J. Indones. Math. Soc.
Vol. 27, No. 02 (2021), pp. 240–248.

ON PURELY HERMITIAN $\mathbb{R}$-COMPLEX FINSLER SPACE WITH (α, β)-METRIC

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Abstract. In the present paper we established some interesting results on purely Hermitian $\mathbb{R}$-complex Finsler space with (α, β)-metrics, Firstly we characterize the conditions for the (α, β)-metric $F = \sqrt{\alpha^2 + \varepsilon\beta^2}$ to be a purely Hermitian. Then determined the fundamental metric tensor, its inverse and determinent of the above metric. Further obtained Chern-Finsler connection coefficients and analysed necessary conditions under which an purely Hermitian $\mathbb{R}$-complex Finsler space with (α, β)-metric to be Berwald, Kähler and strongly Kähler also given some examples.

Key words and Phrases: $\mathbb{R}$-complex Finsler space, Purely Hermitian metric, Connection coefficients, Berwald space

1. INTRODUCTION

In [1, 2, 9, 10, 12, 13, 14, 18, 20], many geometers contributed the field of complex Finsler geometry with reference to the notions of real Finsler geometry. In the very begining Rizza [17] extended the homogeneous property of real Finsler metric to the complex case by defining the function $F : T^{'}M \to \mathbb{R}_+$ with the condition $F(z, \lambda\eta) = |\lambda|F(z, \eta)$, for any $\lambda \in \mathbb{C}$, where (z, η) are complex coordinates. Immediately afterwords S. Kobayashi contributed Kobayashi metric satisfying above said homogeneity property on complex manifold.

Further, in [11, 15] authors reduced the definition of complex Finsler space [16, 19] was extended, reducing homogeneity property to the scalars $\lambda \in \mathbb{R}$, then

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2020 Mathematics Subject Classification: 53B40, 53C56, 53C60.
Received: 29-05-2020, accepted: 16-07-2021.

introduced new class of Finsler space which is named as $\mathbb{R}$-complex Finsler space which is the recent research done in the field of Finsler geometry.

The purpose of this paper is to continue the study of the purely Hermitian $\mathbb{R}$-complex Finsler spaces and also some preliminary properties of the n-dimensional $\mathbb{R}$-complex Finsler spaces. Subsequently, we will focus only on the study of the purely Hermitian complex Finsler spaces, (meaning $g_{i\bar{j}}$ is invertible). Next, we show that any purely Hermitian $\mathbb{R}$-complex Finsler spaces with (α, β)-metric is Berwald. Moreover we prove that any strongly Berwald space is strongly Kähler, by some explicit examples.

2. PRELIMINARIES

Let us define complex coordinates in the form $z = (z^k)_{k=\overline{1,n}}$ and $\eta = (\eta^k)_{k=1,\,n}$ over the n-dimensional complex manifold M. Then the tangent bundle over M is defined by $T_cM = T^{'}M \oplus T^{''}M$, where $T^{'}M$ is the holomorphic tangent bundle and $T^{''}M$ is the conjugate. Since the subbundle $T^{'}M$ is itself a complex manifold its local coordinates defined by $u = (z^k, \eta^k)_{k=\overline{1,n}}$ and are reduced to $(z^{'k}, \eta^{'k})_{k=\overline{1,n}}$ in the form $z^{'k} = z^{'k}(z)$ and $\eta^{'k} = \frac{\partial z^{'k}}{\partial z^l}\eta^l$.

A complex space $(M,\, F)$ is said to be $\mathbb{R}$-complex Finsler space if F is a continuous function defined as a mapping $F : T^{'}M \to \mathbb{R}_+$ and satisfies the following conditions.

i) $L := F^2$ is smooth on $\widetilde{T^{'}M} = T^{'}M \setminus \{0\}$;

ii) $F(z, \eta) \geq 0$, The equality holds if and only if $\eta = 0$;

iii) $F(z,\ \lambda\eta,\ \bar{z},\ \lambda\bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta}), \forall \lambda \in \mathbb{R}$.

For the $\mathbb{R}$-complex Finsler space, the metric tensor generates the below tensor fields:

$$g_{rs} := \frac{\partial^2 L}{\partial \eta^r \partial \eta^s};\ g_{r\bar{s}} := \frac{\partial^2 L}{\partial \eta^r \partial \bar{\eta}^s};\ g_{\bar{r}\,\bar{s}} := \frac{\partial^2 L}{\partial \bar{\eta}^r \partial \bar{\eta}^s}, \tag{1}$$

and they satisfy the properties of homogeneity as follows:

$$\frac{\partial L}{\partial \eta^r}\eta^r + \frac{\partial L}{\partial \bar{\eta}^r}\bar{\eta}^s = 2L;\ g_{rs}\eta^r + g_{s\bar{r}}\bar{\eta}^r = \frac{\partial L}{\partial \eta^s}; \tag{2}$$

$$2L = g_{rs}\eta^r\eta^s + 2g_{r\bar{s}}\eta^r\bar{\eta}^s + g_{\bar{r}\,\bar{s}}\bar{\eta}^r\bar{\eta}^s; \tag{3}$$

$$\frac{\partial g_{rk}}{\partial \eta^s}\eta^s + \frac{\partial g_{rk}}{\partial \bar{\eta}^s}\bar{\eta}^s = 0;\ \frac{\partial g_{r\bar{k}}}{\partial \eta^s}\eta^s + \frac{\partial g_{r\bar{k}}}{\partial \bar{\eta}^s}\bar{\eta}^s = 0. \tag{4}$$

In [3, 15] authors extended the concept of purely Hermitian space to $\mathbb{R}$-complex Finsler space with tensor field $g_{r\bar{s}}(z)$. Suppose metric function F on $\mathbb{R}$-complex Finsler space satiesfies the regularity condition $det(g_{r\bar{s}}) \neq 0$ for any $u \in T^{'}M$ and generates positive definite Levi-form, then such spaces are called $\mathbb{R}$-complex Hermitian Finsler space.

Consider the sections of the complexified tangent bundle of $T^{'}M$. Let $VT^{'}M \subset T^{'}(T^{'}M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$ and $VT^{''}M$ its conjugate. The idea of complex nonlinear connection, briefly (c.n.c) is an instrument in 'linearization' of the geometry of the manifold $T^{'}M$. A (c.n.c) is a supplementary complex subbundle to $VT^{'}M$ in $T^{'}(T^{'}M)$ i.e. $T^{'}(T^{'}M) = HT^{'}M \oplus VT^{'}M$. The horizontal distribution $H_uT^{'}M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of (c.n.c).

The pair $\{\delta_k = \frac{\delta}{\delta z^k},\ \dot{\partial}_k = \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of (c.n.c) which obeys the change rules $\delta_k = \frac{\partial z^{'j}}{\partial z^k}\delta_j^{'}$ and $\dot{\partial}_k = \frac{\partial z^{'j}}{\partial z^k}\partial_j^{'}$. By conjugation everywhere we have obtained an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T_u^{''}(T^{'}M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

In complex Hermitian Finsler space, since complex nonlinear connection depends on fundamental metric function called as Chern-Finsler (c.n.c) and its coefficients are defined by:

$$N_k^i = g^{\bar{m}i}\frac{\partial L^2}{\partial z^k \partial \bar{\eta}^k} = g^{\bar{m}i}(\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial g_{s\bar{m}}}{\partial z^k}\eta^s). \tag{5}$$

Further, Chern-Finsler (c.n.c) generates complex spray in the form:

$$G^i = \frac{1}{2}N_k^i\eta^k = \frac{1}{2}g^{\bar{m}i}(\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial g_{s\bar{m}}}{\partial z^k}\eta^s)\eta^k. \tag{6}$$

Since L is $\mathbb{R}$-homogeneous of degree 2 in the fibre variables that Chern-Finsler (c.n.c) (5) and its induced complex spray (6) satisfy the constraints,

$$(\dot{\partial}_j G^i)\eta^j + (\dot{\partial}_{\bar{r}} G^i)\bar{\eta}^r = 2G^i;\ \ (\dot{\partial}_j N_k^i)\eta^j + (\dot{\partial}_{\bar{r}} N_k^i)\bar{\eta}^r = N_k^i. \tag{7}$$

This shows that G^i and N_k^i are $\mathbb{R}$-homogeneous of degree 2 respectively of degree 1, with respect to η.

Further, the complex spray (6) generates a (c.n.c) by $\overset{c}{N_j^i} := \dot{\partial}_j G^i$, which is called canonical (c.n.c). In the simpler computation gives that $\overset{c}{N_j^i}$, are $\mathbb{R}$-homogeneous of degree 1 and, the complex spray induced of the canonical (c.n.c) is this

$$\overset{c}{G^i} := \frac{1}{2}\overset{c}{N_j^i}\eta^k = G^i - \frac{1}{2}(\dot{\partial}_{\bar{r}} G^i)\bar{\eta}^r. \tag{8}$$

It is obvious that the Chern-Finsler (c.n.c) and the canonical (c.n.c) induce the same complex spray (6), ($\overset{c}{G^i} = G^i$) if and only if the coefficients G^i given in (6) are (2, 0)-homogeneous with respect to η.

Also, we have recovered the Chern-Finsler connection, in an $\mathbb{R}$-complex Hermitian Finsler space [5, 7], which is metrical of (1, 0)-type and it is given by

$$L_{jk}^i = g^{\bar{m}i}(\delta_j g_{k\bar{m}});\ \ C_{jk}^i = g^{\bar{m}i}(\dot{\partial}_j g_{k\bar{m}});\ \ L_{j\bar{k}}^i = C_{j\bar{k}}^i = 0, \tag{9}$$

where δ_j is the frame corresponding to the Chern-Finsler (c.n.c).

Finally, we recall from [4] the definition of an $\mathbb{R}$-complex Hermitian Finsler space with Berwald property. Let (M, F) is Berwald if the local coefficients L_{jk}^i

depend only on the position z. In this case, the local coefficients of the Chern Finsler (c.n.c) have the particular form

$$L^i_{jk} = \dot{\partial}_j N^i_k; \quad N^i_k = L^i_{jk}(z)\eta^j + (\dot{\partial}_{\bar{r}} N^i_j(z)\bar{\eta}^r). \tag{10}$$

which together with 8 lead to

$$(\dot{\partial}_j L^i_{hk})\eta^j + (\dot{\partial}_{\bar{r}} L^i_{hk})\bar{\eta}^r = 0, \tag{11}$$

i.e., the horizontal coefficients L^i_{jk} are $\mathbb{R}$-homogeneous of degree 0 with respect to η. Also, an elementary calculation gives $\delta_k(\dot{\partial}_{\bar{r}} L) = 0$.
Now, we associate to the canonical (c.n.c), a complex linear connection of Berwald type

$$B\Gamma := \left(\overset{c}{N}{}^i_j,\ B^i_{jk} := \dot{\partial}_k \overset{c}{N}{}^i_j,\ B^i_{j\bar{k}} := \dot{\partial}_{\bar{k}} \overset{c}{N}{}^i_j,\ 0,\ 0\right), \tag{12}$$

where $\overset{c}{\delta}_k$ is with respect to $\overset{c}{N}{}^i_j$. $B\Gamma$ is neither h-nor v-metrical. Moreover, it satisfies the following properties

$$B^i_{jk}\eta^j = \overset{c}{N}{}^i_k - (\dot{\partial}_{\bar{r}} \overset{c}{N}{}^i_k)\bar{\eta}^r; \quad B^i_{jk} = B^i_{kj}. \tag{13}$$

3. PURELY HERMITIAN ℝ-COMPLEX FINSLER SPACE

We consider $z \in M$, $\eta \in T'_z M$, $\eta = \eta^r \frac{\partial}{\partial z^r}$. An $\mathbb{R}$-complex Finsler space (M, F) is called $\mathbb{R}$-complex purely Hermitian Finsler space if

$$L = \alpha^2 + \varepsilon\beta^2, \quad \varepsilon = \pm 1 \tag{14}$$

where

$$\begin{aligned} \alpha^2(z,\eta,\bar{z},\bar{\eta}) &= Re\{a_{rs}\eta^r\eta^s + a_{r\bar{s}}\eta^r\bar{\eta}^s\}; \\ \beta(z,\eta,\bar{z},\bar{\eta}) &= Re\{b_r\eta^r\}, \end{aligned}$$

with $a_{rs} = a_{rs}(z)$, $a_{r\bar{s}} = a_{r\bar{s}}(z)$, and $b = b_r(z)dz^r$ is a differential (1, 0)-form. The purely Hermitian function (14) produces two tensor fields g_{rs} and $g_{r\bar{s}}$.

To discuss the Hermitian $\mathbb{R}$-complex Finsler spaces with purely Hermitian metric, we suppose that $a_{r\bar{s}} = 0$. Thus, only the tensor field $g_{r\bar{s}}$ is invertible and it is characterized by the following properties.

Proposition 3.1. *In an $\mathbb{R}$-complex Hermitian Finsler spaces with purely Hermitian metric the following equalities hold:*

$$\begin{gathered} \alpha L_\alpha + \beta L_\beta = 2L,\ \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \\ \alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \quad \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \end{gathered} \tag{15}$$

where

$$L_\alpha := \frac{\partial L}{\partial \alpha},\quad L_\beta = \frac{\partial L}{\partial \beta},\quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta},\quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2},\quad L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}. \tag{16}$$

For a $\mathbb{R}$-complex Hermitian Finsler spaces with $L = \alpha^2 + \varepsilon\beta^2$, we have:

$$L_\alpha = 2\alpha, \quad L_\beta = 2\varepsilon\beta, \quad L_{\alpha\beta} = 0, \quad L_{\alpha\alpha} = 2, \quad L_{\beta\beta} = 2\varepsilon \tag{17}$$

Theorem 3.2. *The fundamental metric tensor of the $\mathbb{R}$-complex purely Hermitian Finsler spaces with (α, β)-metric: $L = \alpha^2 + \varepsilon\beta^2$, $\varepsilon = \pm 1$ is given by*

$$g_{r\bar{s}} = a_{r\bar{s}} + \frac{\varepsilon}{2} b_r b_{\bar{s}} \tag{18}$$

Proof. The invarients of an $\mathbb{R}$-complex purely Hermitian Finsler spaces with (α, β)-metric are given by

$$\rho_0 = 1, \;\; \rho_1 = \varepsilon\beta, \;\; \rho_{-2} = 0, \;\; \rho_{-1} = 0, \;\; \mu_0 = \frac{\varepsilon}{2} \tag{19}$$

Subscripts -2, -1, 0, 1 give us the degree of homogeneity of these invariants we have using obtained the results in (18).

The next aim is to find the formulas for the determinant and the inverse of tensor field $g_{r\bar{s}}$. The solution is obtained by the following lemma from [6], for an arbitrary non-singular Hermitian matrix $(Q_{r\bar{s}})$. □

Lemma 3.3. *Suppose:*
- (Q_{rs}) *is a non-singular* $n \times n$ *complex matrix with inverse* (Q^{rs})*;*
- C_r *and* $C_{\bar{r}} = \bar{C}_r, r = 1, ...n$ *are complex numbers;*
- $C^r := Q^{\bar{s}r} C_{\bar{s}}$ *and its conjugates;* $C^2 := C^r C_r = \bar{C}^r C_{\bar{r}}$*;* $H_{r\bar{s}} := Q_{r\bar{s}} \pm C_r C_{\bar{s}}$*.*

Then

(1) $det(H_{r\bar{s}}) = (1 \pm C^2) det(Q_{r\bar{s}})$,

(2) *whenever* $(1 \pm C^2) \neq 0$, *the matrix* $(H_{r\bar{s}})$ *is invertible and in this case its inverse is* $H^{r\bar{s}} = Q^{\bar{s}r} \mp \frac{1}{1 \mp C^2} C^r C^{\bar{s}}$.

Theorem 3.4. *For the $\mathbb{R}$-complex purely Hermitian Finsler space with the metric $L = \alpha^2 + \varepsilon\beta^2$, $\varepsilon = \pm 1$ the determinant and the inverse of the fundamental metric tensor $g_{r\bar{s}}$ are given by*

(1) $g^{\bar{s}r} = a^{\bar{s}r} + \frac{1}{2+\bar{\omega}} b^r b^{\bar{s}}$,

(2) $det(g_{r\bar{s}}) = \frac{2+\bar{\omega}}{2} det(a_{r\bar{s}})$,

where $\bar{\omega} = b_r b^r$, $b_r = b^{\bar{s}} a_{r\bar{s}}$, $b^r = a^{\bar{s}r} b_{\bar{s}}$.

Proof. Applying lemma (3.3) we set $Q_{r\bar{s}} = a_{r\bar{s}}$ and $C_r = \frac{1}{\sqrt{2}} b_r$. We obtain $Q^{\bar{s}r} = a^{\bar{s}r}$, $C^r = a^{\bar{s}r} \frac{1}{\sqrt{2}} b_{\bar{s}}$, $C^2 = \frac{1}{2}\bar{\omega}$, $1 + C^2 = \frac{2+\bar{\omega}}{2} \neq 0$. So the matrix $H_{r\bar{s}} = g_{r\bar{s}}$ is invertible with $H^{\bar{s}r} = a^{\bar{s}r} + \frac{2}{2+\bar{\omega}} (a^{\bar{k}r} \frac{1}{\sqrt{2}} b_{\bar{k}})(a^{\bar{s}l} \frac{1}{\sqrt{2}} b_{\bar{l}})$ and $det(H_{r\bar{s}}) = \frac{2+\bar{\omega}}{2} det(a_{r\bar{s}})$, thus $g^{r\bar{s}} = a^{r\bar{s}} + \frac{1}{2+\bar{\omega}} b^r b^{\bar{s}}$ and $det(g_{r\bar{s}}) = \frac{2+\bar{\omega}}{2} det(a_{r\bar{s}})$ from here obtained the results (1) and (2). □

Example 3.5. *We consider α as in* [8]*, given by*

$$\alpha^2(z, \eta) = \frac{|\eta|^2 + \epsilon(|z|^2|\eta|^2 - | < z, \eta > |^2)}{(1 + \epsilon|z|^2)^2}, \tag{20}$$

defined over the disk $\triangle_r^n = \{z \in \mathbb{C}^n, |z| < r, r = \sqrt{\frac{1}{|\epsilon|}}\}$ *if* $\epsilon < 0$*, on* $\mathbb{C}^n$ *if* $\epsilon = 0$ *and on the complex projective space* $P^n\mathbb{C}$ *if* $\epsilon > 0$*, where* $| < z,\eta > |^2 =< z,\eta > \overline{< z,\eta >}$*. By computation, we obtain* $a_{ij} = 0$ *and* $a_{i\bar{j}} = \frac{1}{1+\epsilon|z|^2}\left(\delta_{i\bar{j}} - \epsilon\frac{\bar{z}^i z^j}{1+\epsilon|z|^2}\right)$ *and so* $\alpha^2(z,\eta) = a_{i\bar{j}}(z)\eta^i\bar{\eta}^j$*. Thus purely Hermitian metrics which have special properties are determined. They are Kähler with constant holomorphic curvature* $K_\alpha = 4\epsilon$*. Particularly, for* $\epsilon = -1$ *we obtain the Bergman metric on the unit disk* $\triangle^n = \triangle_1^n$*; for* $\epsilon = 0$ *The Euclidean metric on* $\mathbb{C}^n$*, and for* $\epsilon = 1$ *The Fubini-study metric on* $P^n(\mathbb{C})$*. Setting* $\beta(z,\eta) = Re\frac{<z,\eta>}{1+\epsilon|z|^2}$*, where* $b_i = \frac{\bar{z}^i}{1+\epsilon|z|^2}$*, we obtain some example of Purely Hermitian ℝ-complex Finsler metrics*

$$F_\epsilon = \frac{|\eta|^2 + \epsilon(|z|^2|\eta|^2 - | < z,\eta > |^2)}{(1+\epsilon|z|^2)^2} \pm \left(Re\frac{< z,\eta >}{1+\epsilon|z|^2}\right)^2. \tag{21}$$

4. CONNECTION COEFFICINTS AND BERWALD SPACE

The Chern-Finsler connection coefficients (c.n.c) of a ℝ-complex purely Hermitian Finsler space (M,F) with (α,β)-metric is defined by

$$\overset{CF}{N_k^i} = g^{\bar{m}i}\frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^m} = g^{\bar{m}i}(\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial g_{s\bar{m}}}{\partial z^k}\eta^s). \tag{22}$$

After a direct calculus, by using (22) we get

Lemma 4.1. *Let* (M,F) *be an ℝ-complex purely Hermitian Finsler space with* $F = \sqrt{\alpha^2 + \varepsilon\beta^2}$ *and* $a_{ij} = 0$, $\varepsilon = \pm 1$*. Then we have the following expression of Chern-Finsler (c.n.c)*

$$\overset{CF}{N_j^i} = \overset{a}{N_j^i} + \frac{1+\bar{\omega}}{2+\bar{\omega}}\left(\overset{a}{\delta_j}\beta\right) + \frac{1}{2+\bar{\omega}}\frac{\partial l_{\bar{m}}}{\partial z^j}b^i b^{\bar{m}} + 2\beta g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}, \tag{23}$$

where

$$\overset{a}{N_j^i} = a^{\bar{m}i}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l, \quad \overset{a}{\delta_j}\beta = \frac{1}{2}\left(\frac{\partial \bar{b}^r}{\partial z^j}l_{\bar{r}} + \frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r\right), \quad g^{\bar{m}i} = a^{i\bar{m}} + \frac{1}{2+\bar{\omega}}b^i b^{\bar{m}},$$

and so, the spray coefficients are

$$G_j^i = \overset{a}{G_j^i} + \left[\frac{1+\bar{\omega}}{2(2+\bar{\omega})}\left(\overset{a}{\delta_j}\beta\right) + \frac{1}{2(2+\bar{\omega})}\frac{\partial l_{\bar{m}}}{\partial z^j}b^i b^{\bar{m}} + \beta g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}\right]\eta^k. \tag{24}$$

After a direct calculus, we can prove that,

Theorem 4.2. *Let (M,F) be an $\mathbb{R}$-complex purely Hermitian Finsler space with $F=\sqrt{\alpha^2+\varepsilon\beta^2}$ and $a_{ij}=0$, $\varepsilon=\pm1$. If (M,F) is Berwald then*

$$\left.\begin{aligned}(2+\bar{\omega})\left(N^i_j-\overset{a}{N^i_j}\right)l_i&=(2+\bar{\omega})\left(\overset{a}{\delta_j}\beta\right)\bar{\varepsilon}+\bar{\varepsilon}\frac{\partial l_{\bar{m}}}{\partial z^j}b^{\bar{m}}+2\beta(2+\bar{\omega})g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}b_i\\(2+\bar{\omega})\left(N^i_j-\overset{a}{N^i_j}\right)b_i&=(2+\bar{\omega})\left(\overset{a}{\delta_j}\beta\right)\bar{\omega}+\bar{\omega}\frac{\partial l_{\bar{m}}}{\partial z^j}b^{\bar{m}}+2\beta(2+\bar{\omega})g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}b_i\end{aligned}\right\}. \tag{25}$$

Proof. If (M,F) is Berwald. Then $N^i_k=L^i_{jk}(z)\eta^j+(\dot{\partial}_{\bar{h}}N^i_j(z)\bar{\eta}^h)$, which means that N^i_k are $\mathbb{R}$-homogeneous in η and $\bar{\eta}$ of degree 1. Thus, using (23) and we obtained the result (25). □

Lemma 4.3. *The functions b^i and b_i are holomorphic if and only if $\overset{a}{\delta_j}\beta=0$.*

Proof. Since $2(\overset{a}{\delta_j}\beta)=(\frac{\partial\bar{b}^r}{\partial z^j}l_{\bar{r}}+\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)$, the direct implication is immediate. Conversely, the condition $(\overset{a}{\delta_j}\beta)=0$ can be rewritten as

$$\frac{\partial b_i}{\partial z^j}\eta^i-b_i\overset{a}{N^i_j}+\frac{\partial b_{\bar{m}}}{\partial z^j}\bar{\eta}^m=0. \tag{26}$$

Its derivation with respect to $\bar{\eta}$ gives $\frac{\partial b_{\bar{m}}}{\partial z^j}=0$ and so, (26) implies

$$\frac{\partial b_i}{\partial z^j}\eta^i-b_i\overset{a}{N^i_j}=0, \tag{27}$$

which, by derivation with respect to η, it leads to $\frac{\partial b_i}{\partial z^j}=b^{\bar{m}}\frac{\partial a_{i\bar{m}}}{\partial z^j}$. The last relation is equivalently to $b^{\bar{m}}a_{i\bar{m}}=b_i$. This implies $\frac{\partial b^{\bar{m}}}{\partial z^j}=0$, which proves our claim. □

Theorem 4.4. *Let (M,F) be an $\mathbb{R}$-complex purely Hermitian Finsler space with $a_{ij}=0$ and $F=\sqrt{\alpha^2+\varepsilon\beta^2}$, $\varepsilon=\pm1$. If (M,F) is Berwald space and $(N^i_j-\overset{a}{N^i_j})b_i=0$, then $(\overset{a}{\delta_j}\beta)=0$ and $N^i_j=\overset{a}{N^i_j}$.*

Proof. Since under our assumptions, The conditions (25) become

$$(2+\bar{\omega})\left(N^i_j-\overset{a}{N^i_j}\right)l_i = (2+\bar{\omega})\left(\overset{a}{\delta_j}\beta\right)\bar{\varepsilon}+\bar{\varepsilon}\frac{\partial l_{\bar{m}}}{\partial z^j}b^{\bar{m}}+2\beta(2+\bar{\omega})g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}b_i, \tag{28}$$

$$0 = (2+\bar{\omega})\left(\overset{a}{\delta_j}\beta\right)\bar{\omega}+\bar{\omega}\frac{\partial l_{\bar{m}}}{\partial z^j}b^{\bar{m}}+2\beta(2+\bar{\omega})g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}b_i. \tag{29}$$

In view of (27) and (28), now, by using Lemma 4.3 and (25) we get obtain the desired results $N^i_j=\overset{a}{N^i_j}$. □

The next theorem provides the sufficient conditions for an purely Hertmitian $\mathbb{R}$-complex Finsler space $F:=\sqrt{\alpha^2+\varepsilon\beta^2}$, with $a_{ij}=0$ to be Berwald.

Theorem 4.5. *Let (M, F) be an purely Hermitian $\mathbb{R}$-complex Finsler space, with $a_{ij} = 0$. If $\overset{a}{\delta}_j \beta = 0$ then it is a Berwald space and $N^i_j = \overset{a}{N}{}^i_j$. Moreover, if α is Kähler, then F is strongly Kähler.*

Proof. Using Lemma 4.3 and (23), it results $N^i_j = \overset{a}{N}{}^i_j$. Since α^2 is a purely Hermitian metric, it is Berwald and by the relation we obtain that $F := \sqrt{\alpha^2 + \varepsilon\beta^2}$ is also Berwald and $L^i_{kj}(z) = \overset{a}{L}{}^i_{kj}(z)$, where $\overset{a}{L}{}^i_{kj} := \dot{\delta}_k \overset{a}{N}{}^i_j$.
Now, if we suppose that α is Kähler, we have $T^i_{jk} = \overset{a}{L}{}^i_{jk} - \overset{a}{L}{}^i_{kj} = 0$, which proves our claim. □

Finally, we give some explicit examples of purely Hermitian $\mathbb{R}$-complex Finsler metrics which are Berwald or srongly Berwald.

Example 4.6. *On $M = \mathbb{C}^2$ we consider the metric*

$$\alpha^2 = e^{z^1+\bar{z}^1}|\eta^1|^2 + e^{z^2+\bar{z}^2}|\eta^2|^2, \tag{30}$$

and we choose $\varepsilon = e^{z^2}\eta^2$. These imply $a_{ij} = 0$, $(r, s = 1, 2)$, $2\beta = e^{z^2}\eta^2 + e^{\bar{z}^2}\bar{\eta}^2$, $b_1 = b^1 = 0$, $b_2 = e^{z^2}$, $b^2 = e^{-z^2}$ and $\omega = 1$.
With the above tools we construct a $\mathbb{R}$-complex purely Hermitian metric function

$$F = \sqrt{e^{z^1+\bar{z}^1}|\eta^1|^2 + e^{z^2+\bar{z}^2}|\eta^2|^2 + \frac{1}{4}(e^{z^2}\eta^2 + e^{\bar{z}^2}\bar{\eta}^2)^2}, \tag{31}$$

which is a purely Hermitian $\mathbb{R}$-complex Finsler metric having $det(g_{r\bar{s}}) = \frac{2+\bar{\omega}}{2} det(a_{r\bar{s}}) = \frac{3}{2}e^{z^1+\bar{z}^1+z^2+\bar{z}^2} > 0$, $(r, s = 1, 2)$, and $\bar{\omega} = 1 > 0$. A direct computation gives $2(\overset{a}{\delta}_j \beta) = \frac{\partial \bar{b}^2}{\partial z^j} l_{\bar{2}} + \frac{\partial b_{\bar{2}}}{\partial z^j}\bar{\eta}^2 = 0$, $\frac{\partial l_{\bar{m}}}{\partial z^j} = 0$ and $\frac{\partial b_{\bar{m}}}{\partial z^j} = 0$, $(j, m = 1, 2)$.
Substituting these relations into (23), we obtain

$$N^1_1 = \overset{a}{N}{}^1_1 = \eta^1;\ N^1_2 = \overset{a}{N}{}^1_2 = N^2_1 = \overset{a}{N}{}^2_1 = 0;\ N^2_2 = \overset{a}{N}{}^2_2 = \eta^2, \tag{32}$$

and so the metric (31) is Berwald one. Also, due to (32) it is obivious that the metric (30) is Kähler. Thus, by Theorem 4.5, the metric (31) is strongly Berwald.

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