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The Decomposition of a Finitely Generated Module over Some Special Ring

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ABSTRACT

This research aims to give the decompositions of a finitely generated module over some special ring, such as the principal ideal domain and Dedekind domain. One of the main problems with module theory is to analyze the objects of the module. This research was using a literature study on finitely generated modules topics from scientific journals, especially those related to the module theory. And by selective cases we find a pattern to build a conjecture or a hypothesis, by deductive proof, we prove the conjecture and state it as a theorem. The main result in this study is the decomposition of the finitely generated module is a direct sum of the torsion submodule and torsion-free submodule. Since the torsion-free module is always a free module over a principal ideal domain, then the torsion-free submodule is a free module. Last, we generalize the ring, from a principal ideal domain, to a Dedekind domain. We found then the torsion-free submodule became a projective module. Then the decomposition of the finitely generated module is a direct sum of the torsion submodule and the projective submodule. These results should help the researchers to analyze the objects on module theory.

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A. INTRODUCTION

The decomposition of a vector space or matrices is a very important tool to many applied sciences. The regression method for whether prediction to data sciences (Sauri et al., 2021), a Markov chain in stochastic models (Fahmi & Rosli, 2021), and an intervention model have all used the decomposition of matrices or vector space (Utami et al., 2021). In pure mathematics, mathematicians want to know the decomposition of a more general algebraic structure of vector space, named module. The decomposition of a module help mathematician to find the characterizations of an algebraic structure of the module. Previously the study of primeness in submodules such as cyclic prime characterization (Juliana et al., 2021), the characterizations of a prime submodule of matrices (Wardhana et al., 2018), the characterizations of almost prime submodule on a free module (Wardhana et al., 2016), and the characterizations of almost prime submodule on integer modulo module (Wardhana & Astuti, 2014) are all using the decompositions of a finitely generated module over principal ideal domain., , Other than the primeness of a submodules, some study using the decompositions on other area such as, the CMS module characterization (Wardhana et al., 2021), U-complex module (Elfiyanti et al., 2020), F-CS-Rickart modules (Kaewwangsakoon & Pianskool, 2020), Injectivity & Projectivity of a module (Hijriati et al., 2018), uniserial module (Fitriani et al., 2021), linear codes (Irwansyah & Suprijanto, 2018), T-small modules (Sangwirotjanapat & Pianskool, 2018), the characteristics of a weakly prime submodule (Steven & Irawati, 2018), or the characterization of Bezout module (Ali Misri et al., 2013) and generalized Bezout module (Misri et al., 2016). In this study, we will generalize the decomposition, specifically from the principal ideal domain into the Dedekind domain.

In other algebraic structures, there is some study that used the decomposition of a ring. The latest study was to find the characterization of the prime cyclic ideal on Gaussian integer modulo (Misuki et al., 2021) and the characterization of an almost prime ideal on Gaussian ring modulo (Maulana et al., 2019). In this article, we will give some decomposition of a finitely generated module over some special rings, named principal ideal domain and Dedekind domain. As a ring, principal ideal domain very popular ring, in another way, the study of Dedekind domain is a kind rare. But some studies around this ring itself can be found, like by Kusniyanti in 2016 (Kusniyanti et al., 2016). That's why the Decompositions finitely generated module over the Dedekind domain is important.

B. METHODS

This study is based on the characterizations of cyclic prime submodule, and we generalize it into a more advanced module, a finitely generated module (Juliana et al., 2021). This research was using deductive proof based on a literature study in the form of books and scientific journals, especially those related to the module theory. The initial study conducted was to study the cyclic decomposition of the finitely generated module over a principal ideal domain. Then discussed the form of the decomposition of a finitely generated module over some special ring. The special rings consist of two popular rings, which are the principal ideal domain and the Dedekind domain.

C. RESULT AND DISCUSSION

1. The Decomposition Finitely Generated Module over Principal Ideal Domain

First, we will give the decompositions of a finitely generated module over a principal ideal domain. The basic definitions are bellowed.

Definition 1.1. Let M a R-module, and $m \in M$ is called torsion element if there exists non-zero element $r \in R$ such that $rm = 0$. If all elements of M are torsion, then we called M a torsion module. If there are no torsion elements in M, then we called M a torsion-free module.

For a R-module M, all torsion elements of M are denoted by M_{tor}. For example, all torsion elements of $\mathbb{Z}$-module $\mathbb{Z}_4$ and $\mathbb{Z}$-module $\mathbb{Z}$ is $\{0\}$ and $\mathbb{Z}$, then $\mathbb{Z}_4$ is a torsion module and $\mathbb{Z}$ is a torsion-free module.

If m is the torsion element of a module, then all scalar that makes m zero we called the annihilator of m.

Definition 1.2. Let M an R-module, and $x \in M$, then the annihilator of x is $ann(x) = \{r\ di\ R | r\ x = 0\}$, and the annihilator of M is $ann(M) = \{r\ di\ R | r\ M = 0\}$.

Some basic properties from the definitions of annihilator and torsion elements are given by these theorems.

Theorem 1.1. Let M be a module over principal ideal domain R, and $x \in M$. The annihilator $ann(x)$ and $ann(M)$ is an ideal of R, and the generator of $ann(x)$ called order of x, and the generator of $ann(M)$ called order of M.

Proof. See reference (Wardhana et al., 2016)

Teorema 1.2. Let M be a module over principal ideal domain R. The set M_{tor} form a submodule of M.

Proof. See reference (Wardhana et al., 2016)

We call M_{tor} as a torsion submodule of M. There is a submodule that the opposite side of the torsion submodule called the torsion-free submodule.

Definition 1.3. Let M be a module over principal ideal domain R. We called M a torsion-free module if zero is the only torsion element of M.

When we have a torsion submodule, we have the property that the factor module of the module must be torsion-free.

Theorem 1.3. Let M be a module over principal ideal domain R. The module kousien M/M_{tor} is torsion-free.

Proof:

Let $x + M_{tor} \in M/M_{tor}$ is non-zero. If $x + M_{tor}$ a torsion element, then there exist $r \neq 0$ such that $r(x + M_{tor}) = (0 + M_{tor})$. Then we have rx is torsion element, or there exist $k \neq 0$ such that $k(rx) = (kr)x = 0$. This means there exist $kr \in R$ where $kr \neq 0$ such that $(kr)x = 0$, hence x is torsion. We have $x + M_{tor} = 0 + M_{tor}$ (contradiction). So $x + M_{tor}$ not a torsion element, then M/M_{tor} is torsion-free.■

A fascinating result, that the torsion-free submodule must be a free module whenever the ring is principal.

Teorema 1.4. If M finitely generated torsion-free module over principal ideal domain R, then M is free.

Proof. Let M generated by $X = \{x_1, x_2, \dots, x_n\}$. If X is linear-independent then clearly M is a free module, so let assume that X is linear-dependent.

Let P is a collection of the linear-independent subset of X. Clearly $P \neq \emptyset$ since one non-zero torsion element must be linear-independent. Then $(P, \subset)$ is a partially ordered set. Let B a chain of P, Let G is a union set of all sets of B, then G is an upper bound of B. Since B is a chain, then G must be linear-independent.

According to Lema Zorn, P has maksimal element, named Y. Let $Y = \{x_1, x_2, \dots, x_r\}$ with $r < n$. Hence x_j is a linear combination of Y for all $j > r$, since if not then $Y \cup \{x_j\}$ is linear-independent (contradict to Y is the maximal element of P). Then Y also a generator of M.

Hence M is free. ■

And the decomposition of the finitely generated module over the principal ideal is bellowed.

Theorem 1.5. If M is a finitely generated module over principal ideal domain R, then

$$M = M_{tor} \oplus M_{free}$$

Where M_{free} is a free submodule of M and M_{tor} is torsion submodule of M.

Proof. Take natural epimorfism f from M to M/M_{tor}, based on Theorem 1.3 and Theorem 1.4, we have M/M_{tor} is a free module. Let $B' = \{b_1', b_2', \dots, b_n'\}$ bases for M/M_{tor}, with $b_i \in M$ such that $f(b_i) = b_i'$ for all i. It is clear that $Ker(f) = M_{tor}$.

We will show that $B = \{b_1, b_2, \dots, b_n\}$ is linier independent. If $k_1b_1 + k_2b_2 + \cdots + k_nb_n = 0_M$ then $k_1b_1' + k_2b_2' + \cdots + k_nb_n' = 0_{M/M_{tor}}$. Since B' is base, then $k_i = 0$ for all i. Then B is linear independent. It is clear that $spanB$ is a free submodule of M, let $M_{free} = spanB$.

Let $a \in M_{free} \cap Ker(f)$, then $a = c_1b_1 + \cdots + c_nb_n$ and $f(a) = 0$, hence $c_1b_1' + \cdots + c_nb_n' = 0$. So we have $c_1 = \cdots = c_n = 0$, then $M_{free} \cap Ker(f) = \{0\}$.

Let $x \in M$, then we have $f(x) = m_1b_1' + \cdots + m_nb_n'$ hence $x - m_1b_1 - \cdots - m_nb_n \in Ker(f)$. Then we have $x = y + m_1b_1 + \cdots + m_nb_n$ for $y \in Ker(f)$, and $m_1b_1 + \cdots + m_nb_n$ di M_{free}. Hence $M = M_{free} \oplus M_{tor}$.■

2. The Decomposition Finitely Generated Module over Dedekind Domain

Second, we will give the decompositions of a finitely generated module over a Dedekind domain. The basic definitions are given bellow

Definition 2.1. A module P over ring R called projektife module if for every epimorfism $\alpha: M \to N$, and homomorfism $\beta: P \to N$, there exist a homomorfism $\gamma: P \to A$ such that $\alpha\gamma = \beta$.

Definition 2.2. Let D is an integral domain if every ideal of D is a projective module over D, we called D the Dedekind domain.

Whenever the ring is a principal ideal domain, every torsion-free submodule must be a free module. Since the ring is the Dedekind domain, which is a more general form of a principal ideal domain, we have a slightly different result.

Theorem 2.1. Every free module is projective.

Proof. Let A, B be an R-modul and P a free R-module. And let $\alpha: A \to B$ be an abritary epimorfism. If $\beta: P \to B$ is a homomorfism, we will show there exist a homomorfism γ such that $\alpha\gamma = \beta$.

Let X be a base of P, since α epimorfism, then $\forall x \in X, \exists a_x \in A$ such that $\alpha(a_x) = \beta(x)$. Then we have a map $\varphi: X \to A$ such that $\varphi(xa) = a_x$. For every $p \in P, p = \sum_{x \in X} \lambda_x x$ for $\lambda_x \in R$ that not all zero. Now we can construct a linking $\gamma: P \to A$ such that $\gamma\left(\sum_{x \in X} \lambda_x xa\right) = \sum_{x \in X} \lambda_x \varphi(x)$

Then γ is a homomorphism, next, we want to show that $\alpha\gamma(p) = \alpha(\gamma(p))$. We have $\alpha(\gamma(p)) = \alpha\left(\gamma\left(\sum_{x \in X} \lambda_x x\right)\right) = \alpha\left(\sum_{x \in X} \lambda_x \varphi(x)\right) = \alpha\left(\sum_{x \in X} \lambda_x a_x\right) = \sum_{x \in X} \alpha(\lambda_x a_x) = \sum_{x \in X} \lambda_x \alpha(a_x) = \sum_{x \in X} \lambda_x \beta(x) = \beta \sum_{x \in X} \lambda_x x$. Then there exist homomorfism $\gamma : P \to A$ such that $\alpha\gamma = \beta$. Hence P is projective.■

Theorem 2.2 Let $\varphi : M \to P$ is an onto maps where P is projective, then there exists a homomorphism $\phi : P \to M$ such that $\varphi\phi$ is identity map in P and $M = \phi(P) \oplus ker(\varphi)$.

Proof. Let $I : P \to P$ be an identity map. Since φ onto and P projective, then there exists a homomorphism $\alpha : P \to M$ such that $\varphi\alpha = I$. If $x \in P$ and $\alpha(x) = 0$, or in general $\alpha(x) \in$

$ker(\varphi)$, then $0 = \varphi(\alpha(x)) = I(x) = x$. Hence α is surjective. This show us that if $\alpha(x) \in \ker(\varphi)$ then $x = 0$ and $\alpha(x) = 0$, in others word $\alpha(P) \cap \ker(\varphi) = 0$.

Now let $m \in M$, then we have $\varphi\left(m - \alpha(\varphi(m))\right) = \varphi(m) - i(\varphi(m)) = 0$. Hence $M = \alpha(\varphi(m)) + \left(m - \alpha(\varphi(m))\right) \in \alpha(P) + \ker(\varphi)$, then we have $M = \alpha(P) + \ker(\varphi)$. ■

Since P is isomorph to a submodule of M, for convenience, it is safe to call P as a submodule of M.

Theorem 2.3. Let M a R-module. Then there exist R-module free F with submodule G such that $M \approx F/G$

Proof. Let $\{a_\delta | \delta \in I\}$ be the generator of M. Choose free module F with base $\{b_\delta | \delta \in I\}$. Then we have a homomorfism $\eta: F \to M$ with η: $\Sigma\, k_a b_a \to \Sigma\, k_a a_a$. Since $\{a_a\}$ is generated M then η is onto. According to the isomorphism theorem, then $M \approx F/G$, where $G = ker(\eta)$.■

One of the important results is, any projective module must be a submodule of a free module.

Theorem 2.4. Let M be a module over ring R. Module M is projective if and only if M submodule of a free module.

Proof. Let F be a free module such that $F = M \oplus D$. Let $\alpha: A \to B$ be an epimorphism and $\beta: M \to B$ be a homomorphism. Clearly $\gamma: F \to B$ with $\gamma(f) = \gamma(m + d) = \beta(m)$ is a homomorfism since $f = m + d$ unique and β a homomorfism. According to Theorem 2.1, F is projective, hence we have a homomorfism $\gamma': F \to A$ such that $\alpha\gamma' = \gamma$. Let β' be a restriction of γ' in M, then $\alpha\beta' = \beta$. Then M is projective.

Let M be projective, let $\{a_\delta | \delta \in I\}$ be its generator. Choose free module F with base $\{b_\delta | \delta \in I\}$. Let construct a homomorphism from F to M that map $\sum k_a b_a$ to $\sum k_a a_a$, clearly, this homomorphism is onto since $\{a_\delta | \delta \in I\}$ generated M. Hence according to Theorem 2.2, M is a submodule of F. ■

Theorem 2.5. Let P a R-module, then this statement is equivalent

1. P projective
2. Every exact sequence $0 \to M \to B \to P \to 0$ is split.

Proof. According to Theorem 2.4 then P is isomorph to a quotient of a free module, named F/G, then we have exact sequence $0 \to G \to F \to P \to 0$. By assumption, $0 \to G \to F \to P \to 0$ is split, hence P is a component of the direct sum of F, according to Theorem 2.3, then P is projective.

Let P be projective, and $0 \to M \to B \to P \to 0$ is an exact sequence. According to Theorem 2.2, then this sequence is split. ■

From the previous result, we know that a torsion-free module is a free module whenever the ring is a principal ideal domain. But if the ring is not a principal ideal domain, it is no guarantee, hence the decomposition finitely generated module cannot be the same. We will give the decomposition of a finitely generated module if the rings are a more general case, which is a Dedekind domain.

Theorem 2.6. Let M a finitely generated module over Dedekind domain D, then M is projective if and only if M torsion-free.

Proof. Let M finitely generated torsion-free module over Dedekind domain D. Let K be the quotient field of D. Since M isomorphic to $M \otimes_D D$ then we have embedded from M to $M \otimes_D D$.

And we have $M \otimes_D K$is a K-module, and clearly $M \otimes_D K$ is a vector space, hence $M \otimes_D K$ isomorphic to K^r for some natural number $r \geq 0$. Let M is generated by $u_1, u_2, \ldots, u_r$ and f is embendding from M to $M \otimes_D K$. Let $f(u_i) = u_i \otimes p_i/q_i$ and $c = q_1 \ldots q_r$, then cM isomorphic to submodule D^r. Since M is torsion-free, and c is non-zero, then M is isomorphic to cM, hence M isomorphic to a submodule of a free module. Then according to Theorem 2.4, M is a projective module.

Now let M a projective module. According to Theorem 2.4 M a submodule of a free module, hence M is torsion-free. ■

And we now can state the main result of the decompositions of a finitely generated module over the Dedekind domain

Theorem 2.7. Let M a finitely generated module over Dedekind domain D, then $M = tM \oplus P$ where P is a projective submodule of M and tM is M_{tor}.

Proof. Let $0 \to tM \to M \to M/tM \to 0$ be an exact sequence. Since M/tM is torsion-free, then M/tM is projective according to Theorem 2.6. By Theorem 2.5, this sequence is split, the tM is a component of a direct sum of M. Hence $M = tM \oplus P$ for some P, since P isomorphic to M/tM, P is torsion-free, then P is projective. ■

This theorem said that the change on a ring on the module will change its decompositions, in this case, when the ring change from the principal ideal domain into the Dedekind domain, we have a different kind of decomposition.

D. CONCLUSION AND SUGGESTIONS

The decompositions of a finitely generated module is depending on the ring. Whenever the ring is a principal ideal domain, we can always decompose the module into its free submodule and torsion submodule. When the ring is more general, we can decompose the module into free submodule and projective submodule. It is normal to have a question about the decompositions of the module whenever the rings are more general form, like a Noetherian ring or Artinian ring.

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