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SUPERCHARACTERS AND SUPERCLASSES OF CERTAIN ABELIAN GROUPS

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Abstract. Supercharacter theory is developed by P. Diaconis and I. M. Isaacs as a natural generalization of the classical ordinary character theory. Some classical sums of number theory appear as supercharacters which are obtained by the action of certain subgroups of $GL_d(\mathbb{Z}_n)$ on $\mathbb{Z}_n^d$. In this paper we take $\mathbb{Z}_p^d$, p prime, and by the action of certain subgroups of $GL_d(\mathbb{Z}_p)$ we find supercharacter table of $\mathbb{Z}_p^d$.

Key words and Phrases: Supercharacter, Superclass, Ramanujan sum, Kloosterman sum, Character table.

1. Introduction

Let $Irr(G)$ denote the set of all the irreducible complex characters of a finite group G, and let $Con(G)$ denote the set of all the conjugacy classes of G. The identity element of G is denoted by 1 and the trivial character is denoted by 1_G. By definition a supercharacter theory for G is a pair $(\mathcal{X}, \mathcal{K})$ where $\mathcal{X}$ and $\mathcal{K}$ are partitions of $Irr(G)$ and G respectively, $|\mathcal{X}| = |\mathcal{K}|$, $\{1\} \in \mathcal{K}$, and for each $X \in \mathcal{X}$ there is a character σ_X such that $\sigma_X(x) = \sigma_X(y)$ for all $x, y \in K$, $K \in \mathcal{K}$. We call σ_X as supercharacter and each member of $\mathcal{K}$ superclass. We write $Sup(G)$ for the set of all the supercharacter theories of G.

Supercharacter theory of a finite group were defined by Diaconis and Isaacs [3] as a general case of the ordinary character theory. In fact, in a supercharacter theory, characters play the role of irreducible ordinary characters and union of conjugacy classes play the role of conjugacy classes. In [3] it is shown that $\{1_G\} \in \mathcal{X}$

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and if $X \in \mathcal{X}$ then σ_X is a constant multiple of $\sum_{\chi \in X} \chi(1)\chi$, and that we may assume that

$$\sigma_X = \sum_{\chi \in X} \chi(1)\chi.$$

For any finite group, there are two trivial supercharacter theories as follows. In the first case, $\mathcal{X} = \bigcup_{\chi \in Irr(G)} \{\chi\}$ and $\mathcal{K}$ is the set of all conjugacy classes of G. In the second case,

$$\mathcal{X} = \{1_G\} \cup \{Irr(G) - \{1_G\}\}$$

and $\mathcal{K} = \{1\} \cup \{G - \{1\}\}$. In the first case, supercharacters are just irreducible characters and superclasses are conjugacy classes. In the second case, the non-trivial supercharacter is $\rho_G - 1_G$, where ρ_G denotes the regular character of G. These two supercharacter theories of G are denoted by $m(G)$ and $M(G)$ respectively.

It is mentioned in [6] that the set of supercharacter theories of a group form a lattice in the following natural way. $Sup(G)$ can be made to a poset by defining $(\mathcal{X}, \mathcal{K}) \leq (\mathcal{Y}, \mathcal{L})$ if $\mathcal{X} \leq \mathcal{Y}$ in the sense that every part of $\mathcal{X}$ is a subset of some part of $\mathcal{Y}$. In [6] it is shown that this definition is equivalent to $(\mathcal{X}, \mathcal{K}) \leq (\mathcal{Y}, \mathcal{L})$ if $\mathcal{K} \leq \mathcal{L}$. By this definition, $m(G)$ is the least and $M(G)$ is the largest element of $Sup(G)$.

Among construction of supercharacter theories of a finite group G, the following is of great importance which is a lemma by Brauer on character tables of groups. Let A be a subgroup of $\mathbb{A}ut(G)$ and

$$Irr(G) = \{\chi_1 = 1_G, \ldots, \chi_h\}$$
$$Con(G) = \{\mathcal{C}_1 = \{1\}, \ldots, \mathcal{C}_h\}.$$

Suppose that for each $\alpha \in A$, ${\mathcal{C}_i}^\alpha = \mathcal{C}_j$, $1 \leq i \leq h$, and ${\chi_i}^\alpha(g) = \chi_i(g^\alpha)$ for all $g \in G$, $\alpha \in A$. Then the number of conjugacy classes fixed by α equals the number of irreducible characters fixed by α, and more over the number of orbits of A on $Con(G)$ equals the number of orbits of A on $Irr(G)$, [4]. It is easy to see that the orbits of A on $Irr(G)$ and $Con(G)$ yield a supercharacter theory for G. This supercharacter theory of G is called automorphic. In [7] it is shown that all the supercharacter theories of the cyclic group of order p, p prime, are automorphic.

Another aspect of the supercharacter theory of finite groups is to employ the theory to the group $U_n(F)$, the group of $n \times n$ unimodular upper triangular matrices over the Galois field $GF(p^m)$, p prime. Computation of the conjugacy classes and irreducible characters of $U_n(F)$ is still open, but in [1] the author has developed an applicable supercharacter theory for $U_n(F)$. This result is reviewed in [3].

2. Supercharacter table

Let G be a finite group and $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory for G. Suppose

$$\mathcal{X} = \{X_1, X_2, \dots, X_h\}$$

be a partition for $Irr(G)$ with the corresponding supercharacter $\sigma_i = \sum_{\chi \in X_i} \chi(1)\chi$. Let $\mathcal{K} = \{K_1, K_2, \dots, K_h\}$ be the partition of G into superclasses. In fact, $K_1 = \{1\}$ and $X_1 = \{1_G\}$ are union of conjugacy classes of G. The supercharacter table of G corresponding to $(\mathcal{X}, \mathcal{K})$ is the following $h \times h$ array:

Table I: Supercharacter table

	K_1	K_2	$\cdots$	K_j	$\cdots$	K_h
σ_1	$\sigma_1(K_1)$	$\sigma_1(K_2)$	$\cdots$	$\sigma_1(K_j)$	$\cdots$	$\sigma_1(K_h)$
σ_2	$\sigma_2(K_1)$	$\sigma_2(K_2)$	$\cdots$	$\sigma_2(K_j)$	$\cdots$	$\sigma_2(K_h)$
$\vdots$	$\vdots$	$\vdots$				$\vdots$
σ_i	$\sigma_i(K_1)$	$\sigma_i(K_2)$	$\cdots$	$\sigma_i(K_j)$	$\cdots$	$\sigma_i(K_h)$
$\vdots$	$\vdots$	$\vdots$				$\vdots$
σ_h	$\sigma_h(K_1)$	$\sigma_h(K_2)$	$\cdots$	$\sigma_h(K_j)$	$\cdots$	$\sigma_h(K_h)$

Let us set $S = (\sigma_i(K_j))_{i,j=1}^{h}$, and call it the supercharacter table of G.

Recall that a class function on G is a function $f : G \longrightarrow \mathbb{C}$ which is constant on conjugacy classes of G. The set of all the class functions on G, $Cf(G)$ has the structure of a vector space over $\mathbb{C}$ with an orthonormal basis $Irr(G)$ with respect the inner product

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}.$$

Since supercharacters are constant on superclasses, it is natural to call them superclass functions. We have:

$$\langle \sigma_i, \sigma_j \rangle = \frac{1}{|G|} \sum_{k=1}^{h} |K_k| \sigma_i(K_k) \overline{\sigma_j(K_k)}$$

But using the orthogonality of $Irr(G)$ we also can write:

$$\langle \sigma_i, \sigma_j \rangle = \langle \sum_{\chi \in X_i} \chi(1)\chi, \sum_{\varphi \in X_j} \varphi(1)\varphi \rangle = \delta_{ij} \sum_{\chi \in X_i} \chi(1)^2$$

Therefore,

$$\frac{1}{|G|} \sum_{k=1}^{h} |K_k| \sigma_i(K_k) \overline{\sigma_j(K_k)} = \delta_{ij} \sum_{\chi \in X_i} \chi(1)^2.$$

If we set the matrix

$$U = \frac{1}{\sqrt{|G|}} \left[\frac{\sigma_i(K_j)\sqrt{|K_j|}}{\sqrt{\sum_{\chi \in X_i} \chi(1)^2}} \right]_{i,j=1}^{h}$$

then wee see that U is a unitary matrix with the following properties, which are proved in [2]. $U = U^t$, $U^2 = P$ where P is a permutation matrix and $U^4 = I$.

In the course of studying the supercharacter theory of a group G, finding the supercharacter table of G and the matrix U is of great importance. In this paper, we will do this task for certain groups acting on certain sets.

3. Automorphic supercharacter table

In this section we follow the method used in [2] considering the group $G = \mathbb{Z}_n^d$ which is abelian of order n^d. The automorphism group of G is $GL_d(\mathbb{Z}_n)$, the group of $d \times d$ invertible matrices with entries in $\mathbb{Z}_n$. We write elements of G as row vectors $y = (y_1, \ldots, y_d)$ and let the action of $GL_d(\mathbb{Z}_n)$ on G be as follows:

$$y^A = yA \text{ for } A \in GL_d(\mathbb{Z}_n).$$

Irreducible characters of G are of degree 1 and the number of them is equal to $|G|$. For $x \in G$, let us define $\psi_x : G \longrightarrow \mathbb{C}^\times$, by $\psi_x(\zeta) = e\Big(\frac{x \cdot \zeta}{n}\Big)$, where $e(t)$ stands for $e(t) = e^{2\pi i t}$ and $x \cdot \zeta$ is the inner product of two elements x and ζ of G as row vectors in $G = \mathbb{Z}_n^d$. Therefore, $Irr(G) = \{\psi_x | x \in G\}$ and the action of $GL_d(\mathbb{Z}_n)$ on $Irr(G)$ is as follows:

$$\psi_x^A = \psi_{xA^{-t}} \text{ where } A \in GL_d(\mathbb{Z}_n),\ x \in G.$$

Now let Γ be a symmetric subgroup of $GL_d(\mathbb{Z}_n)$, i. e. $\Gamma^t = \Gamma$. Then Γ acts on G and on $Irr(G)$ as above. Let $\mathcal{X}$ be the set of orbits of Γ on $Irr(G)$ and $\mathcal{K}$ be the set of orbits of $GL_d(\mathbb{Z}_n)$ on $Irr(G)$. It is shown in [2] that $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of G. Following the notations used in [2] we identify ψ_x with x and $\psi_x^A = \psi_{xA^{-t}} = xA^{-t}$. Therefore, $\mathcal{X}$ is identified with the set of orbits of $GL_d(\mathbb{Z}_n)$ on G, by $x \longmapsto xA^{-t}$, and $\mathcal{K}$ is identified with the orbits of the action of $GL_d(\mathbb{Z}_n)$ on G by $y \longmapsto yA$.

In [2] using different subgroups of $GL_d(\mathbb{Z}_n)$ the authors provide supercharacter tables for G. For example, the discrete Fourier transform in the case of $\Gamma = \{1\}$, or $\Gamma = \{\pm 1\}$ a group of order 2 are obtain. The Gauss sum is obtained in the case of $G = \mathbb{Z}_p$, p an odd prime, $\Gamma = \langle g^2 \rangle$ where g is a primitive root modulo p. Kloosterman sum in the case $G = \mathbb{Z}_p^2$, p an odd prime and

$$\Gamma = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid 0 \neq a \in \mathbb{Z}_p \right\}.$$

Heilbronn sum, in the case of $G = \mathbb{Z}_p^2$ and $\Gamma = \{x^p \mid 0 \neq x \in \mathbb{Z}_p\}$. The Ramanujan sum in the case of $G = \mathbb{Z}_n$ and $\Gamma = \mathbb{Z}_n^\times$. It is worth mentioning that all the above sum appear as supercharacters.

As a generalization of the group Γ in Kloosterman sum we let

$$\Gamma = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z}_p^\times \right\}$$

a group of order $(p-1)^2$.

Theorem 3.1. *Under the action of Γ on $\mathbb{Z}_p \times \mathbb{Z}_p$ there are four supercharacter and four superclasses.*

Proof. Here $G = \mathbb{Z}_p \times \mathbb{Z}_p$ and orbits of Γ on G are:

$$\begin{array}{ll} Y_1 = \{(0,0)\} & \text{of size } 1 \\ Y_2 = (1,0)\Gamma = \{(a,0) \mid a \in \mathbb{Z}_p^\times\} & \text{of size } p-1 \\ Y_3 = (0,1)\Gamma = \{(0,b) \mid b \in \mathbb{Z}_p^\times\} & \text{of size } p-1 \\ Y_4 = (1,1)\Gamma = \{(a,b) \mid a,b \in \mathbb{Z}_p^\times\} & \text{of size } (p-1)^2 \end{array}$$

Orbits of Γ on $Irr(G)$ are as follows:

$$\begin{array}{ll} X_1 = \{(0,0)\} & \text{of size } 1 \\ X_2 = (1,0)\Gamma = \{(a,0) \mid a \in \mathbb{Z}_p^\times\} & \text{of size } p-1 \\ X_3 = (0,1)\Gamma = \{(0,b) \mid b \in \mathbb{Z}_p^\times\} & \text{of size } p-1 \\ X_4 = (1,1)\Gamma = \{(a,b) \mid a,b \in \mathbb{Z}_p^\times\} & \text{of size } (p-1)^2 \end{array}$$

Now we form the supercharacter table of $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Let σ_i be the supercharacter associated with X_i, with $\sigma_1 = 1$.

We know $\sigma_i = \sum_{\psi_{x_i} \in X_i} \psi_{x_i}$, and for $y \in Y_j$, $\sigma_i(y) = \sum_{\psi_{x_i} \in X_i} \psi_{x_i}(y) = \sum_{x_i \in X_i} e\Big(\frac{x_i \cdot y}{p}\Big)$, $1 \le i \le 4$. Therefore, the following table is calculated:

Table II: Supercharacter table of $\mathbb{Z}_p \times \mathbb{Z}_p$

$\mathbb{Z}_p \times \mathbb{Z}_p$	Y_1	Y_2	Y_3	Y_4
superclass size	1	$p-1$	$p-1$	$(p-1)^2$
σ_1	1	1	1	1
σ_2	$p-1$	-1	$p-1$	-1
σ_3	$p-1$	$p-1$	-1	-1
σ_4	$(p-1)^2$	$-(p-1)$	$-(p-1)$	$-(p-1)$

To find the unitary matrix U we use the formula written down in section 2 to obtain the 4×4 matrix U as follows:

$$U = \frac{1}{p}\begin{bmatrix} 1 & \sqrt{p-1} & \sqrt{p-1} & p-1 \\ \sqrt{p-1} & -1 & p-1 & -\sqrt{p-1} \\ \sqrt{p-1} & p-1 & -1 & -\sqrt{p-1} \\ p-1 & -\sqrt{p-1} & -\sqrt{p-1} & 1 \end{bmatrix}$$

□

At this point it is convenient to consider the general case of $G = \mathbb{Z}_p^d$,

$$\Gamma = \left\{ \begin{bmatrix} a_1 & & & & \\ & a_2 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & a_d \end{bmatrix} \mid a_i \in \mathbb{Z}_p^\times \right\}$$

the diagonal subgroup of order $(p-1)^d$ of $GL_d(\mathbb{Z}_p)$.

Theorem 3.2. *Under the action of* Γ *on* $\mathbb{Z}_p^d$ *there are* 2^d *supercharacters and superclasses.*

PROOF. Orbits of Γ on G are as follows:
$Y_1 = \{(0, 0, \ldots, 0)\}$ is one orbit. Let $y^{(k)} = (1^k, 0^{d-k})$ be a vector of G with k one's in different positions. Then $y^{(k)}\Gamma$ consists of vectors with non-zero entries in exactly k different positions. Therefore the orbit $y^{(k)}$ has size $(p-1)^k$. Since this k positions is taken out of d positions, we have $\binom{d}{k}$ orbits of this shapes each of size $(p-1)^k$. Hence we have $\sum_{k=0}^{d} \binom{d}{k} = 2^d$ orbits of Γ on G. Each orbit has size $(p-1)^k$. Since

$$\sum_{k=0}^{d} \binom{d}{k} (p-1)^k = p^d = |G|,$$

then all the orbits are counted.

Orbits of Γ on $Irr(G)$ have the same setting as above. In this case if ψ_x is a representative of the orbit X of Γ on $Irr(G)$, then we may assume

$$x = x^{(l)} = (1^l, 0^{d-l})$$

is a vector with l ones in different positions, hence:

$$\sigma_X(y) = \sum_{x \in X} \psi_x(y) = \sum_{x \in X} e\Big(\frac{x \cdot y}{p}\Big)$$

and it is computable if the inner product $x \cdot y$ is known.

4. J-Symmetric groups

Let $G = \mathbb{Z}_n^d$ and Γ be a subgroup of $GL_d(\mathbb{Z}_n)$. By [2] we have to assume that Γ is symmetric, i.e. $\Gamma = \Gamma^t$, in order to conclude that the action of Γ on G and on $Irr(G)$ generate the same orbits. Most of the results on supercharacter theory of G holds if we assume Γ is J-Symmetric. Suppose there is a fixed symmetric invertible matrix $J \in GL_d(\mathbb{Z}_n)$ such that $J\Gamma = \Gamma^t J$. As before the action of Γ on G is by $y \longmapsto yA$ and by identifying $\psi_x \in Irr(G)$ with x, the action of Γ on $Irr(G)$ is by $x \longmapsto xA^{-t}$ for $A \in \Gamma$.

If $(\mathcal{X}, \mathcal{Y})$ is the supercharacter theory obtained in this way, then we set

$$\mathcal{X} = \{X_1, X_2, \ldots, X_h\}$$
$$\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_h\}$$

and $\sigma_i = \sigma_{X_i}$, $1 \leq i \leq r$, the unitary matrix U is replaced by

$$U = \frac{1}{\sqrt{n^d}} \left[\frac{\sigma_i(Y_j)\sqrt{|Y_j|}}{\sqrt{|X_i|}} \right]_{i,j=1}^{h}.$$

In this section we consider $G = \mathbb{Z}_p^3$, p a prime, and

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Gamma = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & b \\ 0 & 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_p^{\times}, b, c \in \mathbb{Z}_p \right\}$$

is a subgroup of $GL_3(\mathbb{Z}_p)$ of order $p^2(p-1)^2$.

Theorem 4.1. *Under the action of* Γ *on* $\mathbb{Z}_p^3$ *there are four supercharacters and four superclasses.*

Proof. It is obvious that Γ is a J-Symmetric group.

Orbits of Γ on G are as follows:

$$\begin{aligned} Y_1 &= \{(0,0,0)\} \\ Y_2 &= (0,0,1)\Gamma = \{(0,0,a) \mid a \in \mathbb{Z}_p^{\times}\} \\ Y_3 &= (0,1,0)\Gamma = \{(0,d,b) \mid d \in \mathbb{Z}_p^{\times}, b \in \mathbb{Z}_p\} \\ Y_4 &= (1,0,0)\Gamma = \{(a,b,c) \mid a \in \mathbb{Z}_p^{\times}, b, c \in \mathbb{Z}_p\}. \end{aligned}$$

We have

$$\begin{aligned} |Y_1| &= 1 \\ |Y_2| &= p-1 \\ |Y_3| &= p(p-1) \\ |Y_4| &= p^2(p-1). \end{aligned}$$

Since

$$|Y_1| + |Y_2| + |Y_3| + |Y_4| = p^3$$

we deduce that Y_1, Y_2, Y_3 and Y_4 are orbits of Γ on G. It is easy to see that the orbits of Γ on $Irr(G)$ are as follows:

$$\begin{aligned} X_1 &= \{(0,0,0)\} \\ X_2 &= (1,0,0)\Gamma = \{(a,0,0) \mid a \in \mathbb{Z}_p^{\times}\} \\ X_3 &= (0,1,0)\Gamma = \{(a,b,0) \mid a \in \mathbb{Z}_p, b \in \mathbb{Z}_p^{\times}\} \\ X_4 &= (1,0,0)\Gamma = \{(a,b,c) \mid a, b \in \mathbb{Z}_p, c \in \mathbb{Z}_p^{\times}\}. \end{aligned}$$

We have

$$\begin{aligned} |X_1| &= 1 \\ |X_2| &= p-1 \\ |X_3| &= p(p-1) \\ |X_4| &= p^2(p-1). \end{aligned}$$

Let the supercharacter associated to X_i be σ_i. The following supercharacter table for the group G is constructed:

Table III: Supercharacter table of $\mathbb{Z}_p^3$

$\mathbb{Z}_p^3$ superclass size	Y_1 1	Y_2 $p-1$	Y_3 $p(p-1)$	Y_4 $p^2(p-1)$
σ_1	1	1	1	1
σ_2	$p-1$	$p-1$	$p-1$	-1
σ_3	$p(p-1)$	$p(p-1)$	$-p$	0
σ_4	$p^2(p-1)$	$-p^2$	0	0

The unitary table associated with the above table is:

$$U = \frac{1}{p\sqrt{p}} \begin{bmatrix} 1 & \sqrt{p-1} & \sqrt{p(p-1)} & p\sqrt{p-1} \\ \sqrt{p-1} & p-1 & (p-1)\sqrt{p} & -p \\ \sqrt{p(p-1} & (p-1)\sqrt{p} & -p & 0 \\ p\sqrt{p-1} & -p & 0 & 0 \end{bmatrix}$$

As general case let us consider $G = \mathbb{Z}_p^d$,

$$\Gamma = \left\{ \begin{bmatrix} 1 & a_2 & a_3 & \cdots & a_d \\ 0 & 1 & a_2 & \cdots & a_{d-1} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & a_2 \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \mid a_i \in \mathbb{Z}_p, 2 \leq i \leq d \right\}$$

which is J-symmetric with respect to the $d \times d$ matrix

$$J = \begin{bmatrix} & & & 1 \\ & \mathbf{0} & 1 & \\ & & \cdot & \\ & \cdot & & \\ \cdot & & \mathbf{0} & \\ 1 & & & \end{bmatrix}.$$

We have $|\Gamma| = p^{p-1}$ and it is a p-group.

Theorem 4.2. *Under the action of the J-symmetric group Γ on $\mathbb{Z}_p^d$ there are $1 + (p-1)d$ supercharacters and superclasses.*

PROOF. The orbits of Γ on $G = \mathbb{Z}_p^d$ are grouped as follows:

$$Y_1 = \{(0, 0, \ldots, 0)\}$$
$$Y_2 = (\alpha, 0, \ldots, 0)\Gamma = \{(\alpha, \alpha a_2, \alpha_3, \ldots, \alpha a_d) \mid a_i \in \mathbb{Z}_p\}, \alpha \in \mathbb{Z}_p^\times$$

Hence Y_2 is the union of $p-1$ orbits each of size p^{d-1}. Next,

$$Y_3 = (0, \alpha, 0, \ldots, 0)\Gamma = \{(0, \alpha, \alpha a_2, \alpha a_3, \ldots, \alpha a_{d-1}) \mid a_i \in \mathbb{Z}_p\}, \alpha \in \mathbb{Z}_p^\times.$$

Hence Y_3 is the union of $p-1$ orbits each of size p^{d-2}. If we continue in this way, we obtain

$$Y_d = (0, 0, \ldots, \alpha, 0)\Gamma = \{(0, 0, \ldots, \alpha, \alpha a_2) \mid a_2 \in \mathbb{Z}_p\}$$

has size p and is the union of $p-1$ orbits, and

$$Y_{d+1} = (0,0,\ldots,\alpha)\Gamma = \{(0,0,\ldots,\alpha)\}$$

is the union of $p-1$ orbits each of size 1.

Since $1+(p-1)(p^{d-1}+p^{d-2}+\ldots+1) = p^d = |G|$, all the orbits of Γ on G are counted. Therefore, there are $1+(p-1)d$ orbits.

To find the shapes of the orbits of Γ on $Irr(G)$, we mention that each irreducible character of G has degree 1. We have $Irr(G) = \{\psi_x \mid x \in G\}$ which may be represented by elements x of G under the action $x \longmapsto xA^{-t}$ where $A \in \Gamma$. Therefore, we obtain the following orbits:

$$\begin{aligned}
&X_1 = \{(0,0,\ldots,0)\}\\
&X_2 = (\alpha,0,\ldots,0)\Gamma = \{(\alpha,0,\ldots,0)\}\\
&X_3 = (0,\alpha,0,\ldots,0)\Gamma = \{(a_2,\alpha,0,\ldots,0) \mid a_2 \in \mathbb{Z}_p\}\\
&\vdots\\
&X_{d+1} = (0,0,\ldots,\alpha)\Gamma = \{(a_2,a_3,\ldots,a_d,\alpha) \mid a_i \in \mathbb{Z}_p\}.
\end{aligned}$$

Each set X_i, $2 \le i \le d+1$ is the union of $p-1$ orbits each of size p^{i-2}.

Now if ψ_x is a representative of the orbit X of Γ on $Irr(G)$, we may choose

$$x = (0,0,\ldots,0,\ldots,0) \in X_i.$$

Hence if σ_X is the supercharacter associated to X, then for $y \in Y_j$ we have

$$\sigma_X(y) = \sum_{x\in X} \psi_x(y) = \sum_{x\in X} e\Big(\frac{x\cdot y}{p}\Big).$$

Now, if x and y are taken from orbits such that $x\cdot y = 0$, then

$$\sum_{x\in X} e\Big(\frac{x\cdot y}{p}\Big) = |X| = p^{i-2}$$

provided $X = X_i$. Otherwise if $x\cdot y \neq 0$ then we obtain

$$\sum_{x\in X} e\Big(\frac{x\cdot y}{p}\Big) = 0.$$

In this way, the supercharacter table of G is computed.

Conclusion

There are many open problems concerning supercharacter theories of finite groups. For example, determining finite groups with exactly four supercharacter theories is still open. Know groups such as the dicyclic groups is still an area of research to find their supercharacters.

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