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EXISTENCE AND UNIQUENESS RESULTS FOR SINGULAR BOUNDARY VALUE PROBLEM OF WEIGHTED GENERALIZED FISHER TYPE DIFFERENTIAL EQUATION

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Abstract. In this paper, we give existence and uniqueness results of nontrivial positive solution for the boundary value problem of the weighted and generalized Fisher's type differential equation.

Key words and Phrases: Generalized Fisher's equation, positive solution, fixed point.

1. Introduction

Because of their applications in biology, the study of Fisher's differential equations have received a great deal of attention during the latter two decades; see [11], [12], [13], [15], [16] and [14], and references therein. It was derived in 1937 and takes the form

$$\frac{\partial u}{\partial t} = u\,(1-u) + \frac{\partial^2 u}{\partial x^2}.$$

The generalized Fisher's equation .

$$\frac{\partial u}{\partial t} = u^{\alpha}\left(1-u^{\beta}\right) + \frac{\partial}{\partial x}\left(u^{m}\frac{du}{dx}\right)$$

describes one-dimensional diffusion models for insect and animal dispersal and invasion, where t is time, x is a spatial coordinate, u is a population density, $\frac{\partial u}{\partial t}$ represents the growth of a population and the factor u^m characterizes the diffusion process.

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In this paper, we consider the following singular boundary value problem of the weighted and generalized Fisher's type differential equation

$$\begin{cases} \frac{\partial}{\partial x}\left(h.u^m \frac{\partial u}{\partial x}\right) + a\left(x\right).\frac{\partial u}{\partial x} + b(x).u^\alpha\left(1-u^\beta\right) + f(x,u,u') = 0, \; x > 0, \\ u(0) = 0 = \lim_{x\to+\infty} u(x), \end{cases} \tag{1}$$

where $\alpha,\ \beta \geq 1,\ m \geq 0,\ h \in C\left(\mathbb{R}^+, \mathbb{R}^+\right),\ f \in C\left(\mathbb{R}^+ \times (0,+\infty) \times \mathbb{R}, \mathbb{R}\right),$ a and b are the measurable and positive functions such that for all $x \geq 0$, $h(x) \neq 0$

$$\int_0^{+\infty} \max\left\{\frac{1}{h\left(x\right)}, \frac{a\left(x\right)\gamma^{\frac{-m}{m+1}}\left(x\right)}{p(x)}, b\left(x\right)\right\} dx < \infty, \tag{2}$$

$$p\left(x\right) = \min\left\{h(x)\int_x^\infty \frac{ds}{h(s)}, h(x)\int_0^x \frac{ds}{h(s)}\right\}$$

and

$$\gamma(x) = \min\left\{\frac{\int_0^x \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}}, \frac{\int_x^\infty \frac{ds}{h(s)}}{\int_0^\infty \frac{ds}{h(s)}}\right\}.$$

By using the Krein Rutman Theorem [17] and the homotopy method in the fixed point index theory [1, 6, 7, 10], existence and uniqueness results for the problem (1) are given.

In all this paper, we assume that

$$\begin{cases} \text{there exists a continuous function } \; q : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that} \\ \int_0^{+\infty} q(t)dt < \infty \text{ and for all } (t,x,y) \in \mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^+ \setminus \{0\} \times \mathbb{R} \\ a(t).y + b(t).\left(x^\alpha - x^{\alpha+\beta}\right) + f\left(t,x,y\right) + q\left(t\right) \geq 0 \end{cases} \tag{3}$$

$$\begin{cases} \text{for all } r, R > 0 \text{ with } r \leq R, \text{ there exists a continuous function } g_{r,R} : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that} \\ \int_0^{+\infty} g_{r,R}(t)dt < \infty \;\text{ and} \\ \text{for all } \; (t,x,y) \in \mathbb{R}^+ \setminus \{0\} \times \left[\tilde{\gamma}\left(t\right)\left(r - w^*\right), R\right] \times \left[-R - w^*, R + w^*\right] \\ f\left(t, \rho\left(x\right), \rho'\left(x\right)\frac{y}{p(t)}\right) \leq g_{r,R}\left(t\right) \end{cases} \tag{4}$$

where

$$w^* = \sup_{x\geq 0}\left\{\frac{w(x)}{\tilde{\gamma}(x)}\right\}, \; \tilde{\gamma}(x) = \frac{1}{2}\gamma(x),$$

with

$$w(x) = \int_0^{+\infty} G(x,t)q(t)dt, \; \rho\left(x\right) = x^{\frac{1}{m+1}}$$

and

$$G\left(x,t\right) = \frac{(m+1)}{\int_0^{+\infty} \frac{ds}{h\left(s\right)}}.\begin{cases} \int_x^{+\infty} \frac{ds}{h\left(s\right)} \int_0^t \frac{ds}{h\left(s\right)} & x \geq t \\ \int_t^{+\infty} \frac{ds}{h\left(s\right)} \int_0^x \frac{ds}{h\left(s\right)} & x \leq t. \end{cases}$$

2. Preliminaries

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach space equipped with its norm noted $\|.\|$. A nonempty closed convex subset P of E is said to be a cone if $P \cap (-P) = 0$ and $(tP) \subset P$ for all $t \geq 0$. It is well known that a cone P induces a partial order in the Banach space E. We write for all $x; y{:}\in E$; $x \leq y$ if $y - x \in P$.

The mapping $L : E \to E$ is said to be positive in P if $L(P) \subset P$, and compact if it is continuous and $L(B)$ is relatively compact in E for all bounded subset B of E. The real value

$$r(L) = \sup\{|\lambda| : \ \lambda \in Sp(L)\}$$

denotes the spectral radius of a linear and bounded operator L, where $Sp(L)$ is the spectrum of L, and we have

$$r(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}.$$

The main tool of this work is the following Guo-Krasnoselskii's version of fixed point theorems in a Banach space [10] .

Theorem 2.1. *Let Ω_1, Ω_2 be open bounded subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ is a compact operator such that either:*

(1) $Tu \ngeq u$ for $u \in P \cap \partial\Omega_1$ and $Tu \nleq u$ for $u \in P \cap \partial\Omega_2$, or
(2) $Tu \nleq u$ for $u \in P \cap \partial\Omega_1$ and $Tu \ngeq u$ for $u \in P \cap \partial\Omega_2$,

Then T has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_2)$.

The following Krein Rutman Theorem has been established in [17]:

Theorem 2.2. *Let K be a cone in E and $L : E \to E$ be a linear, positive, and compact operator. Suppose that for some non-zero element $u \in K^*$, the following relation is satisfied:*

$$MLu \geq u, \text{ for some } M > 0.$$

Then L has a non-zero eigenvector $v \in K^*$:

$$Lv = \lambda_0 v,$$

where the positive eigenvalue λ_0 satisfies the inequality $\lambda_0 \geq M^{-1}$.

In what follows, we let E be a Banach space defined as

$$E = \left\{ u \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{x\to\infty} u(x) = \lim_{x\to\infty} p(x).u'(x) = 0 \right\}$$

equipped with the norm

$$\|u\| = \|u\|_0 + \|p.u'\|_0, \ u \in E$$

where $\|u\|_0 = \sup_{x\geq 0} |u(x)|$ and

$$p(x) = \min\left\{ h(x) \int_x^\infty \frac{ds}{h(s)}, h(x) \int_0^x \frac{ds}{h(s)} \right\}.$$

Lemma 2.3. [7] *A non empty subset M of E is relatively compact if the following conditions hold :*

(1) M is bounded in E,
(2) M is locally equicontinuous on $[0,+\infty)$, and
(3) M is equiconvergent at ∞

3. Related Lemmas

Let

$$E^{+}=\{u\in E: u\geq 0\},$$
$$P=\left\{u\in E^{+}: u(0)=0\right\}$$

and

$$K=\{u\in P: u\geq\tilde{\gamma}\,\|u\|\}$$

and

$$C=\{u\in K: p\,|u'|\leq u\}$$

be the cones in E.

For $r>0$, we consider the operator $T_r : K\backslash B(0,r)\rightarrow E$ defined by

$$T_r u(x)=\int_0^{+\infty} G(x,t)\,D(s,u,u')\,dt,$$

where for $(t,x,y)\in\mathbb{R}^{+}\times\mathbb{R}^{+}\backslash\{0\}\times\mathbb{R}$

$$D(t,x,y)=H(t,\rho(x-w),(y-w')\,\rho'(x-w))$$

with

$$H(t,x,y)=a(t).y+b(t).x^{\alpha}(1-x^{\beta})+f(t,x,y)+q(t)$$

and

$$\rho(x)=x^{\frac{1}{m+1}},\ \text{for all } x>0$$

and for $(t,x)\in\mathbb{R}^{+}\times\mathbb{R}^{+}$

$$G(x,t)=\frac{(m+1)}{\int_0^{+\infty}\frac{ds}{h(s)}}.\begin{cases}\int_x^{+\infty}\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)} & x\geq t\\ \int_t^{+\infty}\frac{ds}{h(s)}\int_0^x\frac{ds}{h(s)} & x\leq t.\end{cases}$$

We remark that T_r can be written as

$$T_r u(x)=\int_x^{+\infty}\frac{m+1}{h(t)}\left(-\pi(u)+\int_0^t D(s,u,u')\,ds\right)dt$$

where

$$\pi(u)=\frac{\int_0^{+\infty}\left(\frac{1}{h(t)}\int_0^t D(s,u,u')\,ds\right)dt}{\int_0^{+\infty}\frac{dt}{h(t)}}. \tag{5}$$

Lemma 3.1. *Let $w : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by*

$$w(x) = \int_0^{+\infty} G(x,t)\, q(t)dt.$$

Then

$$\sup_{t\geq 0} \frac{w(t)}{\tilde{\gamma}(t)} = w^* < \infty.$$

Proof. Let $x_0 > 0$ such that

$$\int_{x_0}^{+\infty} \frac{ds}{h(s)} = \int_0^{x_0} \frac{ds}{h(s)}.$$

For all $t \geq 0$,

$$\begin{aligned}
\frac{w(x)}{\tilde{\gamma}(x)} &= 2\begin{cases} \frac{\int_0^{+\infty}\frac{ds}{h(s)}}{\int_0^{x}\frac{ds}{h(s)}} \int_0^{+\infty} G(x,t)\, q(t)dt & \text{if } x \leq x_0 \\ \frac{\int_0^{+\infty}\frac{ds}{h(s)}}{\int_x^{+\infty}\frac{ds}{h(s)}} \int_0^{+\infty} G(x,t)\, q(t)dt & \text{if } x \geq x_0 \end{cases} \\
&\leq 2(m+1)\begin{cases} \int_x^{+\infty}\frac{ds}{h(s)} \int_0^{x} q(t)dt + \int_x^{+\infty}\left(\int_t^{+\infty}\frac{ds}{h(s)}\right) q(t)dt & \text{if } x \leq x_0 \\ \int_0^{x}\left(\int_0^{t}\frac{ds}{h(s)}\right) q(t)dt + \int_0^{x}\frac{ds}{h(s)} \int_x^{+\infty} q(t)dt & \text{if } x \geq x_0 \end{cases} \\
&\leq 2(m+1)\int_0^{+\infty}\frac{ds}{h(s)}\int_0^{+\infty} q(t)dt < \infty.
\end{aligned}$$

This completes the proof. □ □

Remark 3.2. *The positive function $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and for all $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$*

$$0 \leq G(x,t) \leq G(x,x) \leq g_1(x) = 2\tilde{\gamma}(x)\,.(m+1)\int_0^{+\infty}\frac{ds}{h(s)}.$$

Lemma 3.3. *The function G has the following properties:*

1: For all $x, \tau, t \in \mathbb{R}^+$

$$G(\tau,t) \geq \gamma(\tau)\,.G(x,t),$$

where

$$\gamma(\tau) = \min\left\{ \frac{\int_0^{\tau}\frac{ds}{h(s)}}{\int_0^{\infty}\frac{ds}{h(s)}}, \frac{\int_{\tau}^{\infty}\frac{ds}{h(s)}}{\int_0^{\infty}\frac{ds}{h(s)}} \right\}.$$

2: For all $t,\ x,\ y \in \mathbb{R}^+$,

$$|G(x,t) - G(y,t)| \leq c_0\,|x-y|$$

where $c_0 = (m+1)\sup\left\{\frac{1}{h(t)},\ t \geq 0\right\}$.

3: For all $x,\ t \in \mathbb{R}^+$,

$$p(x)\left|\frac{\partial}{\partial x}G(x,t)\right| \leq G(x,t).$$

Proof. 1. For $x,\tau,t\in\mathbb{R}^+$

$$\frac{G(\tau,t)}{G(x,t)}=\begin{cases}\dfrac{\int_\tau^{+\infty}\frac{ds}{h(s)}}{\int_x^{+\infty}\frac{ds}{h(s)}}\geq\dfrac{\int_\tau^{+\infty}\frac{ds}{h(s)}}{\int_0^{+\infty}\frac{ds}{h(s)}}\geq\gamma(\tau) & \text{if } \tau\geq t,\ x\geq t\\ \dfrac{\int_\tau^{+\infty}\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}}{\int_t^{+\infty}\frac{ds}{h(s)}\int_0^x\frac{ds}{h(s)}}\geq\dfrac{\int_\tau^{+\infty}\frac{ds}{h(s)}}{\int_0^{+\infty}\frac{ds}{h(s)}}\geq\gamma(\tau) & \text{if } \tau\geq t,\ x\leq t\\ \dfrac{\int_t^{+\infty}\frac{ds}{h(s)}\int_0^\tau\frac{ds}{h(s)}}{\int_x^{+\infty}\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}}\geq\dfrac{\int_0^\tau\frac{ds}{h(s)}}{\int_0^{\infty}\frac{ds}{h(s)}}\geq\gamma(\tau) & \text{if } \tau\leq t,\ x\geq t\\ \dfrac{\int_0^\tau\frac{ds}{h(s)}}{\int_0^x\frac{ds}{h(s)}}\geq\dfrac{\int_0^\tau\frac{ds}{h(s)}}{\int_0^{\infty}\frac{ds}{h(s)}}\geq\gamma(\tau) & \text{if } \tau\leq t,\ x\leq t\end{cases}$$

Consequently, $\frac{G(\tau,t)}{G(x,t)}\geq\gamma(\tau)$.

2. For $x,y,t\in\mathbb{R}^+$, assume that $x>y$. We have

$$|G(x,t)-G(y,t)|=\frac{(m+1)}{\int_0^\infty\frac{ds}{h(s)}}\begin{cases}\int_0^t\frac{ds}{h(s)}\left(\int_y^x\frac{ds}{h(s)}\right) & \text{if } x>y\geq t\\ \left|\int_x^\infty\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}-\int_0^y\frac{ds}{h(s)}\int_t^\infty\frac{ds}{h(s)}\right| & \text{if } x\geq t\geq y\\ \int_t^\infty\frac{ds}{h(s)}\left(\int_y^x\frac{ds}{h(s)}\right) & \text{if } t\geq x>y\end{cases}.$$

In the case $x\geq t\geq y$, if $\int_x^\infty\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}\geq\int_0^y\frac{ds}{h(s)}\int_t^\infty\frac{ds}{h(s)}$ then

$$\begin{aligned}|G(x,t)-G(y,t)| &= \int_x^\infty\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}-\int_0^y\frac{ds}{h(s)}\int_t^\infty\frac{ds}{h(s)}\\ &\leq \int_t^\infty\frac{ds}{h(s)}\left(\int_y^x\frac{ds}{h(s)}\right)\leq\int_0^\infty\frac{ds}{h(s)}\left(\int_y^x\frac{ds}{h(s)}\right)\end{aligned}$$

and if $\int_x^\infty\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}\leq\int_0^y\frac{ds}{h(s)}\int_t^\infty\frac{ds}{h(s)}$ then

$$\begin{aligned}|G(x,t)-G(y,t)| &= \int_0^y\frac{ds}{h(s)}\int_t^\infty\frac{ds}{h(s)}-\int_x^\infty\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}\\ &\leq \int_0^t\frac{ds}{h(s)}\left(\int_y^x\frac{ds}{h(s)}\right)\leq\int_0^\infty\frac{ds}{h(s)}\left(\int_y^x\frac{ds}{h(s)}\right)\end{aligned}$$

then for $x,y,t\in\mathbb{R}^+$

$$\begin{aligned}|G(x,t)-G(y,t)| &\leq (m+1)\int_y^x\frac{ds}{h(s)}\\ &\leq c_0\,|x-y|.\end{aligned}$$

3. For all $x,\, t\in\mathbb{R}^+$,

$$\frac{p(x)\left|\frac{\partial}{\partial x}G(x,t)\right|}{G(x,t)} = \begin{cases} \frac{p(x)}{h(x)\int_x^\infty \frac{ds}{h(s)}} & \text{if } x\geq t, \\ \frac{p(x)}{h(x)\int_0^x \frac{ds}{h(s)}} & \text{if } x\leq t, \end{cases}$$
$$\leq 1$$

thus

$$p(x)\left|\frac{\partial}{\partial x}G(x,t)\right|\leq G(x,t)$$

completing the proof. □ □

Let $r, R>0$, where $r\leq R$. We denote by $\Theta_{r,R}$ and $\Psi_{r,R}$ the functions defined in $\mathbb{R}^+$ as

$$\begin{cases} \Theta_{r,R}(t) = R\frac{a(t)}{p(t)} + \left(R^{\alpha}+R^{\alpha+\beta}\right)b(t)+g_{r,R}(t)+q(t), \\ \Psi_{r,R}(t) = \frac{a(t)}{p(t)}.\frac{R+w^*}{m+1}\left(\tilde{\gamma}\left(r-w^*\right)\right)^{\frac{-m}{m+1}}+b(t)\left(R^{\frac{\alpha}{m+1}}-\left(\tilde{\gamma}\left(r-w^*\right)\right)^{\frac{\alpha+\beta}{m+1}}\right) \\ \qquad +g_{r,R}(t)+q(t) \end{cases} \tag{6}$$

where $g_{r,R}$ is the function given in (4). We have from (2) that

$$\int_0^{+\infty}\Theta_{r,R}(t)dt<\infty \text{ and } \int_0^{+\infty}\Psi_{r,R}(t)dt<\infty.$$

Lemma 3.4. *Assume that Hypothesis (2), (3) and (4) hold true and let $R>r>w^*$. Then for all $u\in K\cap\partial B(0,r)$*

$$H(t,u,u')\leq\Theta_{r,r}(t), \text{ for all } t\geq 0$$

and for all $u\in K\cap\bar{B}(0,R)\setminus B(0,r)$

$$D(t,u,u')\leq\Psi_{r,R}(t), \text{ for all } t\geq 0.$$

Proof. Let $u\in K\setminus\{0\}$ and $r=\|u\|$. For $t>0$, we have $u(t)\geq\gamma r>0$, and from (4) it follows that for all $t>0$, $H(t,u,u')\geq 0$ and

$$\begin{aligned} H(t,u,u') &\leq a(t).u'(t)+b(t)\left(u^{\alpha}+u^{\alpha+\beta}\right)+f(t,u,u')+q(t) \\ &\leq \|u\|\frac{a(t)}{p(t)}+\left(\|u\|^{\alpha}+\|u\|^{\alpha+\beta}\right)b(t) \\ &\quad +g_{r,r}(t)+q(t) \\ &= \Theta_{r,r}(t). \end{aligned}$$

Now, for $v\in K\cap\bar{B}(0,R)\setminus B(0,r)$, let $u=v-w$.
We have from Assertion (3) of Lemma (3.3) that $p|w'|\leq w\leq w^*$ and then

$$0<\tilde{\gamma}.(r-w^*)\leq u\leq R$$

and

$$-(R+w^*)\leq p.u'\leq R+w^*.$$

Thus, for all $t > 0$

$$\begin{aligned} D(t,v,v') &= H\left(t,(v-w)^{\frac{1}{m+1}},\frac{(v'-w')}{m+1}(v-w)^{\frac{-m}{m+1}}\right) \\ &\le \frac{a(t)}{p(t)}.\frac{R+w^*}{m+1}\left(\tilde{\gamma}(r-w^*)\right)^{\frac{-m}{m+1}}+b(t)\left(R^{\frac{\alpha}{m+1}}-\left(\tilde{\gamma}(r-w^*)\right)^{\frac{\alpha+\beta}{m+1}}\right) \\ &\quad +g_{r,R}(t)+q(t) \\ &= \Psi_{r,R}(t). \end{aligned}$$

This completes the proof. □ □

Let $A_r : E^+ \to E$ be an operator defined as

$$A_r u(x) = \int_x^{+\infty} \frac{m+1}{h(t)}\left(-\pi(u)+\int_0^t g(s,u,u')\,ds\right)dt$$

where

$$\pi_g(u) = \frac{\int_0^{+\infty}\left(\frac{1}{h(t)}\int_0^t g(s,u,u')\,ds\right)dt}{\int_0^{+\infty}\frac{dt}{h(t)}} \tag{7}$$

and

$$g(s,u,u') = H(t,u,u') - q(t).$$

Remark 3.5. *We have from Lemma (3.4) that $g(.,u,u') \in L^1(\mathbb{R}^+)$ for all $u \in E^+\setminus\{0\}$.*

Lemma 3.6. *Assume that Hypothesis (2), (3) and (4) hold true and let $r \ge w^*$ and $v \in K\setminus B(0,r)$. If v is a fixed point of T_r then $u = (v-w)^{\frac{1}{m+1}}$ is a positive solution to the bvp (1).*

Proof. Let $r \ge w^*$ and assume that $v \in K\setminus B(0,r)$ is a fixed point of T_r. We have

$$v(x) = \int_0^{+\infty} G(x,t)\,D(t,v,v')\,dt$$

where for $t > 0$

$$D(t,v,v') = H\left(t,(v-w)^{\frac{1}{m+1}},\frac{(v'-w')}{m+1}(v-w)^{\frac{-m}{m+1}}\right).$$

Let $u = (v-w)^{\frac{1}{m+1}}$. We have for $x \ge 0$

$$\begin{aligned} u^{m+1}(x) &= v(x)-w(x) = \int_0^{+\infty} G(x,t)\left(H(t,u,u')-q(t)\right)dt \\ &= A_r u(x), \end{aligned}$$

where

$$A_r u(x) = \int_x^{+\infty}\frac{m+1}{h(t)}\left(\int_0^t g(s,u,u')\,ds - \pi_g(u)\right)dt$$

$$\pi_g(u) = \frac{\int_0^{+\infty} \left(\frac{1}{h(t)} \int_0^t g(s,u,u')\,ds \right) dt}{\int_0^{+\infty} \frac{dt}{h(t)}}$$

and

$$g(t,u,u') = a(t).u' + b(t)\,u^{\alpha}(1-u^{\beta}) + f(t,u,u').$$

We have from Lemma (3.4) that

$$g(.,u,u') \in L^1\left(\mathbb{R}^+\right).$$

Then

$$\pi_g(u) < \infty$$

and so

$$\int_0^{+\infty} \left| \frac{m+1}{h(t)} \left(\int_0^t g(s,u,u')\,ds - \pi_g(u) \right) \right| < \infty.$$

Furthermore, the relation

$$u^{m+1} = A_r u \tag{8}$$

is equivalent to

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{h(x)}{m+1}\frac{\partial}{\partial x}\left(u^{m+1}\right)\right) = \frac{\partial}{\partial x}\left(\frac{h(x)}{m+1}\frac{\partial}{\partial x}A_r u(x)\right) = -g(x,u,u'),\ x>0, \\ \lim_{x\to+\infty} u(x)^{m+1} = \lim_{x\to+\infty} A_r u(x) = 0. \text{ and} \\ u^{m+1}(0) = \int_0^{+\infty} \frac{m+1}{h(t)}\left(\int_0^t g(s,u,u')\,ds - \pi_g(u)\right) \\ \quad = \int_0^{+\infty} \frac{m+1}{h(t)}\left(\int_0^t g(s,u,u')\,ds - \pi_g(u)\right) = 0. \end{cases}$$

This completes the proof. □ □

Lemma 3.7. *Assume that Hypothesis (2) and (4) hold true. Let $r > w^*$. Then for all and $u \in K\backslash B(0,r)$, $T_r u \geq 0$ and for all $x \geq 0$*

$$p(x)\left|(T_r u)'(x)\right| \leq T_r u(x),$$

and

$$p(x)\left|(T_r u)'(x) - w'(x)\right| \leq \|T_r u\| + w^*,$$

where

$$p(x) = \min\left\{ h(x)\int_x^{\infty} \frac{ds}{h(s)}, h(x)\int_0^x \frac{ds}{h(s)} \right\}.$$

Proof. Let $u \in K\backslash B(0,r)$ and let v be a function defined as

$$v = T_r u = \int_0^{\infty} G(x,t)\,H\left(t, (u-w)^{\frac{1}{m+1}}, \frac{(u'-w')}{m+1}(u-w)^{\frac{-m}{m+1}}\right) dt.$$

we have from (4) that $D(t,u,u') \geq 0$ for all $t \geq 0$. For $x \geq 0$,

$$|v'(x)| = \left| \int_0^{\infty} \frac{\partial}{\partial x} G(x,t).D(t,u,u')\,dt \right| \leq \int_0^{\infty} \left| \frac{\partial}{\partial x} G(x,t) \right| D(t,u,u')\,dt$$

with

$$\frac{p(x)\left|\frac{\partial}{\partial x}G(x,t)\right|}{G(x,t)} = \begin{cases} \frac{p(x)}{h(x)\int_x^{\infty}\frac{ds}{h(s)}} & \text{if } x \geq t, \\ \frac{p(x)}{h(x)\int_0^{x}\frac{ds}{h(s)}} & \text{if } x \leq t, \end{cases}$$
$$\leq 1.$$

Hence, for all $x \geq 0$,

$$\begin{aligned} p(x)\left|v'(x)\right| &\leq p(x)\int_0^{\infty}\left|\frac{\partial}{\partial x}G(x,t)\right| D(t,u,u') \\ &\leq \int_0^{\infty} G(x,t)\, D(t,u,u') = v(x), \end{aligned}$$

and

$$\begin{aligned} p(x)\left|v'(x)-w'(x)\right| &\leq p(x)\left|v'(x)\right| + p(x)\left|w'(x)\right| \\ &\leq \|v\| + w(x) \\ &\leq \|v\| + w^*. \end{aligned}$$

□ □

Lemma 3.8. *Let $r > w^*$ and assume that Hypothesis (2) and (4) hold true. Then the operator $T_r : K\backslash B(0,r) \rightarrow C$ is compact.*

Proof. For $R \geq r$, let $M_{r,R} = T_r(\Omega_{r,R})$, with $\Omega_{r,R} = K \cap \bar{B}(0,R)\backslash B(0,r)$.
1. In fact we show that the set $M_{r,R}$ is a subset of.E.
We have $f \in C(\mathbb{R}^+ \times \mathbb{R}^+ \backslash \{0\} \times \mathbb{R}^+)$ and we see from continuity of the functions $\frac{1}{h}$ and G and so for

$$\frac{\partial}{\partial x}T_r u(x) = -\frac{m+1}{h(x)}\left(-\pi(u) + \int_0^x D(s,u,u')\,ds\right)$$

that $M_{r,R} \subset C^1(\mathbb{R}^+,\mathbb{R})$. Moreover, we have from from Hypothesis (3) that $D(.,u,u') \geq 0$, and from Assertion (3) of Lemma (3.3) that $p\left|w'\right| \leq w$, and from Lemma (3.4) follows that for all $u \in \Omega_{r,R}$

$$D(.,u,u') \leq \Psi_{r,R} \in L^1\left(\mathbb{R}^+\right)$$

where $\Psi_{r,R}$ is the integrable function given in (6).
For $x \in \mathbb{R}^+$

$$\begin{aligned} T_r(u)(x) &= \int_0^{+\infty} G(x,t)D(t,u,u')dt \\ &\leq G(x,x)\int_0^{+\infty} D(t,u,u')dt \\ &= \frac{m+1}{\int_0^{\infty}\frac{ds}{h(s)}}\int_x^{\infty}\frac{ds}{h(s)}\int_0^{x}\frac{ds}{h(s)}\int_0^{+\infty} D(t,u,u')dt \end{aligned}$$

and from Lemma (3.7) we have

$$p(x)\left|\frac{\partial}{\partial x}T_r(u)(x)\right| \leq T_r(u)(x), \text{ for all } x \geq 0$$

then

$$\lim_{x\to\infty} p(x)\left|\frac{\partial}{\partial x}T_r(u)(x)\right| = \lim_{x\to\infty} T_r(u)(x) = 0.$$

2. We show that $M_{r,R}$ is relatively compact.
In first, we show that the set $M_{r,R}$ is bounded. Let $u \in \Omega_{r,R}$ and let $\Psi_{r,R}$ be the integrable function given in (6). We have from Assertion (3) of Lemma (3.3) that $p|w'| \leq w$ then

$$(u-w, p(u'-w')) \in [\tilde{\gamma}(x)(r-w^*), r] \times [-(R+w^*), R+w^*].$$

By using Lemma (3.4) we have that for all $x \geq 0$

$$\begin{aligned} T_r(u)(x) &\leq G(x,x)\int_0^{+\infty} D(t,u,u')dt \\ &\leq (m+1)\int_0^{+\infty}\frac{ds}{h(s)}\int_0^{+\infty}\Psi_{r,R}(t)dt = N_{r,R} \end{aligned}$$

and from Lemma (3.7) we deduce that

$$\|T_r(u)\| \leq 2\|T_r(u)\|_0 \leq 2N_{r,R}$$

proving the boundeness of $M_{r,R}$.
Let $I = [\eta, \zeta]$ be a compact interval in $\mathbb{R}^+$ and let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}^+$, such that $t_1 < t_2$. For all $u \in \Omega$ we have

$$|T_r u(t_2) - T_r u(t_1)| \leq \int_0^{+\infty} |G(t_2,s) - G(t_1,s)|\,\Psi_{r,R}(s)ds$$

and by Assertion 2 of Lemma (3.3)

$$|T_r u(t_2) - T_r u(t_1)| \leq \left(c_0\int_0^{+\infty}\Psi_{r,R}(s)ds\right)|t_2 - t_1|,$$

where

$$c_0 = (m+1)\sup\left\{\frac{1}{h(t)},\ t \geq 0\right\}.$$

Moreover

$$\begin{aligned} \left|p(t_2)\frac{\partial}{\partial x}T_r u(t_2) - p(t_1)\frac{\partial}{\partial x}T_r u(t_1)\right| &\leq \int_0^{+\infty}\left|p(t_2)\frac{\partial}{\partial x}G(t_2,s) - p(t_1)\frac{\partial}{\partial x}G(t_1,s)\right|\Psi_{r,R}(s)ds \\ &\leq |p(t_2)-p(t_1)|\int_0^{+\infty}\left|\frac{\partial}{\partial x}G(t_1,s)\right|\Psi_{r,R}(s)ds \\ &\quad + p(t_2)\int_0^{+\infty}\left|\frac{\partial}{\partial x}G(t_2,s) - \frac{\partial}{\partial x}G(t_1,s)\right|\Psi_{r,R}(s)ds. \end{aligned}$$

As

$$\left|\frac{\partial}{\partial x}G(t_2,s)-\frac{\partial}{\partial x}G(t_1,s)\right| = \frac{m+1}{\int_0^{+\infty}\frac{d\tau}{h(\tau)}}\begin{cases}\left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right|\int_0^s\frac{d\tau}{h(\tau)} & t_2>t_1\geq s\\ \frac{1}{h(t_2)}\int_0^s\frac{ds}{h(s)}+\frac{1}{h(t_1)}\int_s^{+\infty}\frac{d\tau}{h(\tau)} & t_1\leq s\leq t_2\\ \left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right|\int_s^{+\infty}\frac{d\tau}{h(\tau)} & t_1<t_2\leq s\end{cases}$$

$$\leq (m+1)\begin{cases}\left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right| & t_2>t_1\geq s \text{ or } t_1<t_2\leq s\\ \frac{1}{h(t_2)}+\frac{1}{h(t_1)} & t_1\leq s\leq t_2\end{cases}$$

and

$$\left|\frac{\partial}{\partial x}G(t_1,s)\right| = \frac{m+1}{\int_0^{+\infty}\frac{d\tau}{h(\tau)}}\begin{cases}\frac{1}{h(t_1)}\int_0^s\frac{d\tau}{h(\tau)} & t_1\geq s\\ \frac{1}{h(t_1)}\int_s^{+\infty}\frac{d\tau}{h(\tau)} & t_1\leq s\end{cases}$$

$$\leq (m+1)\sup_{x\geq 0}\left(\frac{1}{h(x)}\right)$$

then

$$\left|p(t_2)\frac{\partial}{\partial x}T_ru\,(t_2)-p(t_1)\frac{\partial}{\partial x}T_ru\,(t_1)\right|\leq c_1\left|p(t_2)-p(t_1)\right|+c_2\left|\frac{1}{h(t_2)}-\frac{1}{h(t_1)}\right|+c_3\int_{t_1}^{t_2}\Psi_{r,R}(s)ds$$

where

$$c_1=(m+1)\sup_{x\geq 0}\left(\frac{1}{h(x)}\right)\int_0^{+\infty}\Psi_{r,R}(s)ds,\ c_2=2\,(m+1)\sup_{x\in[\eta,\zeta]}(p(x))\int_0^{+\infty}\Psi_{r,R}(s)ds$$

and

$$c_3=2\,(m+1)\sup_{x\in[\eta,\zeta]}(p(x))\sup_{x\geq 0}\left(\frac{1}{h(x)}\right).$$

Because that $p,\frac{1}{h}$ and $t\to\int_0^t\Psi_{r,R}(s)ds$ are uniformly continuous on compact intervals, the above estimates prove that $M_{r,R}$ is equicontinuous on compact intervals.

Finally, let $u\in\Omega_{r,R}$. By using Lemma (3.7) we have for $x\geq 0$

$$\left|p(x)\frac{\partial}{\partial x}T_r(u)\,(x)\right|\leq\left|T_r(u)\,(x)\right|\leq G(x,x)\int_0^{+\infty}\Psi_{r,R}(s)ds.$$

With the fact that

$$\lim_{x\to+\infty}G(x,x)=0$$

the equiconvergence of $M_{r,R}$ holds. By Lemma (2.3), we deduce that $M_{r,R}$ is relatively compact.

3. We show that T_r is continuous in $\Omega_{r,R}$.

Let $(u_n)_n$ be a sequence in $\Omega_{r,R}$ such that

$$\lim_{n\to\infty}u_n=u\in\Omega_{r,R}.$$

For all $t>0$, we have

$$u\,(t)-w\,(t)\geq\tilde{\gamma}\,(t)\,.\,(r-w^*)>0$$

and

$$\begin{aligned} |T_r(u_n)(t) - T_r(u)(t)| &\leq \int_0^{+\infty} G(t,s) |D(s,u_n,u_n') - D(s,u,u')| \, ds \\ &\leq c \int_0^{+\infty} |D(s,u_n,u_n') - D(s,u,u')| \, ds \end{aligned}$$

where

$$c = (m+1) \int_0^{+\infty} \frac{ds}{h(s)}$$

and by using Assertion (3) of Lemma (3.3) we obtain

$$\begin{aligned} p(t) \left| \frac{\partial}{\partial t} T_r(u_n)(t) - \frac{\partial}{\partial t} T_r(u)(t) \right| &\leq p(t) \int_0^{+\infty} \left| \frac{\partial}{\partial t} G(t,s) \right| |D(s,u_n,u_n') - D(s,u,u')| \, ds \\ &\leq \int_0^{+\infty} G(t,s) |D(s,u_n,u_n') - D(s,u,u')| \, ds \\ &\leq c \int_0^{+\infty} |D(s,u_n,u_n') - D(s,u,u')| \, ds \end{aligned}$$

leading to

$$\|T_r(u_n) - T_r(u)\| \leq 2c \int_0^{+\infty} |D(s,u_n,u_n') - D(s,u,u')| \, ds.$$

Let $t > 0$. Because of $f \in C(\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{0\} \times \mathbb{R})$,$\rho \in C^1(\mathbb{R}^+ \setminus \{0\})$ and

$$\tilde{\gamma}(t) r > \tilde{\gamma}(t) w^* \geq w(t) > 0,$$

we have that the function $(x,y) \to D(t,x,y)$ defined by

$$\begin{aligned} D(t,x,y) &= a(t)(y - w'(t)) \rho'(x - w(t)) + b(t) \left(\rho^{\alpha}(x - w(t)) - \rho^{\alpha+\beta}(x - w(t)) \right) \\ &\quad + f(t, \rho(x - w(t)), (y - w'(t)) \rho'(x - w(t))) \end{aligned}$$

is continuous in $[\tilde{\gamma} r, +\infty) \times \mathbb{R}$, and then

$$\lim_{n \to \infty} |D(t,u_n(t),u_n'(t)) - D(t,u(t),u'(t))| = 0 \text{ } a.e. \text{ in } \mathbb{R}^+.$$

Moreover, we have

$$|D(t,u_n(t),u_n'(t)) - D(t,u(t),u'(t))| \leq 2\Psi_{r,R}(t)$$

where $\Psi_{r,R} \in L^1(\mathbb{R}^+)$ is the function given in (6). Then the Lebesgue dominated convergence theorem guarantees that

$$\lim_{n \to \infty} \|T_r(u_n) - T_r(u)\| = 0$$

which shows the continuity of T_r.
4. Finally, we prove that $T_r(K \backslash B(0,r)) \subset C$.

Set $v = T_r u$, $u \in K\backslash B(0,r)$, and let $\tau \in \mathbb{R}^+$. Assertion 1 of Lemma (3.3) gives

$$\begin{aligned} v(\tau) &= \int_0^{+\infty} G(\tau,s)H(s,u,u')ds \\ &\geq \int_0^{+\infty} \gamma(\tau)\,.G(x,s)H(s,u(s))ds \\ &= \gamma(\tau)\,v(x) \end{aligned}$$

this is for all $x \in \mathbb{R}^+$, then

$$v(\tau) \geq \gamma(\tau)\,\|v\|_0$$

and with Lemma (3.7) holds

$$v(\tau) \geq \gamma(\tau)\,\|p.v'\|_0$$

leading to

$$v(\tau) \geq \tilde{\gamma}(\tau)\,\|v\|$$

then $T_r(K\backslash B(0,r)) \subset K$, and with Lemma (3.7) it follows that

$$T_r(K\backslash B(0,r)) \subset C$$

completing the proof. □ □

Lemma 3.9. *For $\varphi \in L^1(\mathbb{R}^+, \mathbb{R}^+)$, let $L_\varphi : E \to E$ be a linear operator defined as*

$$L_\varphi u(t) = \int_0^{+\infty} G(t,s)\varphi(s)u(s)ds. \tag{9}$$

Then L_φ is compact and $L_\varphi(E^+\backslash\{0\}) \subset C\backslash\{0\}$. Moreover

$$r(L_\varphi) > 0.$$

Proof. First, it's clear that L_φ is compact and $L_\varphi(E) \subset E$, and from Assertion (1) of Lemma (3.3), we have that for all $u \in E^+\backslash\{0\}$,

$$L_\varphi u \geq \gamma\,\|L_\varphi u\|_0$$

and from Assertion (3) of Lemma (3.3) that

$$L_\varphi u \geq \tilde{\gamma}\,\|L_\varphi u\|$$

then $L_\varphi(E^+\backslash\{0\}) \subset K\backslash\{0\}$, and Lemma (3.7) leads $L_\varphi(E^+\backslash\{0\}) \subset C\backslash\{0\}$. Now we show that $r(L_\varphi) > 0$. Let $u = \tilde{\gamma}$. Since $L_\varphi(E^+) \subset K$, we have

$$L_\varphi u \geq u.\,\|L_\varphi u\| > 0$$

then from Lemma (2.2) we deduce that L_φ has a positive eigenvalue $\lambda \geq M^{-1} = \|L_\varphi \tilde{\gamma}\|$. Thus

$$r(L_\varphi) \geq \lambda \geq \|L_\varphi(\tilde{\gamma})\| > 0.$$

□ □

Let $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be a function defined as

$$\eta(t,x,y) = a(t)y + b(t)x^{\alpha}\left(1 - x^{\beta}\right).$$

In what follows, we consider the following Hypothesis ;

$$\left\{ \begin{array}{c} \text{there exist } R_0 > w^*,\ 0 < \theta_1 < \theta_2 \text{ and } \sigma_1, \sigma_2 \in L^1\left(\mathbb{R}^+\right) \text{ such that} \\ \sup_{t\geq 0}\left\{\int_{\theta_1}^{\theta_2} G(t,s)\left(\sigma_1\left(s\right)\left(\tilde{\gamma}\left(s\right)\right)^N - \sigma_2\left(s\right) - b(s)\right)ds\right\} > 1 \\ \text{and for all } (t,x,y) \in [\theta_1,\theta_2] \times [R_0,\infty) \times \mathbb{R} \\ f(t,\rho\left(x\right), \frac{y}{p(t)}\rho'\left(x\right)) + h(t) \geq \sigma_1\left(t\right)x^N - \sigma_2\left(t\right)\left|y\right|^N \end{array} \right. \tag{10}$$

where

$$N = \max\left\{1, \frac{\alpha+\beta}{m+1}\right\}.$$

Lemma 3.10. *Assume that Hypothesis (10) holds true and let $r > w^*$. Then there exists $R > \max\{r, R_0\}$ such that for all $u \in C\backslash B\left(0,R\right)$,*

$$\left\|T_r u\right\|_0 > \left\|u\right\|_0.$$

Proof. In the contrary, we suppose that for all $n \geq \left[\max\left(\frac{R_0}{\min_{t\in[\theta_1,\theta_2]}\tilde{\gamma}(t)} + w^*, r\right)\right] + 1$, there exists $u_n \in C\backslash B\left(0,n\right)$, $t_n \geq 0$ such that

$$\begin{aligned} u_n\left(t_n\right) &= \left\|u_n\right\|_0 \geq \left\|T_r u_n\right\|_0 &(11)\\ &\geq T_r u_n\left(t\right) = \int_0^{+\infty} G(t,s)D(s,u_n,u_n')ds &(12)\\ &\geq \int_{\theta_1}^{\theta_2} G(t,s)D(s,u_n,u_n')ds,\ \forall t \geq 0. &(13) \end{aligned}$$

As $u_n - w \geq \min_{t\in[\theta_1,\theta_2]}\tilde{\gamma}\left(t\right)\left(n - w^*\right) \geq R_0$ for all $t \in [\theta_1,\theta_2]$, then we have from Hypothesis (10) that for $s \in [\theta_1,\theta_2]$

$$D(s,u_n,u_n') \geq \eta(s,\rho\left(u_n - w\right),(u_n'-w')\rho'(u_n-w)) + \sigma_1\left(s\right)\left(u_n - w\right)^N - \sigma_2\left(s\right)\left(p(s)\left|u_n' - w'\right|\right)^N$$

with

$$\begin{aligned} \eta(s,\rho\left(u_n - w\right),(u_n' - w')\rho'(u_n - w)) &= D(s,u_n,u_n') - f\left(s,\rho\left(u_n - w\right),(u_n' - w')\rho'(u_n - w)\right) - h(s) \\ &= \frac{a(s)}{p(s)}p\left(s\right)(u_n' - w')\rho'(u_n - w) + b(s)\rho^{\alpha}\left(u_n - w\right)\left(1 - \rho^{\beta}\left(u_n - w\right)\right) \end{aligned}$$

From Assertion (3) of Lemma (3.3) we have $p\left|w'\right| \leq w$, then

$$-\left(w + u_n\right) \leq p(s)(u_n' - w') \leq u_n + w$$

and so

$$\eta(s,\rho\left(u_n - w\right),(u_n'-w')\rho'(u_n-w)) \geq -\frac{a(s)}{p(s)}(w+u_n)\rho'(u_n-w) + b(s)\rho^{\alpha}\left(u_n - w\right)\left(1 - \rho^{\beta}\left(u_n - w\right)\right).$$

By dividing in (11) by $\|u_n\|^N$ we obtain for $t \geq 0$

$$\begin{aligned} v_n(t_n) &= \frac{u_n(t_n)}{\|u_n\|} \geq \frac{u_n(t_n)}{\|u_n\|^N} \\ &\geq \int_{\theta_1}^{\theta_2} G(t,s) \frac{\eta(s, \rho(u_n - w), (u_n' - w')\rho'(u_n - w)) + \sigma_1(s)(u_n - w)^N - \sigma_2(s)(u_n + w)^N}{\|u_n\|^N} ds \\ &\geq \int_{\theta_1}^{\theta_2} G(t,s) \frac{-\frac{a(s)}{p(s)}(w + u_n)\rho'(u_n - w)}{\|u_n\|^N} ds \\ &\quad + \int_{\theta_1}^{\theta_2} G(t,s) \frac{b(s)\rho^\alpha(u_n - w)\left(1 - \rho^\beta(u_n - w)\right)}{\|u_n\|^N} ds \\ &\quad + \int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(v_n - \frac{w}{\|u_n\|}\right)^N - \sigma_2(s)\left(v_n + \frac{w}{\|u_n\|}\right)^N ds. \end{aligned}$$

As $1 \geq v_n \geq \tilde{\gamma}$, we have

$$\begin{aligned} \int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(v_n - \frac{w}{\|u_n\|}\right)^N - \sigma_2(s)\left(v_n + \frac{w}{\|u_n\|}\right)^N ds &\geq \int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(\tilde{\gamma} - \frac{w}{\|u_n\|}\right)^N ds \\ &\quad - \int_{\theta_1}^{\theta_2} G(t,s)\sigma_2(s)\left(1 + \frac{w}{\|u_n\|}\right)^N ds \end{aligned}$$

and then

$$\lim_{n\to\infty} \int_{\theta_1}^{\theta_2} G(t,s)\sigma_1(s)\left(v_n - \frac{w}{\|u_n\|}\right)^N - \sigma_2(s)\left(v_n + \frac{w}{\|u_n\|}\right)^N ds \geq \int_{\theta_1}^{\theta_2} G(t,s)\left(\sigma_1(s)(\tilde{\gamma})^N - \sigma_2(s)\right) ds.$$

Since $N = \max\left\{1, \frac{\alpha+\beta}{m+1}\right\}$, then

$$\lim_{n\to\infty} \frac{\rho(u_n - w)}{\|u_n\|^N} = \lim_{n\to\infty} \frac{(u_n - w)^{\frac{1}{m+1}}}{\|u_n\|^N} = \lim_{n\to\infty} \frac{(u_n)^{\frac{1}{m+1}}}{\|u_n\|^N} = 0,$$

$$\lim_{n\to\infty} \frac{(w + u_n)\rho'(u_n - w)}{\|u_n\|^N} = \lim_{n\to\infty} \frac{\left(\frac{u_n}{\|u_n\|^N}\right)\left(\frac{u_n}{\|u_n\|}\right)^{\frac{-m}{m+1}}}{(m+1)\|u_n\|^{\frac{m}{m+1}}} = 0,$$

$$\lim_{n\to\infty} \frac{\rho^\alpha(u_n - w)}{\|u_n\|^N} = \frac{(u_n - w)^{\frac{\alpha}{m+1}}}{\|u_n\|^N} = 0$$

and

$$\begin{aligned} \lim_{n\to\infty} \frac{\rho^{\alpha+\beta}(u_n - w)}{\|u_n\|^N} &= \frac{(u_n - w)^{\frac{\alpha+\beta}{m+1}}}{\|u_n\|^N} = \begin{cases} 0 & if\ \frac{\alpha+\beta}{m+1} < 1 \\ (v_n)^{\frac{\alpha+\beta}{m+1}} & if\ \frac{\alpha+\beta}{m+1} \geq 1 \end{cases} \\ &\leq 1. \end{aligned}$$

Then we have

$$1 \geq \lim_{n\to\infty} v_n(t_n) \geq \int_{\theta_1}^{\theta_2} G(t,s)\left(\sigma_1(s)(\tilde{\gamma})^N - \sigma_2(s) - b(s)\right) ds$$

this is for all $t \geq 0$, and with (10) we obtain the following contradiction

$$1 \geq \sup_{t\geq 0}\left\{\int_{\theta_1}^{\theta_2} G(t,s)\left(\sigma_1(s)(\tilde{\gamma})^N - \sigma_2(s) - b(s)\right) ds\right\} > 1.$$

Then there exists $R > R_0$ such that for all $u \in \partial B(0,R) \cap C$,

$$\|T_r u\|_0 > \|u\|_0 .$$

This completes the proof. □ □

4. Main results

4.1. Existence results.

Theorem 4.1. *Assume that Hypothesis (2), (3), (4) and (10) hold true.*
If there exist $r > w^*$, λ_1, $\lambda_2 \geq 0$ and ϕ_1, $\phi_2 \in L^1(\mathbb{R}^+)$ such that for all $t > 0$ and all $(x,y) \in [\tilde{\gamma}(t)(r - w^*), r] \times [-r - w^*, r + w^*]$

$$f(t,\rho(x), \frac{y}{p(t)}\rho'(x)) + q(t) \leq \lambda_1\phi_1(t)x + \lambda_2\phi_2(t)y - \eta(t,\rho(x), \frac{y}{p(t)}\rho'(x)), \quad (14)$$

and there exists $i \in \{1,2\}$ such that

$$0 < \lambda_i < \frac{1}{r(L_\psi)}$$

where

$$\psi = \frac{\lambda_1\phi_1 + 2\lambda_2\phi_2}{\lambda_i}$$

then bvp (1) admits at least one positive solution.

Proof. Fisrt, we show that for all $u \in \partial B(0,r) \cap C$

$$T_r u \ngeq u.$$

Suppose in the contrary that there exist $u \in \partial B(0,r) \cap C$ and $t > 0$ such that

$$T_r u \geq u$$

then for $t \geq 0$

$$u(t) \leq T_r u(t) = \int_0^{+\infty} G(t,s)D(s,u,u')ds.$$

We have $pw' \geq -w$ and

$$(u - w, p(u' - w')) \in [\tilde{\gamma}(t)(r - w^*), r] \times [-r - w^*, r + w^*]$$

and from the condition (14) follows that for all $s > 0$,

$$\begin{aligned} f(s,\rho(u-w),(u'-w')\rho'(u-w)) + q(t) \quad \leq \quad & \lambda_1\phi_1(s)(u-w) + \lambda_2\phi_2(s)p(s)(u'-w') \\ & -\eta(s,\rho(u-w),(u'-w')\rho'(u-w)) \end{aligned}$$

leading to

$$D(s,u,u') \leq \lambda_1\phi_1(s)(u-w) + \lambda_2\phi_2(s)\,p(s)(u'-w'),$$

then

$$\begin{aligned} u(t) &\leq \int_0^{+\infty} G(t,s)\left[\lambda_1\phi_1(s)(u-w) + \lambda_2\phi_2(s)\,p(s)(u'-w')\right]ds \\ &\leq \int_0^{+\infty} G(t,s)\left[(\lambda_1\phi_1(s) + \lambda_2\phi_2(s))\,u - (\lambda_1\phi_1(s) - \lambda_2\phi_2(s))\,w\right]ds \\ &\leq \int_0^{+\infty} G(t,s)\left(\lambda_1\phi_1(s) + 2\lambda_2\phi_2(s)\right)u(s)\,ds. \end{aligned}$$

Let $i \in \{1,2\}$ such that $0 < \lambda_i < \frac{1}{r(L_\psi)}$, with

$$\psi = \frac{\lambda_1\phi_1 + 2\lambda_2\phi_2}{\lambda_i}.$$

We have for $t \geq 0$

$$u(t) \leq \lambda_i \int_0^{+\infty} G(t,s)\psi(s)\,u(s)\,ds = L_\psi u$$

leading from Lemma (2.2) for $M = \lambda$, to the following contradiction

$$\lambda_i^{-1} \leq \lambda_0 \leq r(L_\psi) < \lambda_i^{-1}.$$

Then for all $u \in \partial B(0,r) \cap C$

$$T_r u \ngeq u.$$

Moreover, as Hypothesis (10) is verified, then Lemma (3.10) guarantees that there exists $R > r$ such that for all $u \in \partial B(0,R) \cap C$

$$\|T_r u\|_0 > \|u\|_0\,,$$

which means that

$$T_r u \nleq u.$$

Thus, it follows from Theorem (2.1) that T_r admits a fixed point $v \in C$ such that $r \leq \|v\| \leq R$. Hence by Lemma (3.6) , $u = (v-w)^{\frac{1}{m+1}}$ is a positive solution to the bvp (1). □ □

Let Λ be the following positive value

$$\Lambda = \left(\frac{1}{2\sup_{x\geq 0}\int_0^{+\infty} G(x,s)\left[(\delta_1(s) + 2^N\delta_2(s)) + 2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s)\right]ds}\right)^{\frac{1}{N-1}}.$$

Theorem 4.2. *Assume that Hypothesis (2), (3), (4) and (10) hold true. If there exist* $r_0 \geq w^* + 1$ *and* $\delta_1, \delta_2 \in L^1(\mathbb{R}^+, \mathbb{R}^+) \setminus \{0\}$ *such that*

$$\sup_{x\geq 0} \int_0^{+\infty} G(x,s) \left(\left(\delta_1(s) + 2^{N_O}\delta_2(s)\right) + 2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s) \right) ds < \frac{1}{2(1+w^*)^{N_0-1}} \tag{15}$$

and for all $t > 0$, *and all* $(x,y) \in [\tilde{\gamma}(t), r_0] \times [-r_0 - w^*, r_0 + w^*]$

$$f(t, \rho(x), \frac{y}{p(t)}\rho'(x)) + q(t) \leq \delta_1(t) x^{N_0} + \delta_2(t) |y|^{N_0}, \tag{16}$$

where

$$N_0 = \max\left\{1, \frac{\alpha}{m+1}\right\}.$$

then bvp (1) admits at least one positive solution.

Proof. Let $r \in [1 + w^*, r_0]$ be a real number such that

$$r < \Lambda, \tag{17}$$

where

$$\Lambda = \left(\frac{1}{2\sup_{x\geq 0} \int_0^{+\infty} G(x,s) \left[\left(\delta_1(s) + 2^{N_0}\delta_2(s)\right) + 2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)} + b(s) \right] ds} \right)^{\frac{1}{N_0-1}}.$$

We show that for all $u \in \partial B(0,r) \cap C$

$$T_r u \ngeq u.$$

In the contrary, we assume that there exists $u \in \partial B(0,r) \cap C$ such that $T_r u \geq u$ and let $t \geq 0$. We have

$$u(t) \leq T_r u(t) = \int_0^{+\infty} G(t,s) D(s,u,u') ds. \tag{18}$$

We have

$$(u-w, p(u'-w')) \in [\tilde{\gamma}(t)(r-w^*), r] \times [-r-w^*, r+w^*] \subset [\tilde{\gamma}(t), r_0] \times [-r_0-w^*, r_0+w^*]$$

and from the condition (16) follows that for all $s > 0$,

$$\begin{aligned} f(s, \rho(u-w), (u'-w')\rho'(u-w)) + q(t) &\leq \delta_1(s)(u-w)^{N_0} + \delta_2(s)(p(s)|u'-w'|)^{N_0} \\ &\leq \delta_1(s)(u-w)^{N_0} + \delta_2(s)(u+w)^{N_0} \\ &\leq \left(\delta_1(s) + 2^{N_0}\delta_2(s)\right) u^{N_0}. \end{aligned}$$

and

$$\begin{aligned}\eta(s,\rho\left(u-w\right),\left(u'-w'\right)\rho'\left(u-w\right)) &= a(s)\left(u'-w'\right)\rho'\left(u-w\right)+b(s)\rho^{\alpha}\left(u-w\right)\left(1-\rho^{\beta}\left(u-w\right)\right)\\ &\leq \frac{a(s)}{p(s)}\left(u+w\right)\rho'\left(u-w\right)+b(s)\rho^{\alpha}\left(u-w\right)\left(1-\rho^{\beta}\left(u-w\right)\right)\\ &\leq \frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}\left(2u\right)\rho'\left(r-w^{*}\right)+b(s)\rho^{\alpha}\left(r\right).\end{aligned}$$

As $r\geq 1+w^{*}$, we have $\rho'\left(r-w^{*}\right)\leq 1$. Then

$$\eta(s,\rho\left(u-w\right),\left(u'-w'\right)\rho'\left(u-w\right))\leq \frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}\left(2r\right)+b(s)\rho^{\alpha}\left(r\right),$$

leading to

$$D\left(s,u,u'\right)\leq\left(\delta_1\left(s\right)+2^{N_0}\delta_2\left(s\right)\right)r^{N_0}+\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}\left(2r\right)+b(s)\rho^{\alpha}\left(r\right),$$

and by dividing (18) by r^{N_0} we obtain

$$\frac{u\left(t\right)}{r^{N_0}}\leq\int_0^{+\infty}G(t,s)\left[\left(\delta_1\left(s\right)+2^{N_0}\delta_2\left(s\right)\right)+2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}\left(r\right)^{1-N_0}+b(s)\left(r\right)^{\frac{\alpha}{m+1}-N_0}\right]ds$$

and with the fact that $N_0=\max\left\{1,\frac{\alpha}{m+1}\right\}$ we have

$$\frac{u\left(t\right)}{r^{N_0}}\leq\sup_{x\geq 0}\left(\int_0^{+\infty}G(x,s)\left[\left(\delta_1\left(s\right)+2^{N_0}\delta_2\left(s\right)\right)+2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}+b(s)\right]ds\right)$$

this is for all $t\geq 0$, then

$$\begin{aligned}\frac{\|u\|_0}{r^{N_0}} &\leq \sup_{x\geq 0}\left(\int_0^{+\infty}G(x,s)\left[\left(\delta_1\left(s\right)+2^{N_0}\delta_2\left(s\right)\right)+2\frac{a(s)\tilde{\gamma}^{-\frac{m}{m+1}}}{p(s)}+b(s)\right]ds\right)\\ &= \frac{1}{2\Lambda^{N_0-1}}.\end{aligned}$$

Since $\|u\|_0\geq\|pu'\|_0$, we have

$$\frac{r}{2r^{N_0}}\leq\frac{1}{2\Lambda^{N_0-1}}$$

giving

$$r\geq\Lambda$$

which contradicts (17). Then for all $u\in\partial B\left(0,r\right)\cap C$

$$T_r u\ngeq u.$$

Moreover, Lemma (3.10) guarantees that there exists $R>r$ such that for all $u\in\partial B\left(0,R\right)\cap C$

$$\|T_r u\|_0>\|u\|_0,$$

which means that

$$T_r u\nleq u,$$

and Theorem (2.1) guarantees that T_r admits a fixed point $v \in C$ such that $r \leq \|v\| \leq R$. Hence by Lemma (3.6) , $u = (v-w)^{\frac{1}{m+1}}$ is a positive solution to the bvp (1). □ □

4.2. **Uniqueness results.** For $0 < \theta < \theta'$, $\psi \in L^1(\mathbb{R}^+)$ and $r > 0$, let

$$\lambda_\psi(\theta,\theta') = \left(\sup_{t\geq 0}\int_\theta^{\theta'} G(t,s)\psi(s)\right)^{-1}$$

and $\sum_r = \left\{(u-w)^{\frac{1}{m+1}},\ u \in K\backslash B(0,r)\right\}$.

Theorem 4.3. *Assume that Hypothesis (2), (3) and (4) hold true.*
If there exist $r, \theta, \theta' > 0$, $c \geq 0$, and $\phi \in L^1(\mathbb{R}^+)$ such that

$$c < \frac{1}{r(L_\phi)},$$

for all $(x, p(t)y) \in [\tilde{\gamma}(t)r, +\infty) \times [-r, r]$

$$D(t,x,y) \geq \lambda_\psi(\theta,\theta')\psi(t)x, \text{ for } t \in [\theta,\theta'] \tag{19}$$

and for all $x_1, x_2 \in [\tilde{\gamma}(t)r, +\infty)^2$, $y_1, y_2 \in \mathbb{R}$ and all $t > 0$

$$|D(t,x_1,y_1) - D(t,x_2,y_2)| \leq c\phi(t)|x_1 - x_2|. \tag{20}$$

Then bvp (1) has a unique positive solution in $\sum_r$.

Proof. The case $c = 0$ is obvious, so we suppose that $c > 0$:

Existence. Let $v \in K \cap \partial B(0,r)$ and consider the sequence $(u_n)_n$ defined by

$$\begin{cases} u_{n+1} = T_r u_n \\ u_0 = v. \end{cases}$$

We show that $(u_n)_n \subset C\backslash B(0,r)$.
Let $u \in P\backslash B(0,r)$. We have from Lemma (3.8) that $T_r(C\backslash B(0,r)) \subset C\backslash\{0\}$. For $t \geq 0$, we have $u \geq \tilde{\gamma}(t)r$, $p(t)|u'| \leq r$ and from the condition (19)

$$\begin{aligned} T_r u &\geq \lambda(\theta)\int_\theta^{\theta'} G(t,s)\psi(s)u(s)\,ds \\ &\geq r.\lambda(\theta)\int_\theta^{\theta'} G(t,s)\psi(s)\tilde{\gamma}(s)\,ds \end{aligned}$$

this is for all $t \geq 0$, then

$$\|T_r u\| \geq r.\lambda(\theta)\sup_{t\geq 0}\left(\int_\theta^{\theta'} G(t,s)\psi(s)\tilde{\gamma}(s)\,ds\right) = r.$$

Then $T_r(C\backslash B(0,r)) \subset C\backslash B(0,r)$, which means that $(u_n)_n \subset C\backslash B(0,r)$.

Now, by (20) we have for all $n \geq 1$

$$\begin{aligned} |u_{n+1} - u_n| &= |T_r u_n - T_r u_{n-1}| \leq \int_0^{+\infty} G(t,s) \left| D(s, u_n, u_n') - D\left(s, u_{n-1}, u_{n-1}'\right) \right| ds \\ &\leq cL_\phi |u_n - u_{n-1}| . \end{aligned}$$

Then, for all $n \geq 0$

$$|u_{n+1} - u_n| \leq c^n L_\phi^n |u_1 - u_0|$$

and from Lemma (3.3) we have

$$\begin{aligned} p\left|u_{n+1}' - u_n'\right| &= p\left|(T_r u_n)' - (T_r u_{n-1})'\right| \leq p(t) \int_0^{+\infty} \left|\frac{\partial}{\partial t} G(t,s)\right| \left| D(s, u_n, u_n') - D\left(s, u_{n-1}, u_{n-1}'\right) \right| ds \\ &\leq \int_0^{+\infty} G(t,s) \left| D(s, u_n, u_n') - D\left(s, u_{n-1}, u_{n-1}'\right) \right| ds \\ &\leq cL_\phi |u_n - u_{n-1}| . \end{aligned}$$

leads

$$p\left|u_{n+1}' - u_n'\right| \leq c^n L_\phi^n |u_1 - u_0|$$

then

$$\|u_{n+1} - u_n\| \leq 2c^n \left\| L_\phi^n |u_1 - u_0| \right\|_0 .$$

Therefore, for $m > n \geq 1$,

$$\begin{aligned} \|u_m - u_n\| &\leq \|u_m - u_{m-1}\| + \|u_{m-1} - u_{m-2}\| + ... + \|u_{n+1} - u_n\| \\ &\leq 2c^{m-1} \left\| L_\phi^{m-1} |u_1 - u_0| \right\|_0 + 2c^{m-2} \left\| L_\phi^{m-2} |u_1 - u_0| \right\|_0 + ... + 2c^n \left\| L_\phi^n |u_1 - u_0| \right\|_0 \\ &= 2\left(S_{m-1} - S_{n-1}\right), \end{aligned}$$

where

$$S_n = \sum_{n=0}^{n=+\infty} c^n \left\| L_\phi^n w \right\|_0, \text{ with } w = |u_1 - u_0| .$$

Since $c < (r(L_\phi))^{-1}$, we have that

$$\lim_{n\to\infty} \sqrt[n]{c^n \left\| L_\phi^n w \right\|_0} \leq c \lim_{n\to\infty} \sqrt[n]{\left\| L_\phi^n \right\|} = c.r(L_\phi) < 1,$$

then $(S_n)_n$ converges and

$$\lim_{n\to\infty} \|u_m - u_n\| \leq 2 \lim_{n\to\infty} (S_{m-1} - S_{n-1}) = 0.$$

Therefore, the sequence $(u_n)_n$ is also a cauchy sequence and the completeness of E leads to $\lim_{n\to\infty} u_n = u \in C$, with

$$\|u\| \geq r > 0.$$

At the end, passing to the limit in $u_{n+1} = T_r u_n$, and by continuity of T_r in $C \backslash B(0,r)$, we obtain $u = T_r u$, and $u \in C\backslash B(0,r)$ is a fixed point of T_r, and so, $v = (u-w)^{\frac{1}{m+1}}$ is a positive solution to the bvp (1) in $\sum_r$.

Uniqueness. If $u_1,\ u_2 \in K\backslash B(0,r)$ are two solutions of (1) with $u_1 \neq u_2$, then $u_1,\ u_2$ are fixed points of T_r where

$$r = \min\{\|u_1\|, \|u_2\|\}.$$

For all $t \in \mathbb{R}$,

$$u_i(t) \geq \tilde{\gamma}(t).r, \text{ for } i \in \{1,2\}$$

and so

$$\begin{aligned} |u_1 - u_2| &= |T_r u_1 - T_r u_2| \leq \int_0^{+\infty} G(t,s)\left|D(s,u_1,u_1') - D(s,u_2,u_2')\right| ds \qquad (21) \\ &\leq c\int_0^{+\infty} G(t,s)\phi(s)\left|u_1 - u_2\right| ds. \end{aligned}$$

Let L_ϕ be the operator defined in Lemma (3.9) by

$$L_\phi u(t) = \int_0^{+\infty} G(t,s)\phi(s)\, u(s)ds$$

and let $w = L_\phi |u_1 - u_2|$. The inequality (21) leads

$$w \leq cL_\phi w.$$

As $|u_1 - u_2| \in E^+\backslash\{0\}$, then by Lemma (3.9) we have $w \in K\backslash\{0\}$. Then by Theorem (2.2) we deduce that L_ϕ admits a positive eigenvalue λ_0 such that

$$\lambda_0 \geq c^{-1}$$

leading to the following contradiction

$$r(L_\phi) \geq \lambda_0 \geq c^{-1} > r(L_\phi).$$

The proof is complete. □ □

In what follows, we assume that

$$\left\{\begin{array}{c} \alpha+\beta \leq m+1,\ a \in C^1(\mathbb{R}^+,\mathbb{R}^+),\ a(t) \leq \min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty}\frac{ds}{h(s)}\right) \\ e(t) = \max\left(b(t), \frac{1}{h(t)}, |a'(t)|\right) \in L^1_{\gamma^{-1}}(\mathbb{R}^+,\mathbb{R}^+) \text{ and} \\ \text{there exists a function } F \text{ sucth that for all } (t,x,y) \in \mathbb{R}^+\times\mathbb{R}^+\backslash\{0\}\times\mathbb{R} \\ f(t,x,y) = F(t,x) \text{ and} \\ c(t)x + b(t)x^\alpha\left(1-x^\beta\right) + F(t,x) + q(t) \geq 0 \end{array}\right. \qquad (22)$$

where

$$c(t) = \frac{-a(t)}{h(t)\min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty}\frac{ds}{h(s)}\right)} - a'(t)$$

and

$$L^1_{\gamma^{-1}}\left(\mathbb{R}^+,\mathbb{R}\right) = \left\{u : \mathbb{R}^+ \to \mathbb{R} \text{ such that } \int_0^{+\infty} |u(s)|\,\gamma^{-1}(s)ds < \infty\right\}$$

and consider the Banach space $E_0 = \{u \in C(\mathbb{R}^+),\ \lim_{t\to\infty} u(t) = 0\}$ equipped with its sup-norm $\|.\|_0$ and the cone

$$K_0 = \{u \in E_0,\ u(0) = 0 \text{ and } u \geq \gamma \|u\|_0\}.$$

Let $\tilde{T}_r : K_0 \to E_0$ be an operator defined by

$$\tilde{T}_r u(t) = \int_0^{+\infty} J(t,s)\rho(u-w)\,ds + \int_0^{+\infty} G(t,s)D_0(s,u)ds$$

where for $t,s \in \mathbb{R}^+$, $x > 0$,

$$J(t,s) = -\frac{\partial}{\partial s}(G(t,s).a(s))$$

and

$$D_0(s,x) = b(s)\rho^{\alpha}(x-w)\left(1-\rho^{\beta}(x-w)\right) + F(s,\rho(x-w)) + q(s).$$

Lemma 4.4. *Assume that the condition (22) holds true.*
The function $J : \mathbb{R}^+ \times \mathbb{R}^+$ is continuous and verifies for all $x,t \in \mathbb{R}^+$,

$$|J(x,t)| \leq \mu(t)G(x,t),$$

and

$$\frac{J(t,s)}{G(t,s)} \geq c(t)$$

where

$$\mu(t) = \frac{1}{h(t)} + |a'(t)|.$$

Moreover, for all $u \in C^1(\mathbb{R}^+) \cap K_0$, $\tilde{T}_r u = T_r u$.

Proof. Beacause that a, a', $\frac{1}{h} \in L^1(\mathbb{R}^+)$, J is continuous. Moreover, for $x,t \in \mathbb{R}^+ \setminus \{0\}$

$$J(x,t) = -\frac{m+1}{\int_0^{+\infty}\frac{ds}{h(s)}}\begin{cases} \int_x^{+\infty}\frac{ds}{h(s)}\left[\frac{a(t)}{h(t)} + a'(t)\int_0^t \frac{ds}{h(s)}\right] & \text{if } x \geq t \\ \int_0^x \frac{ds}{h(s)}\left[-\frac{a(t)}{h(t)} + a'(t)\int_t^{+\infty}\frac{ds}{h(s)}\right] & \text{if } x \leq t \end{cases}$$

leading to

$$|J(x,t)| \leq \frac{m+1}{\int_0^{+\infty}\frac{ds}{h(s)}}\begin{cases} \int_x^{+\infty}\frac{ds}{h(s)}\left[\frac{a(t)}{h(t)} + |a'(t)|\int_0^t \frac{ds}{h(s)}\right] & \text{if } x \geq t \\ \int_0^x \frac{ds}{h(s)}\left[\frac{a(t)}{h(t)} + |a'(t)|\int_t^{+\infty}\frac{ds}{h(s)}\right] & \text{if } x \leq t \end{cases}.$$

We have from the condition (22) that $a(t) \leq \min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty}\frac{ds}{h(s)}\right)$, then

$$\begin{aligned} |J(x,t)| &\leq \frac{m+1}{\int_0^{+\infty}\frac{ds}{h(s)}}\begin{cases} \int_x^{+\infty}\frac{ds}{h(s)}\int_0^t\frac{ds}{h(s)}\left[\frac{1}{h(t)} + |a'(t)|\right] & \text{if } x \geq t \\ \int_0^x\frac{ds}{h(s)}\int_t^{+\infty}\frac{ds}{h(s)}\left[\frac{1}{h(t)} + |a'(t)|\right] & \text{if } x \leq t \end{cases} \\ &= G(x,t)\left[\frac{1}{h(t)} + |a'(t)|\right] \end{aligned}$$

and

$$\frac{J(x,t)}{G(x,t)} = \begin{cases} \left[-\frac{a(t)}{h(t)\int_0^t \frac{ds}{h(s)}} - a'(t)\right] & \text{if } x \geq t \\ \left[\frac{a(t)}{h(t)\int_t^{+\infty} \frac{ds}{h(s)}} - a'(t)\right] & \text{if } x \leq t \end{cases}$$

$$\geq -\frac{a(t)}{h(t)\min\left(\int_0^t \frac{ds}{h(s)}, \int_t^{+\infty} \frac{ds}{h(s)}\right)} - a'(t).$$

Now, let $u \in C^1(\mathbb{R}^+) \cap K_0$. By integrating by parts we obtain

$$\begin{aligned}\int_0^{+\infty} G(t,s)a(s)\left(\rho\left(u-w\right)\right)' ds &= \left[G(t,s)a(s)\rho\left(u-w\right)\right]_0^\infty - \int_0^{+\infty} \frac{\partial}{\partial s}\left(G(t,s)a(s)\right)\rho\left(u-w\right)ds \\ &= \int_0^{+\infty} J(t,s)\rho\left(u-w\right)ds,\end{aligned}$$

then

$$\begin{aligned}\tilde{T}_r u(t) &= \int_0^{+\infty} J(t,s)\rho\left(u-w\right)ds + \int_0^{+\infty} G(t,s)D_0(s,u)ds \\ &= \int_0^{+\infty} G(t,s)a(s)\left(\rho\left(u-w\right)\right)' ds + \int_0^{+\infty} G(t,s)D_0(s,u)ds \\ &= \int_0^{+\infty} G(t,s)D(s,u,u')ds\end{aligned}$$

hence $\tilde{T}_r u = T_r u$. □ □

Theorem 4.5. *Assume that Hypothesis (4), (10) and (22) hold true. If there exist* $r \geq w^* + 1$, $\lambda > 0$ *and* $\sigma \in L^1_{\gamma^{-1}}(\mathbb{R}^+, \mathbb{R}^+) \setminus \{0\}$ *such that for all* $(t,x) \in \mathbb{R}^+ \setminus \{0\} \times [\gamma(t)r, r]$

$$|F(t,\rho(x)) - F(t,\rho(y))| \leq \lambda\sigma(t)|x-y|, \tag{23}$$

and

$$\lambda < \frac{1}{r(L_\delta)} \tag{24}$$

where

$$\delta = \gamma^{-1}(s)\left(\frac{\mu(s)+2b(s)}{\lambda} + \sigma(s)\right)$$

then bvp (1) admits a unique positive solution in $\sum_r$.

Proof. The case $c = 0$ is obvious, so we suppose that $c > 0$:

Existence. From Hypothesis (10) and Lemma (3.10) we have that there exists $R > r$ such that for all $u \in C$, if $\|u\| \geq R$ then

$$\|T_r u\|_0 > \|u\|_0 .$$

Let $v \in C$ such that $\|v\|_0 \geq R$ and consider the sequence $(u_n)_n$ defined by

$$\begin{cases} u_{n+1} = \tilde{T}_r u_n \\ u_0 = v. \end{cases}$$

For all $n \geq 1,\ u_n \in C^1(\mathbb{R}^+)$, then $u_{n+1} = \tilde{T}_r u_n = T_r u_n \in C$ and since $\|v\| \geq \|v\|_0 \geq R$, we have

$$\|u_{n+1}\|_0 = \|T_r u_n\|_0 > \|u_n\|_0 \geq R > r,\ \forall n \geq 1$$

.

then $(u_n)_n \subset K_0 \backslash B(0, r)$.

Now, by (23) we have for all $n \geq 1$

$$|u_{n+1} - u_n| = \left|\tilde{T}_r u_n - \tilde{T}_r u_{n-1}\right| \leq \int_0^{+\infty} G(t,s)\mu(s)\left|\rho(u_n - w) - \rho(u_{n-1} - w)\right| ds \tag{25}$$

$$+ \int_0^{+\infty} G(t,s)\left|D_0(s, u_n) - D_0(s, u_{n-1})\right| ds. \tag{26}$$

Let $v_n = u_n - w$, $n \geq 1$. We have

$$\begin{aligned}
|\rho(v_n) - \rho(v_{n-1})| &\leq \frac{1}{m+1}(\gamma r - \tilde{\gamma} w^*)^{\frac{-m}{m+1}} |v_n - v_{n-1}| \\
&\leq \frac{\gamma^{-1}}{m+1}\left(r - \frac{1}{2}w^*\right)^{\frac{-m}{m+1}} |v_n - v_{n-1}| \\
&\leq \gamma^{-1}|v_n - v_{n-1}| \\
|\rho^\alpha(v_n) - \rho^\alpha(v_{n-1})| &\leq \frac{\alpha}{m+1}\left(r - \frac{1}{2}w^*\right)^{\left(\frac{\alpha}{m+1} - 1\right)} |v_n - v_{n-1}| \\
&\leq \gamma^{-1}|v_n - v_{n-1}| \\
|\rho^\alpha(v_n) - \rho^\alpha(v_{n-1})| &\leq \frac{\alpha + \beta}{m+1}\left(r - \frac{1}{2}w^*\right)^{\left(\frac{\alpha+\beta}{m+1} - 1\right)} |v_n - v_{n-1}| \\
&\leq \gamma^{-1}|v_n - v_{n-1}|
\end{aligned}$$

leading to

$$\begin{aligned}
|v_{n+1} - v_n| &\leq \int_0^{+\infty} G(t,s)\left(\mu(s) + 2b(s) + \lambda\sigma(s)\right)|v_n - v_{n-1}| ds \qquad (27) \\
&= \lambda L_\delta(|v_n - v_{n-1}|)
\end{aligned}$$

Then, for all $n \geq 0$

$$|u_{n+1} - u_n| \leq \lambda^n L_\delta |u_1 - u_0|.$$

Therefore, for $m > n \geq 1$,

$$\begin{aligned} |u_m - u_n| &\leq |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + ... + |u_{n+1} - u_n| \\ &\leq \lambda^{m-1} {L_\delta}^{m-1} |u_1 - u_0| + \lambda^{m-2} {L_\delta}^{m-2} |u_1 - u_0| + ... + \lambda^n L_\delta^n |u_1 - u_0| \end{aligned}$$

then

$$\begin{aligned} \|u_m - u_n\|_0 &\leq \lambda^{m-1} \left\|L_\delta^{m-1} w\right\|_0 + \lambda^{m-2} \left\|L_\delta^{m-2} w\right\|_0 + ... + \lambda^n \|L_\delta^n w\|_0 \\ &= S_{m-1} - S_{n-1}, \end{aligned}$$

where

$$S_n = \sum_{n=0}^{n=+\infty} c^n \|L_\delta^n w\|_0, \text{ with } w = |u_1 - u_0|.$$

Since $\lambda < (r(L_\delta))^{-1}$, we have that

$$\lim_{n\to\infty} \sqrt[n]{c^n \|L_\delta^n w\|_0} \leq c \lim_{n\to\infty} \sqrt[n]{\|L_\delta^n\|} = c.r(L_\delta) < 1,$$

then $(S_n)_n$ converges and

$$\lim_{n\to\infty} \|u_m - u_n\|_0 = \lim_{n\to\infty} S_{m-1} - S_{n-1} = 0.$$

Therefore, the sequence $(u_n)_n$ is also a cauchy sequence and the completeness of E leads to $\lim_{n\to\infty} u_n = u \in C$, with

$$\|u\|_0 \geq r > 0.$$

At the end, passing to the limit in $u_{n+1} = \tilde{T}_r u_n$, and by continuity of T_r in $C \backslash B(0, r)$, we obtain

$$u = \tilde{T}_r u = T_r u$$

and u is a fixed point of T_r in $C \backslash B(0, r)$, which means that $v = (u - w)^{\frac{1}{m+1}}$ is a positive solution of bvp (1) in $\sum_r$.

Uniqueness. If u_1, $u_2 \in K_0 \backslash B(0, r)$ are two solutions of (1) with $u_1 \neq u_2$, then u_1, u_2 are fixed points of T_r.

Then u_1, $u_2 \in C^1(\mathbb{R}^+)$ and by Lemma (4.4) we have that u_1, u_2 are fixed points of $\tilde{T}_r$.

For all $t \in \mathbb{R}$,

$$u_i(t) \geq \gamma(t).r, \text{ for } i \in \{1, 2\}$$

and so

$$\begin{aligned} |u_1 - u_2| &= \left|\tilde{T}_r u_1 - \tilde{T}_r u_2\right| \leq \int_0^{+\infty} G(t,s)\mu(s) |\rho(u_1 - w) - \rho(u_2 - w)| \, ds \qquad (28) \\ &+ \int_0^{+\infty} G(t,s) |D_0(s, u_1) - D_0(s, u_2)| \, ds. \qquad (29) \end{aligned}$$

Let $v_i = u_i - w$, $i \in \{1, 2\}$. We have

$$\begin{aligned}
|\rho(v_1) - \rho(v_2)| &\leq \frac{1}{m+1}(\gamma r - \tilde{\gamma} w^*)^{\frac{-m}{m+1}} |v_1 - v_2| \\
&\leq \frac{\gamma^{-1}}{m+1}\left(r - \frac{1}{2}w^*\right)^{\frac{-m}{m+1}} |v_1 - v_2| \\
&\leq \gamma^{-1}|v_1 - v_2| \\
|\rho^\alpha(v_1) - \rho^\alpha(v_2)| &\leq \frac{\alpha}{m+1}(\gamma r - \tilde{\gamma} w^*)^{\left(\frac{\alpha}{m+1}-1\right)} |v_1 - v_2| \\
&\leq \gamma^{-1}|v_1 - v_2| \\
\left|\rho^{\alpha+\beta}(v_1) - \rho^{\alpha+\beta}(v_2)\right| &\leq \frac{\alpha+\beta}{m+1}(\gamma r - \tilde{\gamma} w^*)^{\left(\frac{\alpha+\beta}{m+1}-1\right)} |v_1 - v_2| \\
&\leq \gamma^{-1}|v_1 - v_2|
\end{aligned}$$

leading to

$$\begin{aligned}
|v_1 - v_2| &\leq \int_0^{+\infty} G(t,s)\gamma^{-1}(s)\left(\mu(s) + 2b(s) + \lambda\sigma(s)\right)|v_1 - v_2|\, ds \qquad (30) \\
&= \lambda L_\delta(|v_1 - v_2|)
\end{aligned}$$

where L_δ is the operator defined in Lemma (3.9) by

$$L_\delta u(t) = \int_0^{+\infty} G(t,s)\delta(s)\, u(s)ds$$

where

$$\delta(s) = \gamma^{-1}(s)\left(\frac{\mu(s) + 2b(s)}{\lambda} + \sigma(s)\right)$$

and let $w = L|u_1 - u_2|$. The inequality (30) leads

$$w \leq \lambda L_\delta w.$$

As $|u_1 - u_2| \in E^+ \backslash \{0\}$, then by Lemma (3.9) we have $w \in K_0 \backslash \{0\}$. Then by Theorem (2.2) we deduce that L_δ admits a positive eigenvalue λ_0 such that

$$\lambda_0 \geq \lambda^{-1}$$

leading to the following contradiction

$$r(L) \geq \lambda_0 \geq \lambda^{-1} > r(L).$$

This completes the proof. □ □

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