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SOME IDENTITIES INVOLVING MULTIPLICATIVE (GENERALIZED) $(\alpha, 1)$-DERIVATIONS IN SEMIPRIME RINGS

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Abstract. Let R be a semiprime ring, I a nonzero ideal of R and α be an automorphism of R. A map $F : R \longrightarrow R$ is said to be a multiplicative (generalized) $(\alpha, 1)$-derivation associated with a map $d : R \longrightarrow R$ such that $F(xy) = F(x)\alpha(y) + xd(y)$, for all $x, y \in R$. In the present paper, we shall prove that R contains a nonzero central ideal if any one of the following holds: $(i)\, F[x, y] \pm \alpha[x, y] = 0, (ii)\, F(x \circ y) \pm \alpha(x \circ y) = 0, (iii)\, F[x, y] = [F(x), y]_{\alpha,1}, (iv)\, F[x, y] = (F(x) \circ y)_{\alpha,1}, (v)\, F(x \circ y) = [F(x), y]_{\alpha,1}$ and $(vi)\, F(x \circ y) = (F(x) \circ y)_{\alpha,1}$, for all $x, y \in I$.

Key words and Phrases: Semiprime rings, Multiplicative (generalized) $(\alpha, 1)$-derivations, Ideal.

1. INTRODUCTION

Let R be an associative ring with center Z. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and symbol $x \circ y$ denotes for the anti-commutator $xy + yx$. Recall, a ring R is prime ring if $xRy = 0$ implies $x = 0$ or $y = 0$ and R is semiprime ring if $xRx = 0$ implies $x = 0$. Let α and β be automorphisms of R. For any $x, y \in R$, $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$. By considering $\beta = 1$, where 1 is an identity mapping on R, we have $[x, y]_{\alpha,1} = x\alpha(y) - yx$ and $(x \circ y)_{\alpha,1} = x\alpha(y) + yx$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The concept of a derivation was extended to generalized derivation by Bresar [2]. An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

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Inspired by the work of Martindale III [11], Daif [5] introduced the concept of multiplicative derivations. Accordingly, a map $d : R \longrightarrow R$ is called a multiplicative derivation of R if $d(xy) = d(x)y+xd(y)$ holds for all $x, y \in R$. Of course, these maps are not necessarily additive. Then the complete description of these maps was given by Goldman and Semrl [9]. Further, Daif and Tammam-El-Sayiad [7] extended the notion of multiplicative derivation to multiplicative generalized derivation of R if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d is derivation on R. Recently, the definition of multiplicative generalized derivation was extended to multiplicative (generalized)-derivation by Dhara and Ali [8] as follows: a map $F : R \longrightarrow R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d can be any map(not necessarily additive nor a derivation).

Chang [4] introduced the notion of a generalized(α, β)-derivation of a ring R and investigated some properties of such derivations. let α, β be mappings of R into itself. An additive mapping $F : R \longrightarrow R$ is called a generalized (α, β)-derivation of R such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$ where α and β are automorphisms on R. A mapping $F : R \longrightarrow R$ is said to be a multiplicative (generalized) (α, β)-derivation if there exists a map d on R such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$. Obviously every generalized (α, β)-derivation is a multiplicative (generalized) (α, β)-derivation. In 1992, Daif [6], proved a result that if R is a semiprime ring, I be a non-zero ideal of R and d is a derivation of R such that $d([x, y]) = \pm [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. Quadri [12] extended the result of Daif by replacing derivation d with a generalized derivation in a prime ring. Recently, shauliang [10] studied the identities related to generalized (α, β) derivation on prime rings. Asma Ali et al.[1] studied the identities related to multiplicative (generalized) (α, β)-derivations in semiprime rings. In this line of investigation, in the present paper we shall prove that R contains a non-zero central ideal if any one of the following holds: $(i)\, F[x, y] \pm \alpha[x, y] = 0, (ii)\, F(x \circ y) \pm \alpha(x \circ y) = 0, (iii)\, F[x, y] = [F(x), y]_{\alpha,1}, (iv)\, F[x, y] = (F(x) \circ y)_{\alpha,1}, (v)\, F(x \circ y) = [F(x), y]_{\alpha,1}, (vi)\, F(x \circ y) = (F(x) \circ y)_{\alpha,1}$, for all $x, y \in I$.

Throughout the present paper, we shall make use of the following basic identities without any specific mention:

(i) $[x, yz] = y[x, z] + [x, y]z$,
(ii) $[xy, z] = [x, z]y + x[y, z]$,
(iii) $x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$,
(iv) $xy \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$,
(v) $[xy, z]_{\alpha,1} = x[y, z]_{\alpha,1} + [x, z]y = x[y, \alpha(z)] + [x, z]_{\alpha,1}y$,
(vi) $[x, yz]_{\alpha,1} = y[x, z]_{\alpha,1} + [x, y]_{\alpha,1}\alpha(z)$,
(vii) $(x \circ (yz))_{\alpha,1} = (x \circ y)_{\alpha,1}\alpha(z) - y[x, z]_{\alpha,1} = y(x \circ z)_{\alpha,1} + [x, y]_{\alpha,1}\alpha(z)$,
(viii) $((xy) \circ z)_{\alpha,1} = x(y \circ z)_{\alpha,1} - [x, z]y = (x \circ z)_{\alpha,1}y + x[y, \alpha(z)]$.

2. MAIN RESULTS

In order to prove our main theorems, we shall need the following lemma.

Lemma 2.1. ([13, *Lemma* 2.1]) *Let R be a semiprime ring and I is a nonzero two sided ideal of R and $a \in R$ such that $axa = 0$ for all $x \in I$, then $a = 0$.*

Theorem 2.2. *Let R be a semiprime ring, I a nonzero ideal of R and α is an automorphism of R. Suppose that F is multiplicative (generalized) $(\alpha, 1)$-derivation on R associated with the map d on R. If $F[x, y] \pm \alpha[x, y] = 0$ holds for all $x, y \in I$, then d is commuting on I.*

PROOF. By the hypothesis, we have

$$F[x, y] \pm \alpha[x, y] = 0 \text{ for all } x, y \in I. \tag{2.1}$$

Replacing y by yx in (2.1), we obtain that

$$F([x, y]x) \pm \alpha([x, y]x) = 0 \text{ for all } x, y \in I,$$

and so

$$F([x, y])\alpha(x) + [x, y]d(x) \pm \alpha([x, y])\alpha(x) = 0 \text{ for all } x, y \in I.$$

Using the hypothesis, we obtain

$$[x, y]d(x) = 0 \text{ for all } x, y \in I. \tag{2.2}$$

Replacing y by ry in (2.2), we get

$$r[x, y]d(x) + [x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R.$$

Using (2.2), we obtain

$$[x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.3}$$

Replacing y by yx in (2.3), we get

$$[x, r]yxd(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.4}$$

Right multiplying (2.3) by x, we have

$$[x, r]yd(x)x = 0 \text{ for all } x, y \in I, r \in R. \tag{2.5}$$

Subtracting (2.4) from (2.5), we get

$$[x, r]y[x, d(x)] = 0 \text{ for all } x, y \in I, r \in R.$$

Replacing r by $d(x)$ in the last equation, we have

$$[x, d(x)]y[x, d(x)] = 0 \text{ for all } x, y \in I.$$

That is

$$[x, d(x)]I[x, d(x)] = 0 \text{ for all } x \in I.$$

By lemma 2.1, we conclude that $[x, d(x)] = 0$ for all $x \in I$. Therefore d is commuting on I.

Theorem 2.3. *Let R be a semiprime ring, I a nonzero ideal of R and α is an automorphism of R. Suppose that F is multiplicative (generalized) $(\alpha, 1)$-derivation on R associated with the map d. If $F(x \circ y) \pm \alpha(x \circ y) = 0$ holds for all $x, y \in I$, then d is commuting on I.*

Proof. By the hypothesis, we have

$$F(x \circ y) \pm \alpha(x \circ y) = 0 \text{ for all } x, y \in I. \tag{2.6}$$

Replacing y by yx in (2.6), we obtain that

$$F((x \circ y)x) \pm \alpha((x \circ y)x) = 0 \text{ for all } x, y \in I,$$

and so

$$F((x \circ y))\alpha(x) + (x \circ y)d(x) \pm \alpha((x \circ y))\alpha(x) = 0 \text{ for all } x, y \in I.$$

Using the hypothesis, we obtain

$$(x \circ y)d(x) = 0 \text{ for all } x, y \in I. \tag{2.7}$$

Replacing y by ry in (2.7), we find that

$$r(x \circ y)d(x) + [x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R.$$

Using (2.7), we obtain

$$[x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.8}$$

Using the same arguments as used in the proof of Theorem 2.2, we get the required result.

Theorem 2.4. *Let R be a semiprime ring, I a nonzero ideal of R and α is an automorphism of R. Suppose that F is multiplicative (generalized) $(\alpha, 1)$-derivation on R associated with the map d. If $F[x, y] = [F(x), y]_{\alpha,1}$ holds for all $x, y \in I$, then d is commuting on I.*

Proof. By the hypothesis, we have

$$F[x, y] = [F(x), y]_{\alpha,1} \text{ for all } x, y \in I. \tag{2.9}$$

Replacing y by yx in (2.9), we obtain that

$$F([x, y]x) = y[F(x), x]_{\alpha,1} + [F(x), y]_{\alpha,1}\alpha(x) \text{ for all } x, y \in I,$$

and so

$$F([x, y])\alpha(x) + [x, y]d(x) = y[F(x), x]_{\alpha,1} + [F(x), y]_{\alpha,1}\alpha(x) \text{ for all } x, y \in I.$$

Using the hypothesis, we obtain

$$[x, y]d(x) = y[F(x), x]_{\alpha,1} \text{ for all } x, y \in I. \tag{2.10}$$

Replacing y by ry in (2.10), we find that

$$r[x, y]d(x) + [x, r]yd(x) = ry[F(x), x]_{\alpha,1} \text{ for all } x, y \in I, r \in R.$$

Using (2.10), we get

$$[x, r]yd(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.11}$$

Using similar argument as used in the proof of Theorem 2.2, we get the required result.

Theorem 2.5. *Let R be a semiprime ring, I a nonzero ideal of R and α is an automorphism of R. Suppose that F is multiplicative (generalized) $(\alpha, 1)$-derivation on R associated with the map d. If $F[x,y] = (F(x) \circ y)_{\alpha,1}$ holds for all $x, y \in I$, then d is commuting on I.*

PROOF. By the hypothesis, we have

$$F[x,y] = (F(x) \circ y)_{\alpha,1} \text{ for all } x, y \in I. \tag{2.12}$$

Replacing y by yx in (2.12), we obtain that

$$F([x,y]x) = (F(x) \circ y)_{\alpha,1}\, \alpha(x) - y[F(x), x]_{\alpha,1} \text{ for all } x, y \in I$$

and so,

$$F([x,y])\, \alpha(x) + [x,y]\, d(x) = (F(x) \circ y)_{\alpha,1}\, \alpha(x) - y[F(x), x]_{\alpha,1} \text{ for all } x, y \in I.$$

Using the hypothesis, we obtain

$$[x,y]\, d(x) = -y[F(x), x]_{\alpha,1} \text{ for all } x, y \in I. \tag{2.13}$$

Replacing y by ry in (2.13), we get

$$r[x,y]\, d(x) + [x,r]\, yd(x) = -ry[F(x), x]_{\alpha,1} \text{ for all } x, y \in I, r \in R.$$

Using (2.13), we get

$$[x,r]\, yd(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.14}$$

Arguing in the similar manner as in Theorem 2.2, we get the result.

Theorem 2.6. *Let R be a semiprime ring, I a nonzero ideal of R and α is an automorphism of R. Suppose that F is multiplicative (generalized) $(\alpha, 1)$-derivation on R associated with the map d. If $F(x \circ y) = [F(x), y]_{\alpha,1}$ holds for all $x, y \in I$, then d is commuting on I.*

PROOF. By the hypothesis, we have

$$F(x \circ y) = [F(x), y]_{\alpha,1} \text{ for all } x, y \in I. \tag{2.15}$$

Replacing y by yx in (2.15), we obtain that

$$F((x \circ y)x) = y[F(x), x]_{\alpha,1} + [F(x), y]_{\alpha,1}\, \alpha(x) \text{ for all } x, y \in I,$$

and so

$$F((x \circ y))\, \alpha(x) + (x \circ y)\, d(x) = y[F(x), x]_{\alpha,1} + [F(x), y]_{\alpha,1}\, \alpha(x) \text{ for all } x, y \in I.$$

Using the hypothesis, we obtain

$$(x \circ y)\, d(x) = y[F(x), x]_{\alpha,1} \text{ for all } x, y \in I. \tag{2.16}$$

Replacing y by ry in (2.16), we get

$$r(x \circ y)\, d(x) + [x,r]\, yd(x) = ry[F(x), x]_{\alpha,1} \text{ for all } x, y \in I, r \in R.$$

Using (2.16), we have

$$[x, r]\, yd\,(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.17}$$

Arguing in the similar manner as in Theorem 2.2, we get the result.

Theorem 2.7. *Let R be a semiprime ring, I a nonzero ideal of R and α is an automorphism of R. Suppose that F is multiplicative (generalized) $(\alpha, 1)$-derivation on R associated with the map d. If $F\,(x \circ y) = (F\,(x) \circ y)_{\alpha,1}$ holds for all $x, y \in I$, then d is commuting on I.*

Proof. By the hypothesis, we have

$$F\,(x \circ y) = (F(x) \circ y)_{\alpha,1} \text{ for all } x, y \in I. \tag{2.18}$$

Replacing y by yx in (2.18), we obtain that

$$F\,((x \circ y)\, x) = (F(x) \circ y)_{\alpha,1}\, \alpha\,(x) - y\,[F(x), x]_{\alpha,1} \text{ for all } x, y \in I,$$

and so

$$F\,((x \circ y))\, \alpha\,(x) + (x \circ y)\, d\,(x) = (F(x) \circ y)_{\alpha,1}\, \alpha\,(x) - y\,[F(x), x]_{\alpha,1} \text{ for all } x, y \in I.$$

Using the hypothesis, we obtain

$$(x \circ y)\, d\,(x) = -y\,[F(x), x]_{\alpha,1} \text{ for all } x, y \in I. \tag{2.19}$$

Replacing y by ry in (2.19), we find that

$$r\,(x \circ y)\, d\,(x) + [x, r]\, yd\,(x) = -ry\,[F(x), x]_{\alpha,1} \text{ for all } x, y \in I, r \in R.$$

Using (2.19), we have

$$[x, r]\, yd\,(x) = 0 \text{ for all } x, y \in I, r \in R. \tag{2.20}$$

Arguing in the similar manner as in Theorem 2.2, we get the result.

Corollary 2.8. *Let R be a semiprime ring. Suppose that F, d is a multiplicative (generalized) $(\alpha, 1)$-derivation of R. If any one of the following holds:*

(i) $F\,[x, y] \pm \alpha\,[x, y] = 0$
(ii) $F\,(x \circ y) \pm \alpha\,(x \circ y) = 0$
(iii) $(iii)\, F\,[x, y] = [F(x), y]_{\alpha,1}$
(iv) $F\,[x, y] = (F(x) \circ y)_{\alpha,1}$
(v) $F\,(x \circ y) = [F(x), y]_{\alpha,1}$
(vi) $F\,(x \circ y) = (F(x) \circ y)_{\alpha,1}\, \forall x, y \in R$

then d is commuting on R.

3. Example

In this, we construct an example to the condition (i) of corollary 2.8 so that the semiprimeness condition of the ring is essential.

Example 1. Let $\mathbb{Z}$ be the set of integers and $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, $I = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Let us define $F, d, \alpha : R \longrightarrow R$ by F$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & c \end{pmatrix}$, d$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$, $\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$. It is easy to verify that I is an ideal on R, F is multiplicative (generalized) $(\alpha, 1)$-derivation associated with the map d, α is an automorphism on R. We see that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ R $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nonzero element of R. It implies that R is not semiprime.

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