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Analysis of the Stability of Bessel, Legendre and Euler Differential Equations

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Abstract

The Bessel, Legendre and Euler differential equations discussed in this paper are second-level differential equations. These three equations become a system with two equations. The equilibrium point of all three of these equations is at the point $(0,0)$. These three equations are locally asymptotically stable at the equilibrium point $(0,0)$.

Keywords: Bessel; Legendre; Euler differential equations; equilibrium point; locally asymptotically stable.

Introduction

The Bessel differential equation is a second level differential equation developed by German mathematician and astronomer Friedrich Wilhelm Bessel in 1824 [1]. The solution of this differential equation is called the cylinder function or the Bessel function [2].

The second level differential equation which will be discussed next is Legendre differential equation. This equation was discovered by Adrien-Marie Legendre in 1782 [3]. Adrien-Marie Legendre is a scientist in physics and mathematics. Actually what Adrien-Marie Legendre discovered was not only a second-order differential equation, but a level *n*, which is better known as the Legendre polynomial [4].

Another second level differential equation that will be discussed in this paper is the Euler differential equation. This equation was introduced by Leonhard Euler, starting in 1740 [5]. Leonhard Euler is a scientist in the fields of mathematics, physics, astronomer, geography, logic and engineer [6]. The development of this equation by other scientists is quite a lot, for example the Cauchy-Euler equation, the Hermite-Euler Polynomial and the Frobenius-Euler Polynomial.

Analysis of Stability

Bessel Differential Equation Analysis of Stability

The Bessel differential equation is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0$$

where α is a constant, often called the order of the Bessel differential equation or simply abbreviated order [7].

Let $z_1=y, z_2=\frac{dy}{dx}$ so that

$$\frac{dz_1}{dx}=z_2$$

$$\frac{dz_2}{dx}=-\frac{1}{x}z_2-\frac{(x^2-\alpha^2)}{x^2}z_1 \quad (1)$$

The equilibrium point is $z_2 = 0$ and $-\frac{1}{x}z_2 - \frac{(x^2-\alpha^2)}{x^2}z_1 = 0$. Because $z_2 = 0$, then $z_1 = 0$. So the equilibrium point of the Bessel differential equation is $(z_1, z_2) = (0,0)$.
Let

$$f_1 = \frac{dz_1}{dx} = z_2$$

$$f_2 = \frac{dz_2}{dx} = -\frac{1}{x}z_2 - \frac{\left(x^2-\alpha^2\right)}{x^2}z_1$$

We get

$$\frac{df}{dz} = \begin{pmatrix} \frac{df_1}{dz_1} & \frac{df_1}{dz_2} \\ \frac{df_2}{dz_1} & \frac{df_2}{dz_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\left(x^2-\alpha^2\right)}{x^2} & -\frac{1}{x} \end{pmatrix}$$

Eigenvalue of the equilibrium point $(z_1, z_2) = (0,0)$ is

$$\lambda_1 = \frac{-\frac{1}{x} + \sqrt{\frac{-4x^2+4\alpha^2+1}{x^2}}}{2} \quad \text{and} \quad \lambda_2 = \frac{-\frac{1}{x} - \sqrt{\frac{-4x^2+4\alpha^2+1}{x^2}}}{2}$$

If $x = \frac{1}{4}\sqrt{4\alpha^2+1}$ or $x = -\frac{1}{4}\sqrt{4\alpha^2+1}$ and $x \neq 0$, then λ_1 and λ_2 the eigenvalue is negative, so the equilibrium point of the Bessel differential equation, $(0,0)$, locally asymptotically stable. To facilitate the simulation in Equation (1) is changed to

$$\frac{dz_1}{dx} = z_2$$

$$\frac{dz_2}{dx} = -\frac{1}{c}z_2 - \frac{\left(c^2-\alpha^2\right)}{c^2}z_1,$$

where $\alpha = 1$, $c = 3$ and initial value $z_{1_0} = z_{2_0} = 2$.

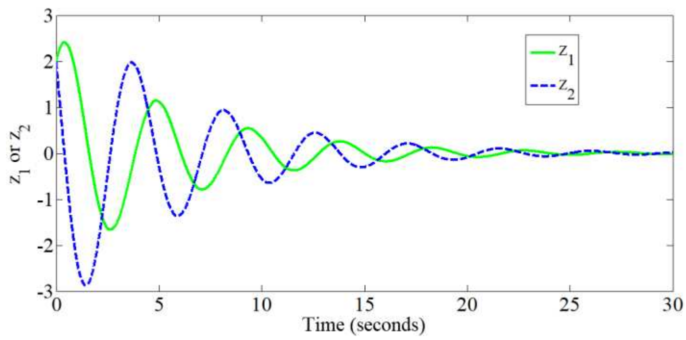

Figure 1. Trajectory of the Bessel Differential Equation.

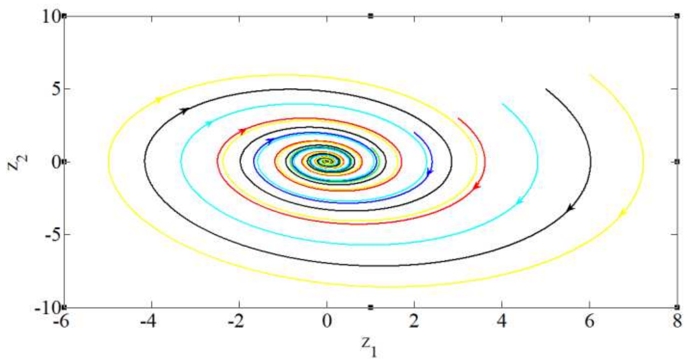

Figure 2. Phase Portrait of the Bessel Differential Equation.

Analysis of the Legendre Differential Equation Stability

The Legendre differential equation is

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+l(l+1)y=0,$$

where l is a known number, often called the Legendre differential equation or simply abbreviated order [8].

Let $w_1=y$, $w_2=\frac{dy}{dx}$, so that

$$\frac{dw_1}{dx}=w_2$$

$$\frac{dw_2}{dx}=\frac{2x}{1-x^2}w_2-\frac{l(l+1)}{1-x^2}w_1. \tag{2}$$

The equilibrium point is $w_2=0$ and $\frac{2x}{1-x^2}w_2-\frac{l(l+1)}{1-x^2}w_1=0$. Because $w_2=0$, then $w_1=0$. So the equilibrium point of the Legendre differential equation is $(w_1,w_2)=(0,0)$.

Let

$$f_1=\frac{dw_1}{dx}=w_2$$

$$f_2=\frac{dw_2}{dx}=\frac{2x}{1-x^2}w_2-\frac{l(l+1)}{1-x^2}w_1$$

We obtain

$$\frac{df}{dw}=\begin{pmatrix}\frac{df_1}{dw_1} & \frac{df_1}{dw_2}\\ \frac{df_2}{dw_1} & \frac{df_2}{dw_2}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\frac{l(l+1)}{1-x^2} & \frac{2x}{1-x^2}\end{pmatrix}$$

Eigenvalue of the equilibrium point $(w_1,w_2)=(0,0)$ is

$$\lambda_1=\frac{x}{1-x^2}+\frac{1}{2(1-x^2)}\sqrt{\left[4-4l(l+1)\right]x^2+4l(l+1)}$$ and

$$\lambda_2=\frac{x}{1-x^2}-\frac{1}{2(1-x^2)}\sqrt{\left[4-4l(l+1)\right]x^2+4l(l+1)}$$

If $x=\sqrt{\dfrac{l.(l+1)}{[l(l+1)-1]}}$ or $x=-\sqrt{\dfrac{l.(l+1)}{[l(l+1)-1]}}$, and $x<0$, $x\neq -1$, then λ_1 and λ_2 the eigenvalue is negative, so the equilibrium point of the Legendre differential equation, $(0,0)$, locally asymptotically stable. To facilitate the simulation in Equation (2) is changed to

$$\frac{dw_1}{dx}=w_2$$
$$\frac{dw_2}{dx}=\frac{2c}{1-c^2}w_2-\frac{l(l+1)}{1-c^2}w_1,$$

where $l=-0.1$, $c=3$ and initial value $w_{1_0}=w_{2_0}=2$.

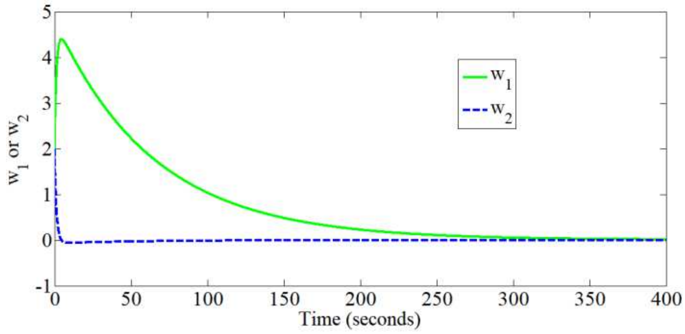

Figure 3. Trajectory of the Legendre Differential Equation.

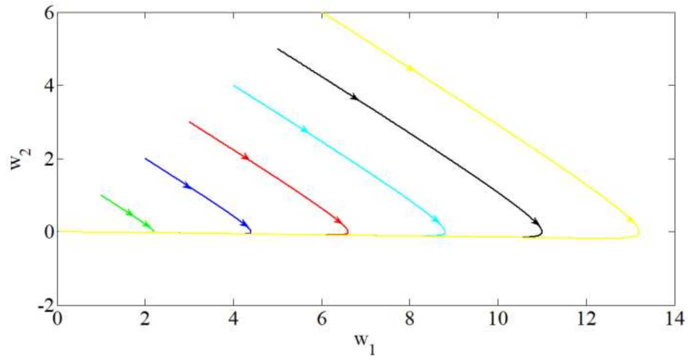

Figure 4. Phase Portrait of the Legendre Differential Equation.

Analisis Kestabilan Persamaan Diferensial Euler

The Euler differential equation is

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=0,$$

where a and b any constant [9].

Let $v_1=y$, $v_2=\dfrac{dy}{dx}$, so that

$$\frac{dv_1}{dx}=v_2$$
$$\frac{dv_2}{dx}=-\frac{a}{x}v_2-\frac{b}{x^2}v_1. \tag{3}$$

The equilibrium point is $v_2=0$ and $-\dfrac{a}{x}v_2-\dfrac{b}{x^2}v_1=0$. Because $v_2=0$, then $v_1=0$. So the equilibrium point of the Euler differential equation is $(v_1,v_2)=(0,0)$.

Let

$$f_1 = \frac{dv_1}{dx} = v_2$$
$$f_2 = \frac{dv_2}{dx} = -\frac{a}{x}v_2 - \frac{b}{x^2}v_1$$

We have

$$\frac{df}{dv} = \begin{pmatrix} \frac{df_1}{dv_1} & \frac{df_1}{dv_2} \\ \frac{df_2}{dv_1} & \frac{df_2}{dv_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b}{x^2} & -\frac{a}{x} \end{pmatrix}$$

Eigenvalue of the equilibrium point $(v_1, v_2) = (0,0)$ is

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2x} \quad \text{and} \quad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2x}$$

If $a^2 = b$ and $x \neq 0$, then λ_1 and λ_2 the eigenvalue is negative, so the equilibrium point of the Euler differential equation, $(0,0)$, locally asymptotically stable. To facilitate the simulation in Equation (3) is changed to

$$\frac{dv_1}{dx} = v_2$$
$$\frac{dv_2}{dx} = -\frac{a}{c}v_2 - \frac{b}{c^2}v_1,$$

where $a = 1$, $b = 2$, $c = 3$ and initial value $v_{1_0} = v_{2_0} = 2$.

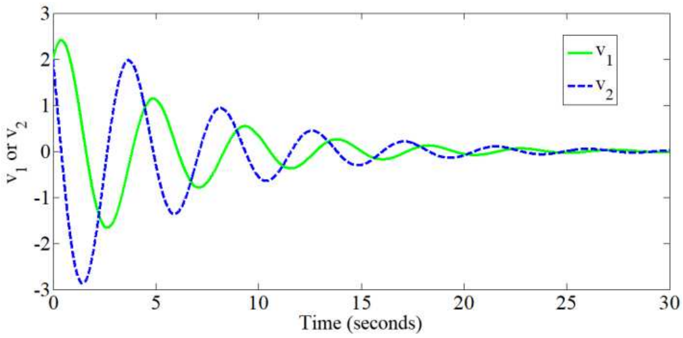

Figure 5. Trajectory of the Euler Differential Equation.

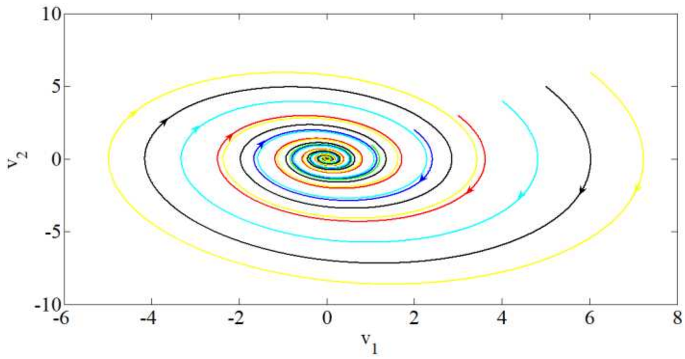

Figure 6. Phase Portrait of the Euler Differential Equation.

Conclusion

The three differential equations Bessel, Legendre and Euler are second order differential equations. These three equations have an equilibrium point at $(0,0)$ and all three are locally asymptotically stable at that equilibrium point.

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