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TECHNICAL PAPER

Linear Subspaces of Solutions Applied to Hirota Bilinear Equations

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Abstract - Linear subspace of solution is applied to Boussinesq and Kadomtseve-Petviashvili (KP) equations using Hirota bilinear transformation. A sufficient and necessary condition for the existence of linear subspaces of exponential travelling wave solutions to Hirota bilinear equations is applied to show that multivariate polynomials whose zeros form a vector space can generate the desire Hirota bilinear equations with given linear subspaces of solutions and formulate such multivariate polynomials by using multivariate polynomials which have one and only one zero.

Keywords: Hirota bilinear form, solton solution, N-wave solution, linear subspaces.

Introduction

The bilinear form which was discovered by Hirota has played a vital role in the study of integrable nonlinear systems. The formalism is perfectly suitable for obtaining not only multi-soliton solutions but also several types of many nonlinear evolution equations. Moreover, it has been used for the study of the algebraic structure of evolution equations and extension of integrable systems (Hirota, 2004; Hieteranta, 2005). Beside the solitons and algebraic structure of evolution equations (Ma, 2004), (Jimbo and Miwa, 1983) another class of interesting multiple exponential wave solutions is linear combinations of exponential waves, which implies the existence of linear subspaces of solutions. It is also shown that, the kind of nonlinear equations can possess such a linear superposition principle, and a sufficient criterion for its existence was given for Hirota bilinear equations in (Ma and Fan, 2011).

We would in this paper like to extend the work of Ma and Fan (2011) to describe Hirota bilinear equations which possesses linear subspaces of exponential travelling wave solutions. The involved exponential wave solutions may or may not satisfy the corresponding dispersion relation. It is also known that bilinear equations are the nearest neighbors to linear equations and this explain why Hirota bilinear equations has advantage over the other methods. And we expect our resulting theory to exhibit such common features. The paper is organized as follows: in section 2 we will briefly discuss the linear superposition principle for exponential travelling waves and the established sufficient and necessary criterion for the existence of linear subspaces of exponential travelling wave solutions to Hirota bilinear equations.

In section 3 after analyzing the zeros of a kind of multivariate polynomials ,we will show that multivariate polynomials whose zeroes form s vector can generate the desired Hirota bilinear equations with given ;linear subspaces of solutions, and formulate such multivariate polynomials by using multivariate polynomials which have one and only one zero. An application is made to Boussinesq and KP equations. Concluding remarks will be given in section 5.

Linear superposition principle

Recall that the Hirota bilinear operator is defined by the following rule (see, for example, Man and Fan, 2011; and Ma *et al.*, 2010)

$$\begin{aligned} D_{x_1}^{n_1}...D_{x_m}^{n_1} f.g &= (\partial_{x_1} - \partial_{x_1'})^{n_1}...(\partial_{x_m} - \partial_{x_m'})^{n_m} f(x_1,...,x_m) g(x_1',...,x_m') \,|_{x_1'=x_1...x_m^1=x_m} \\ &= \partial_{x_1}^{n_1}....\partial_{x_m}^{n_m} f(x_1+x_1',...,x_m+x_m') g(x_1-x_1',...,x_m-x_m') \,|_{x_1'=...=x_m=0} \end{aligned} \tag{2.1}$$

Where $n_1,\ldots,n_m$ are arbitrary nonnegative integers.
Let p be a polynomial in m variables, satisfying,

$$\underbrace{p(0,..,0)=0}_{m} \tag{2-2}$$

which means that p has no constant term. Hence the corresponding Hirota bilinear equation can be written as

$$p(D_x)f.f = p(D_{x_1},...,D_{x_m})f.f = 0 \tag{2.3}$$

Note that a term of odd degree in p produces zero in the resulting Hirota bilinear equation, and so we assume that p is an even polynomial, I. e

$$p(-x_1,...,-x_m) = p(x_1,...,x_m) \tag{2.4}$$

Various evolution equations can be written in Hirota bilinear equations through a dependent variable transformation, see (Airy, Stokes, Boussinesq and Raleigh, 1991).
Introducing an N wave variables by fixing $N \in \mathbb{N}$ we have

$$\eta_i = k_{1,i}x_1 + \ldots + k_{M,i}x_M, \qquad 1 \le i \le N \tag{2.5}$$

And N exponential wave functions

$$f_i = e^{\eta_i} = e^{k_{1,i}x_1+\ldots+k_{M,i}x_M} \quad ,1 \le i \le N \tag{2.6}$$

Where $k_{j,i}{}'s$ are constant. Recall from bilinear identity that

$$p(D_{x_1},\ldots,D_{x_M})e^{\eta_i}.e^{\eta_j} = p(k_{1,i}x_1 + \ldots + k_{M,i}x_M)e^{\eta_i}.e^{\eta_j} \tag{2.7}$$

It is easily seen from (2.2) that every of the wave functions f_i, $1 \le i \le N$, gives a solution to the Hirota bilinear equation (2.3)

Let us now consider a linear combination of the form

$$f = \varepsilon_1 f_1 + \ldots + \varepsilon_N f_N = \varepsilon_1 e^{\eta_1} + \ldots + \varepsilon_1 e^{\eta_N} \tag{2.8}$$

Where ε_i, $1 \le i \le N$ are arbitrary constant.

In establishing a linear superposition principle for the exponential waves e^{η_i}, $1 \le i \le N$ in order that the linear combination (2.8) gives a solution to Hirota bilinear equation (2.3), the following computation was made using (2.7), (2.2) and (2.4)

$$\begin{aligned}
p(D_{x_1},\ldots,D_{x_M})f.f &= \sum_{i,j=1}^{N} \varepsilon_i\varepsilon_j p(D_{x_1},\ldots,D_{x_M})e^{\eta_i}.e^{\eta_j} \\
&= \sum_{i,j=1}^{N} \varepsilon_i\varepsilon_j p(k_{1,i}-k_{1,j},\ldots,k_{M,i}-k_{M,j})e^{\eta_i+\eta_j} \\
&= \sum_{1\le i<j\le N}^{N} \varepsilon_i\varepsilon_j [p(k_{1,i}-k_{1,j},\ldots,k_{M,i}-k_{M,j}) + p(k_{1,j}-k_{1,i},\ldots,k_{M,j}-k_{M,i})]e^{\eta_i+\eta_j} \\
&= \sum_{1\le i<j\le N}^{N} 2\varepsilon_i\varepsilon_j [p(k_{1,i}-k_{1,j},\ldots,k_{M,i}-k_{M,j})
\end{aligned}$$

Hence (2.8) solves the Hirota bilinear equation (2.3) if and only if

$$p(k_{1,i}-k_{1,j},\ldots,k_{M,i}-k_{M,j}) = 0, \quad 1 \le i < j \le N \quad \text{is satisfied} \tag{2.9}$$

Observe that (2.9) gives a system of nonlinear algebraic equations on the wave related numbers $k_{i,j}{}'s$ as soon as the polynomial p is given. The above discussions is summarized in a paper by (Ma and Fan, 2011) in a theorem. The theorem tells us exactly when a linear superposition of exponential wave solutions solves a given Hirota bilinear equation. The theorem elaborated more on interrelation between Hirota bilinear equation and the linear superposition principle for exponential waves. Let us give some special examples in (1+1) and a (2+1)- dimensional equations respectively

$$\left.\begin{aligned} \eta_i &= k_i x + \omega_i t \\ \eta_i &= k_i x + l_i y + \omega_i t \end{aligned}\right\} \quad 1 \le i \le N \tag{2.10}$$

The first example to be considered is the following polynomial:

$$p(x,t) = t^2 - x^2 - x^4 \tag{2.11}$$

The corresponding condition (2.9) gives

$$p(k_i-k_j,\omega_i-\omega_j) = \omega_i^2 - 2\omega_i\omega_j + \omega_j^2 - k_i^2 + 2k_ik_j - k_j^2 - k_i^4 + 4k_i^3k_j - 6k_i^2k_j^2 + 4k_ik_j^3 - k_j^4 = 0 \tag{2.12}$$

and the outcome of the Hirota bilinear equation is

$$(D_t^2 - D_x^2 - D_x^4)f.f = 0 \tag{2.13}$$

which will give

$$f_{tt}f - f_t^2 - f_{xx}f + f_x^2 - f_{xxxx}f + 4f_{xxx}f_x - 3f_{xx}^2 = 0$$

Under the transformation $u = 2(\ln f)_{xx}$

This equation correspond to

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0 \tag{2.14}$$

Based on the linear superposition principle for the exponential wave given in theorem (1), solving the system (2.12) on the wave related numbers leads an N-wave solution to the nonlinear equation (2.14)

$$u = 2(\ln f)_{xx}, \quad f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + \sqrt{k_i^2 + k_i^4}\, t} \tag{2.15}$$

Where the ε_i's and k_i's are arbitrary constants. Each exponential wave f_i in the solution satisfies the corresponding nonlinear dispersion relation i. e

$$p(k_i,\omega_i)=0, \qquad 1\le i\le N$$

The next example is the polynomial in (2+1) dimension.

$$p(x,y,t)=tx+x^4\pm y^2, \tag{2.16}$$

From equation (2.9) we have

$$p(k_i-k_j,l_i-l_j,\omega_i-\omega_j,)=k_i\omega_i-k_j\omega_i+\omega_ik_j+k_i^4-4k_i^3k_j+6k_i^2k_j^2-4k_ik_j^3+k_i^4\pm[l_i^2-2l_il_j+l_j^2]=0,$$

and the resulting Hirota bilinear equation reads

$$(D_xD_t+D_x^4\pm D_2^y)f.f=0 \tag{2.17}$$

Which is equivalent to

$$f_{xx}f-f_xf_t+f_{xxxx}f-4f_{xxx}f_x+3f_{xx}^2\pm(f_{yy}f-f_y^2)=0$$

Under the transformation $u=2(\ln f)_{xx}$ this equation is mapped into

$$(u_t+6uu_x+u_{xxxx})_x\pm u_{yy}=0 \tag{2.18}$$

Based on the linear superposition principle for the exponential wave given in theorem (1), solving the above system on the wave related numbers leads an N-wave solution to the nonlinear equation (2.18)

$$u=2(\ln f)_{xx},\ f=\sum_{i=1}^N\varepsilon_if_i=\sum_{i=1}^N\varepsilon_ie^{k_ix+k_i^2y}\quad or\ \sum_{i=1}^N\varepsilon_ie^{k_ix+k_i^2y-2k_i^3t}$$

Where the e_i's and k_i's are arbitrary constants. Each exponential wave f_i in the solution f satisfies the corresponding nonlinear dispersion relation i. e

$$p(k_i,l_i,\omega_i)=0, \qquad 1\le i\le N$$

It is directly to prove that

$$p(D_x,D_t)(e^{\varsigma}f).(e^{\eta}g)=e^{\varsigma+\eta}p(D_x+k_1-k_2,D_t+\omega_1-\omega_2)f.g$$

Where

$$\zeta=k_1x-\omega_1y\quad ,\eta=k_2x-\omega_2t \text{ and}$$

$$p(D_x,D_y,D_t)(e^{\varsigma}f).(e^{\eta}g)=e^{\varsigma+\eta}p(D_x+k_1-k_2,D_t-\omega_1+\omega_2,D_y+l_1-l_2)f.g$$

where,

$$\zeta=k_1x+l_1y-\omega_1t\quad ,\eta=k_2x+l_2y-\omega_2t$$

And p is a polynomial in the indicated variables. Taking,

$$\varsigma=\eta=\varsigma_0=k_0x-\omega_0t\quad and\ \varsigma=\eta=\varsigma_0=k_0x+l_0y-\omega_0t$$

The above identity yields

$$p(D_x,D_t)(e^{\varsigma_0}f).(e^{\varsigma_0}g)=e^{2\varsigma_0}p(D_x,D_t)f.g \qquad and$$
$$p(D_x,D_y,D_t)(e^{\varsigma_0}f).(e^{\varsigma_0}g)=e^{2\varsigma_0}p(D_x,D_y,D_t)f.g$$

Hence we can get a new class of multiple exponential wave solutions by $f'=e^{\varsigma_0}f$ where *f* is an original multiple exponential wave solution like any of (2.14) and (2.18); and such will be shown to form a new linear subspaces of solutions. Bilinear equations with given linear subspaces of solutions To established the given result the following theorem will be used with proof found in (Ma *et al.*, 2012). Theorem: (structure of Hirota bilinear equation)

Let $let\ M,M'\in N\ and\ A(a_{ij})_{M,N}$ be a constant matrix of rank n. suppose that $Q(y_1,...,y_N)$ is a multivariate polynomial in y $=(y_1,...y_M)^T$ possesses only one zero y = y_0 then,

$$p(x_1,...,x_M)=Q((Ax+y_0)^T),x=(x_1,...,x_M)^T$$

Is a multivariate polynomial where zeros form an n-dimensional subspaces and the corresponding Hirota bilinear equation

$$p(D_{x_1},...,D_{x_1})f.f=Q((AD_x+y_0)^T)f.f=0$$

Possesses a linear subspace of solution defined by

$$f=\sum_{i=1}^{N}\varepsilon_i e^{k_{1,i}x_1+\ldots+k_{M,i}x_M}, \quad N>1,$$

where,

$$A(k_{1,i}-k_{1,j},\ldots,k_{M,i}-k_{M,j})^T=0, \quad 1\le i\ne j\le N$$

And the ε_i 's are arbitrary constants.

In the following we will present two examples to throw more light on the algorithm of the above theorem

The first example has

$$Q(y_1,y_2)=y_1^2-(y_2+1)^2, \quad y_0=\begin{pmatrix}0\\-1\end{pmatrix}, \quad A=\begin{pmatrix}2&0\\3&-1\end{pmatrix}, \quad x=(x,t)$$

The associated multivariate polynomial is

$$p(x,t)=6xt-5x^2-t^2$$

And the corresponding Hirota bilinear equation reads

$$(6D_xD_t-5D_x^2-D_t^2)f.f=0,$$

This bilinear equation possesses the linear subspaces of solution defined by

$$f=\sum_{i=1}^{N}\varepsilon_i f_i=e^{k_0x-\omega_0}\sum_{i=1}^{N}\varepsilon_i e^{k_ix-k_it}$$

where the ε_i 's and k_i 's are arbitrary constants, but k_0 *and* ω_0 are arbitrary fixed constants. Obviously all exponential waves f_i in the solution satisfy the corresponding nonlinear dispersion relation if and only if

$$e^{k_0x-\omega_0t}$$

satisfies the corresponding dispersion relation.

The second example is

$$Q(y_1,y_2)=2y_1^2+y_2 \quad ,y_0=\begin{pmatrix}0\\0\end{pmatrix}, \quad A=\begin{pmatrix}1&0&1\\2&3&4\end{pmatrix}, \quad x=(x,y,t)^T$$

The associated multivariate polynomial is

$$p(x,y,t)=2x^3-6x^2t+2t^3+2x+3y+4t$$

And the corresponding Hirota bilinear equation reads

$$(2D_x^3+6D_x^2D_t++2D_t^3+2D_x+3D_y+4D_t)f.f=0$$

This bilinear equation possesses the linear subspaces of solution defined by

$$f=\sum_{i=1}^{N}\varepsilon_i f_i=e^{k_0x+l_0y-\omega_0t}\sum_{i=1}^{N}\varepsilon_i e^{-\omega_ix-\frac{2}{3}\omega_iy-\omega_it}, \quad N\ge1$$

where the ε_i 's and l_i 's *and* ω_i 's are arbitrary constants, but k_0, *lo and* ω_0 are arbitrary fixed constants. Obviously all exponential waves f_i in the solution f satisfy the corresponding nonlinear dispersion relation if and only if

$$e^{k_0x+l_0y-\omega_0t}$$

satisfies the corresponding nonlinear dispersion relation.

Parameterization

Following the idea of (Ma and Fan, 2011) we can compute Hirota bilinear equations with linear subspaces of solutions by using parameterization of wave numbers and frequencies. The problem is how to construct a multivariate polynomial $P(x_1,\ldots,x_M)$ with no constant term such that

$$P(k_{1,1}-k_{1,2},\ldots,k_{M,1}-k_{M,2})=0 \tag{4.1}$$

For the sets of constants $k_{1,i},\ldots,k_{M,i}, i=1,2.$

Let us first introduce the weights of independent variables

$$(w(x_1),\ldots,w(x_M))=(n_1,\ldots,n_M) \tag{4.2}$$

Where each weight $w(x_i)=n_i$ is an integer, and then form a polynomial $P(x_1,\ldots,x_M)$ being homogenous in same weight.

Second for I = 1, 2 we parameterize the constants $k_{1,i},...,k_{M,i}, i = 1,2.$ consisting of wave numbers of frequencies using a parameter k_i as follows

$$b_{j+i} = b_i k_i^{n_i} \qquad 1 \leq j \leq M \tag{4.3}$$

Where b_j's are constant to be determined, to balance the system (4.1). then putting the parameterize constants into (4.1), we collect terms by powers of parameters k_1 and k_2, and set the coefficients of each power to zero, to obtain algebraic equations on the constant b_j's and the coefficients of the polynomial $P(x_1,...,x_M)$.

Finally solve the resulting algebraic equations to determine the polynomial $P(x_1,...,x_M)$ and the parameterization. Now based on (2.20) the resulting parameterization really tells that the obtained Hirota bilinear equation possesses the linear subspaces of solutions defined by

$$f = \sum_{i=1}^{N} e_i f_i = e^{k_{1,0}x_1 + ...+ k_{M,0}x_M} \sum e_i e^{b_1 k_i^{n_1} x_1 + ...+ b_M k_i^{n_M} x_M}, \; N \geq 1 \tag{4.4}$$

Where the e_i's and k_i's are arbitrary constants but the $k_{i,0}$'s are arbitrary fixed constants.
In the following we give some illustrative example in 1+1 and 2+1-dimensions which apply the above parameterization and achieved by using one parameter.

Example 1. $(w(x), w(t)) = (1,1)$

Let us introduce the weights of independent variables

$$(w(x), w(t)) = (1,1) \tag{4.5}$$

Then, a general even polynomial being homogenous in weight 2 is

$$P = c_1 x^2 + c_2 xt + c_3 t^2 \tag{4.6}$$

Following the parameterization of wave numbers and frequency in (4.3) the wave variables read

$$h_i = k_i x + b_i k_i t \qquad 1 \leq i \leq N$$

Where k_i $1 \leq i \leq N$ are arbitrary constant but b_1 is a constant to be determined. In this example the corresponding Hirota bilinear equation $P(D_x, D_t) f.f = 0$ has the linear subspace of N-wave solutions define by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = e^{k_0 x - \omega_0 t} \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i t}, \; N \geq 1 \tag{4.7}$$

Where k_0 and w_0 are arbitrary fixed constants and b_1 satisfies

$$c_3 b_1^2 + c_2 b_1 + c_1 = 0 \tag{4.8}$$

$$\text{With } b_1 = \frac{-c_2 \pm \sqrt{c_1^2 - 4c_3 c_1}}{2c_3} \tag{4.9}$$

Example 2 $(w(x), w(y), w(t)) = (1,3,2)$

Let us infroduce the weights of independent variable

$$(w(x), w(y), w(t)) = (1,3,2) \tag{4.10}$$

Then a general polynomial of weights 3 will be

$$p = c_1 x^3 + c_2 xt + c_3 y \tag{4.11}$$

Following the parameterization of wave numbers and frequencies in (4.3) the N-wave variables is

$$h_i = k_i x + b_1 k_i^3 y + b_2 k_i^2 t, \qquad 1 \leq i \leq N$$

where k_i $1 \leq i \leq N$ are arbitrary constant but b_1 and b_2 are constants to be determined.

The computation of the corresponding Hirota bilinear equation $P(D_x, D_y, D_t) f.f = 0$ has the linear subspaces of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = e^{k_0 x + l_0 y - \omega_0 t} \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i^3 y - b_2 k_i^2 t}, \qquad (4.12)$$

Where $k_0, l_0,$ *and* w_0 are arbitrary constant and b_1 and b_2 satisfy

$$\begin{cases} c_3 b_1 + 2c_1 = 0 \\ c_2 b_2 - c_1 = 0 \end{cases} \qquad (4.13)$$

And the wave solution defined by (4.12) with

$$b_1 = \frac{-2c_1}{c_3}, \quad b_2 = \frac{c_1}{c_2} \qquad (4.14)$$

Examples without the dispersion relation

Example. 1. $(w(x), w(t)) = (1, -1)$

Let us introduce the weights of independent variables

$$(w(x), w(t)) = (1, -1) \qquad (4.15)$$

Then, a general even polynomial being homogenous in weight 1 is

$$P = c_1 x + c_2 x^2 t + c_3 x^3 t^2 \qquad (4.16)$$

Following the parameterization of wave numbers and frequency in (4.3) the wave variables read

$$h_i = k_i x + b_1 k_i^{-1} t \qquad 1 \le i \le N \qquad (4.17)$$

Where k_i $1 \le i \le N$ are arbitrary constant but b_1 is a constant to be determined. In this example the corresponding Hirota bilinear equation

$$(c_1 D_x + c_2 D_x^2 D_t + c_3 D_x^3 D_t^3) f.f = 0 \qquad (4.18)$$

has the linear subspace of N-wave solutions define by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = e^{k_0 x - \omega_0 t} \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i^{-1} t}, \quad N \ge 1 \qquad (4.19)$$

Where e_i and k_i are arbitrary, and l_0 and w_0 are arbitrary fixed constants and b_1 satisfy

$$c_3 b_1^2 + c_2 b_1 + c_1 = 0 \qquad (4.20)$$

With $$b_1 = \frac{-c_2 \pm \sqrt{c_2^2 - 4c_3 c_1}}{2c_3} \qquad (4.21)$$

Example 2

$(w(x), w(y), w(t)) = (1, -2, 3)$

Let us infroduce the weights of independent variable

$$(w(x), w(y), w(t)) = (1, -2, 3) \qquad (4.22)$$

Then the even polynomial being homogenous in weights 2 will be

$$p = c_1 x^2 + c_2 x^4 y + c_3 xyt \qquad (4.23)$$

Following the parameterization of wave numbers and frequencies in (4.3) the N-wave variables is

$$h_i = k_i x + b_1 k_i^{-2} y + b_2 k_i^3 t, \qquad 1 \le i \le N$$

Where k_i $1 \le i \le N$ are arbitrary constant but b_1 and b_2 are a constant to be determined.

Now a direct computation of the corresponding Hirota bilinear equation of the form $P(D_x, D_y, D_t) f.f = 0$

$(c_1 D_x^2 + c_2 D_x^4 D_y + c_3 D_x D_y D_t) f.f = 0$ and has a linear subspaces of N-wave solution defined by

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = e^{k_0 x + l_0 y - \omega_0 t} \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i^{-2} y - b_2 k_i^3 t}, \tag{4.24}$$

Where e_i and k_i are arbitrary constants and b_2 but l_0, y_0, w_0 are arbitrary fixed constants and b_1 and b_2 satisfy

$$\left.\begin{aligned} c_3 b_1 b_2 + 2c_2 b_1 &= 0 \\ c_2 b_1 - c_1 &= 0 \end{aligned}\right\} \tag{4.25}$$

And has N- wave solution defined by (4.2) with

$$b_1 = \frac{-2c_2}{c_3}, \quad b_1 = \frac{c_1}{c_2}$$

Conclusions

Hirota bilinear equations that possesses the linear superposition principle for exponential wave solution is analyzed and also examined on how to construct multivariate polynomials which generate such Hirota bilinear equations using some examples. In particular we show that multivariate polynomials whose zeros form a vector space could generate the desire Hirota bilinear equations. The related multivariate polynomials were formulated by using multivariate polynomials which have one and only one zero. Though the linear superposition principle does not apply to nonlinear differential equations, in general, it is known that Hirota bilinear equations possess the linear superposition principle among exponential wave solutions. This gives the existence of linear subspace of solution.

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