(t) \geq lr(t) \exp\left\{\int_{h(t)}^{t} u(s)\,ds\right\}, \ t \geq t_0, \ u(t) = 0, t < t_0 \quad (2.08)$$

   has a nonnegative locally integrable solution;

3. Equation (2.06) has a nonoscillatory solution if

$$\lim_{t\to\infty} \sup \int_{h(t)}^{t} lr(s)ds > \frac{1}{e} \quad (2.09)$$

$$\text{If } \lim_{t\to\infty} \sup \int_{h(t)}^{t} lr(s)ds \leq \frac{1}{e} \quad (2.10)$$

Then all solutions of equation (2.06) are oscillatory

We assume that (A), (B) and (C) hold and consider only such solutions equation of (2.03) for which the following condition holds.

$$1 + x(t) > 0 \quad (2.11)$$

We begin with the following lemma.
Lemma 2.2: Suppose that

$$\int_0^\infty \frac{lr(t)}{1+s(t)}dt = \infty. \quad (2.12)$$

And $x(t)$ is nonoscillatory solution of equation (2.03). Then $\lim_{t\to\infty} x(t) = 0$
Proof: Suppose first $x(t) > 0, t \geq t_1$, then there exists $t_2 \geq t_1$ such that

$$h(t) \geq t_1, \ g(t) \geq t_1 \quad (2.13)$$

for $t \geq t_2$. Denote

$$u(t) = -\frac{x'(t)}{x(t)}, \qquad t \geq t_2. \quad (2.14)$$

Then $u(t) \geq 0, \ t \geq t_2$. Substitute

$$x(t) = x(t_2)\exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\}, \ t \geq t_2 \quad (2.15)$$

in equation (2.03). After multiplying by $\frac{\exp\left\{\int_{t_0}^{t} u(s)ds\right\}}{c}$ we get the following equation

$$u(t) = \frac{lr(t)}{1+g(t)} \exp\left\{\int_{h(t)}^{t} u(s)ds\right\} \frac{[1+s(t)]\left[1+c\exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\}\right]}{1+s(t)\left[1+c\exp\left\{-\int_{t_2}^{g(t)} u(s)ds\right\}\right]} \quad (2.16)$$

where $h(t) \leq t, \ g(t) \leq t, \quad t \geq t_2$ and $c = (t_2)$.
Hence,

$$u(t) \geq \frac{lr(t)}{(1+s(t))} \frac{(1+s(t))}{(1+c)(1+s(t))} = \frac{lr(t)}{(1+c)(1+s(t))} \quad (2.17)$$

Then by equation (2.12), $\int_{t_2}^{\infty} u(t)dt = \infty$

Now suppose $-1 < x(t) < 0, \ t \geq t_1$. Then there exists $t_2 \geq t_1$ such that equation (2.13) holds for. $t \geq t_2$. Suppose $u(t)$ is denoted by equation (2.14) and $c = x(t_2)$ Then $u(t) \geq 0, -1 < c <$. Substitute equation (2.15) into equation (2.03), thus equation (2.16) yields

$$u(t) \geq \frac{lr(t)}{(1+s(t))} \frac{(1+c)(1+s(t))}{(1+s(t))} = \frac{lr(t)(1+c)}{(1+s(t))} \quad (2.18)$$

Then again $\int_{t_2}^{\infty} u(t)dt = \infty$.
Equation (2.15) implies that $\lim_{t\to\infty} x(t) = 0$.
Next we consider the food-limited equation with several delays

$$N(t)' = \sum_{k=1}^{\infty} r_k(t)N(t)\frac{k - N(h_k(t))}{k + s_k(t)N_k(g)} \quad (2.19)$$

where the parameters of this equation satisfy (A) and (B), $k > 0$, and the initial function ψ satisfies (C).
Let $x(t) = \frac{N(t)}{k} - 1$, then,

$$x'(t) = -\sum_{k=1}^{m} r_k(t)x(h_k(t))\frac{1+x(t)}{1+s_k(t)[1+g_k(t)]}, \quad t \geq t_0 \tag{2.20}$$

Together with (2.20), we consider for each $t_0 \geq 0$ an initial value problem

$$x'(t) = -\sum_{k=1}^{m} r_k(t)x(h_k(t))\frac{1+x(t)}{1+s_k(t)[1+g_k(t)]}, \quad t \geq t_0 \tag{2.21}$$

$x(t) = \phi(t),\ t < t_0,\ \ x(t_0) = x_0.$
Now consider the linear delay differential equation

$$x'(t) + \sum_{k=1}^{\infty} r_k(t)x(h_k(t)) = 0, \ \ t \geq 0 \tag{2.22}$$

We state without proof the following lemma; see Grammatikopoulos *et al.* (2003):
Lemma 2.3: Let A, B hold for equation (2.22). Then the following hypotheses are equivalent:

1. The differential inequality has an eventually positive solution.

$$x'(t) + \sum_{k=1}^{\infty} r_k(t)x(h_k(t)) \leq 0, \ \ t \geq 0 \tag{2.23}$$

2. There exists $t_0 \geq 0$ such that the inequality has a nonnegative locally integrable solution.

$$u(t) \geq \sum_{k=1}^{m} r_k(t)exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\},\ t \geq t_0,\ u(t) = 0,\ t < t_0 \tag{2.24}$$

3. Equation (2.22) has a non-oscillatory solution if

$$\lim_{t\to\infty} sup \sum_{k=1}^{m} \int_{min\, h_{k(t)}}^{t} r_k(s)\,ds < \frac{1}{e} \tag{2.25}$$

If

$$\lim_{t\to\infty} inf \sum_{k=1}^{m} \int_{min\, h_{k(t)}}^{t} r_k(s)\,ds > \frac{1}{e} \tag{2.26}$$

then all the solutions of equation (2.22) are oscillatory. We assume that A, B and C hold, and consider only such solutions of equation (2.20) for which the following condition holds

$$1 + x(t) > 0 \tag{2.27}$$

We begin with the following Lemma:
Lemma 2.4: Suppose

$$\sum_{k=1}^{m} \int_0^{\infty} \frac{r_k(t)}{1+s_k(t)} dt = \infty \tag{2.28}$$

And $x(t)$ is a non-oscillatory solution of equation (2.20). Then $\lim_{t\to\infty} x(t) = 0$
Proof: Suppose first $x(t) > 0, t \geq t_1$. Then there exists $t_2 \geq t_1$ such that

$$h_k(t) \geq t_1, \quad g_k(t) \geq t \tag{2.29}$$

for $t \geq t_2$. Denote

$$u(t) = \frac{x'(t)}{x(t)}, \quad t \geq t_2 \tag{2.30}$$

Then $u(t) \geq 0,\ \ t \geq t_2$. Substitute

$$x(t) = x(t_2)exp\left\{-\int_{t_2}^{t} u(s)\,ds\right\}, t \geq t_2 \tag{2.31}$$

Into equation (2.20) and after some transformations, we have the following equation

$$u(t) = \sum_{k=1}^{\infty} \frac{r_k(t)}{1+s_k(t)} exp\left\{-\int_{h_k(t)}^{t} u(s)\,ds\right\} \frac{[1+s_k(t)][1+c\,exp\{-\int_{t_2}^{t} u(s)\,ds\}]}{1+s_k(t)[1+c\,exp\{-\int_{t_2}^{g_k(t)} u(s)\,ds\}]} \tag{2.32}$$

where $h_k(t) \leq t,\ g(t \leq t),\ \ t \geq t_2$, and $c = x(t_2) > 0$
Hence,

$$u(t) \geq \sum_{k=1}^{m} \frac{r_k(t)}{1+s_k(t)} \frac{1+s_k(t)}{(1+c)(1+s_k(t))} = \frac{r_k(t)(1+c)}{1+s_k(t)} \tag{2.33}$$

Then, by (2.28) $\sum_{k=1}^{m} \int_{t_2}^{\infty} u(t)dt = \infty.$

Now suppose $-1 < x(t) < 0,\ t \geq t_1$. Then there exists $t_2 \geq t_1$ such that equation (2.29) holds for $t \geq t_2$. Suppose $u(t)$ is given as in equation (2.30) and $c = x(t_2)$. Then $u(t) \geq 0,\ \ -1 < c < 0$. Substitute equation (2.31) into equation (2.20), thus equation (2.32) yields

$$u(t) \geq \sum_{k=1}^{m} \frac{r_k(t)}{1+s_k(t)} \frac{(1+c)(1+s_k(t))}{1+s_k(t)} = \frac{r_k(t)(1+c)}{1+s_k(t)} \tag{2.34}$$

Then again by $\sum_{k=1}^{m} \int_0^{\infty} u(t)dt = \infty.$
Equation (2.34) implies $\lim_{t\to\infty} x(t) = 0.$

Main Results

Theorem 2. 1: Suppose equation (2.12) holds and for some $\epsilon > 0$, all solutions of the linear equation

$$x'(t) + (1-\epsilon)\frac{lr(t)}{1+s(t)}x(h(t)) = 0 \tag{3.01}$$

are oscillatory. Then all solutions of equation (2.03) are oscillatory.

Proof: Suppose $x(t)$ is an eventually positive solution of equation (2.03). Lemma (2.2) implies that there exists $t_1 \geq 0$ such that $0 < x(t) < \epsilon$ for $t \geq t_1$. We suppose equation (2.13) holds for $t \geq t_2 \geq t_1$. For $t \geq t_2$, we have

$$\frac{[1+s(t)](1+x(t))}{1+s(t)[1+x(g(t))]} \geq \frac{1+s(t)}{1+s(t)(1+\epsilon)} \geq \frac{1+s(t)}{(1+\epsilon)(1+s(t))} \geq \frac{1}{1+\epsilon} \geq 1-\epsilon \quad (3.02)$$

Equation (2.03) implies,

$$x'(t) + (1-\epsilon)\frac{lr(t)}{1+s(t)}x(h(t)) \leq 0, \quad t \geq t_2 \quad (3.03)$$

Lemma 2.1 yields that equation (3.01) has a non-oscillatory solution, which is a contradiction.

Now suppose $-\epsilon < x(t) < 0$ for $t \geq t_1$ and equation (2.13) holds for $t \geq t_2 \geq t_1$. Then for $t \geq t_2$

$$\frac{[1+s(t)](1+x(t))}{1+s(t)[1+x(g(t))]} \geq \frac{1+s(t)(1-\epsilon)}{1+s(t)} = 1-\epsilon \quad (3.04)$$

Hence, equation (2.18) has a non-oscillatory solution and we again obtain a contradiction which completes the proof. Now consider the delay food-limited equation (2.01), where the parameters of this equation satisfy conditions (A) and (B), $k > 0$ and the initial function ψ satisfies (C). There exists a unique solution of (2.01) with the initial condition

$$N(t) = \psi(t), t < t_0, \quad N(t_0) = y_0 \quad (3.05)$$

We also assume that the following additional condition holds: (D) $\psi(t) \geq 0, t < t_0, y_0 > 0$.

Suppose *y* is a positive solution of equation (2.02) and define *x* as $x = \frac{y}{k} - 1$. Then *x* is a solution of equation (2.03) such that $1 + x(t) > 0$. Hence oscillation of *y* about *k* is equivalent to oscillation of *x* about 0.

Suppose *N* is a positive solution of equation (2.01) and define *y* as $y = N|N|^{l-1}$. Then *y* is a solution of equation (2.02), and oscillation of N about $k^{\frac{1}{l}}$ is equivalent to oscillation of *y* about *k* which is equivalent to oscillation of *x* about 0. Hence, suppose equation (2.12) holds and for some $e > 0$, all solutions of the linear equation

$$x'(t) + (1-\varepsilon)\frac{lr(t)}{1+s(t)}x(h(t)) = 0 \quad (3.06)$$

are oscillatory. Then all solutions of equation (2.01) are oscillatory about $k^{\frac{1}{l}}$.

To establish conditions necessary for the oscillations of food-limited equations with delay we now consider the linear delay differential equation of the form

$$x'(t) + \sum_{k=1}^{m} r_k x(h_k(t)) = 0, \quad t \geq 0 \quad (3.07)$$

Theorem 2.2: Suppose (3.07) holds and for some $\varepsilon > 0$, all solutions of the linear equation

$$x'(t) + (1-\varepsilon)\sum_{k=1}^{m}\frac{lr_k(t)}{1+s_k(t)}x(h_k(t)) = 0 \quad (3.08)$$

are oscillatory. Then all solutions of equation (2.20) are oscillatory about *k*.

Proof: Suppose $x(t)$ is an eventually positive solution of equation (2.20). Lemma 2.4 implies that there exists $t_1 \geq 0$ such that $0 < x(t) < \varepsilon$ for $t \geq t_1$. We suppose equation (2.28) holds for $t \geq t_2 \geq t_1$. For $t \geq t_2$, we have

$$\frac{[1+s_k(t)](1+x(t))}{1+s_k(t)[1+x(g_k(t))]} \geq \frac{1+s_k(t)}{1+s_k(t)(1+\varepsilon)} \geq \frac{1+s_k(t)}{(1+\varepsilon)(1+s_k(t))} = \frac{1}{1+\varepsilon} = 1-\varepsilon \quad (3.09)$$

Equation (2.20) implies that

$$x'(t) + (1-\varepsilon)\sum_{k=1}^{m}\frac{lr_k(t)}{1+s_k(t)}x(h_k(t)) \leq 0, t \geq t_2. \quad (3.10)$$

Lemma 2.3 yields that (2.19) has a non-oscillatory solution, we have a contradiction.

Now suppose $-\varepsilon < x(t) < 0$ for $t \geq t_1$ and equation (2.26) holds for $t \geq t_2 \geq t_1$. Then for $t \geq t_2$

$$\frac{[1+s_k(t)](1+x(t))}{1+s_k(t)[1+x(g_k(t))]} \geq \frac{(1+s_k(t))(1-\varepsilon)}{(1+s_k(t))} = 1-\varepsilon \quad (3.11)$$

Hence, equation (2.19) has a non-oscillatory solution and we again obtain a contradiction which completes the proof. Therefore when all solutions of equation (2.20) are oscillatory about 0, then all solution of equation (2.19) are oscillatory about *k*.

Now consider the delay food-limited equation (2.19), where the parameters of this equation satisfy conditions (A) and (B), *k > 0* and the initial function ψ satisfies (C). There exists a unique solution of equation (2.19) with the initial condition equation (2.23), we also assume that condition (D) holds.

Suppose *N* is a positive solution of equation (2.19) and define x as $x(t) = \frac{N(t)}{k} - 1$, then $x(t)$ is a solution of equation (2.20) such that $1 + x(t) > 0$.

Hence oscillation or non-oscillation of N about *k* is equivalent to oscillation of *x* about 0.

Example:

Here we are going to give an example to illustrate the applicability and significance of our results. Consider the generalized food-limited equation

$$N'(t) = r(t)N(t)\frac{k-N(h(t))}{k+s(t)N(g(t))},\ t \geq 0,\ h(t) \leq t,\ g(t) \leq t \qquad (4.01)$$

where $r(t) = ln^2 t$, $S(t) = t^2 - 1$ and $h(t) = t - sint$, $g(t) = t - cost$. Then the food-Limited equation (4.01) satisfies condition A and B, that is,

(a). $r(t)$ and $s(t)$ are Lébesgue measurable locally essentially bounded functions, $r(t) \geq 0$ and $s(t) \geq 0$.

(b). $h, g: [0, \infty) \rightarrow \mathbb{R}$ are Lébesque measurable functions, $h(t) \leq t$, $g(t) \leq t$ $\lim_{t\rightarrow\infty} h(t) = \infty$ and $\lim_{t\rightarrow\infty} h(t) = \infty$.

Also Lemma 2.1 and Lemma 2.2 are satisfied, that is, $\int_0^\infty \frac{lr(t)}{1+s(t)}dt = \int_0^\infty \frac{lLn^2 t}{t^2}dt = -\frac{l}{x}[Lnx + 1] = \infty$

Conclusions

We have established the oscillation results for all the solutions of the Delay Food-Limited equations (1.13) and (1.14) by transforming the equation to a non-linear delay differential equation and using the property of the scalar delay differential equation and the methods of Berezansky and Braverman (2003) as posed by them to obtained our result. Also an example was constructed to establish the applicability of our results

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