FORCED STRING OSCILLATIONS UNDER THE INFLUENCE OF CONTINUOUS FORCE AND FORCES OF IMPULSE NATURE

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ANNOTATION:

The string oscillations of finite length with fixed ends, under the influence of a continuous force and forces of impulsive nature arising at some fixed moments of time are investigated.

The mathematical model of such an oscillatory process is a string oscillation equation with homogeneous boundary conditions, inhomogeneous initial conditions and the condition of impulse action at fixed points of time.

This work gives the solution of this problem in the form of a series on eigenfunctions of the Sturm-Louiville problem, the conditions under which the obtained series converges uniformly are specified.

Keywords: String, string oscillations, continuous force, impulse effects, impulse differential equations, eigenvalues, eigenfunctions, uniform convergence.

INTRODUCTION:

Consider the problem of forced oscillations of a string with fixed ends, oscillating under the influence of an external continuous force, and forces of impulsive nature acting on the process of string oscillation at fixed moments of time.

Mathematically, this problem is the problem of solving a linear inhomogeneous equation of string oscillation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad t \neq t_k , \quad (1)$$

at boundary conditions

$$u(0,t) = u(l,t) = 0, \quad (2)$$

with initial conditions

$$u(x,0) = \varphi_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = \varphi_1(x), \; 0 \le x \le l \quad (3)$$

and impulse conditions of the type

$$\Delta \frac{\partial u}{\partial t}\Big|_{t=t_k} = \frac{\partial u(x,t_k+0)}{\partial t} - \frac{\partial u(x,t_k-0)}{\partial t} = I_k(x), \; 0 \le x \le l , \quad (4)$$

where $t_k > 0$, $k \in N$, the moments of impulse actions, and $I_k(x)$, $k \in N$, corresponding to t_k, $k \in N$, the values of impulse actions. Concerning the moments of impulse actions, assume that $t_k > t_m$ at $k > m$ and $t_k \to +\infty$ at $k \to +\infty$.

The solution of the problem (1)-(4) is a continuous on $[0,l]\times[0,+\infty)$ function $u(x,t)$, twice continuously differentiable with respect to both arguments for $t \in \bigcup_{k=0}^{\infty}(t_k, t_{k+1})$ and having a continuous right-hand derivative for t at $t = t_k, k \in N$, satisfying equation (1) at $t \neq t_k, k \in N$, boundary conditions (2), initial conditions (3), and at $t = t_k, k \in N$ impulse conditions (4).

The solution of equation (1) satisfying the conditions (2)-(4) is sought in the series:

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) \cdot \sin\frac{n\pi}{l} x$$

(5)

by its own functions $\lambda_n = \sin\frac{n\pi x}{l}, n = 1, 2, \cdots$, Sturm-Louiville tasks

$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0$

Substituting (5) into equation (1) for $t \neq t_k, k = 1, 2, \ldots$, we have

$$\sum_{n=1}^{\infty}\left[T_n'' + \omega_n^2 T_n(t)\right]\sin\frac{n\pi}{l} x = f(x,t), \quad t = t_k, \quad k = 1, 2, \ldots$$

(6)

where $\omega_n = \frac{n\pi a}{l}$

Decompose the function $f(x,t)$ of the Sturm-Louiville task in the interval $(0,l)$ into a Fourier series of sines:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi}{l} x,$$

(7)

where

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi}{l} x dx$$

(8)

By comparing the expansions (6) and (7) for the same function $f(x,t)$, we obtain a second-order ordinary differential equation with respect to $T_n(t)$:

$$T_n''(t) + \omega_2^n T_n(t) = f_n(t), \quad t \neq t_k,$$

(9)

where $n, k = 1, 2, \cdots$.

From the initial condition (3) and (7), it follows

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin\frac{n\pi}{l} x = \varphi_0(x),$$

(10)

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} T_n'(0)\sin\frac{n\pi}{l} x = \varphi_1(x),$$

(11)

and from the impulse condition (4), we obtain

$$\Delta\frac{\partial u}{\partial t}(x,\mathrm{t})\Big|_{t=t_k} = \sum_{n=1}^{\infty}\Delta T_n(\mathrm{t})\Big|_{t=t_k}\cdot\sin\frac{n\pi}{l} x = I_k(x)$$

(12)

Now multiplying both parts of equations (10)-(12) by $\sin\frac{k\pi}{l} x, \quad k \in \mathrm{N}$, and integrating the result over the segment $[0,l]$ taking into account the orthogonality of the system function $\left\{\sin\frac{k\pi}{l} x\right\}$ on the segment $[0,l]$, we have

$$\mathrm{T}_n(0) = \frac{2}{l}\int_0^l \varphi_0(x)\sin\frac{n\pi}{l} x dx,$$

$$T_n'(0) = \frac{2}{l}\int_0^l \varphi_1(x)\sin\frac{n\pi}{l} x dx,$$

$$\Delta\mathrm{T}_n(\mathrm{t})\Big|_{t=t_k} = \frac{2}{l}\int_0^l I_k(x)\sin\frac{n\pi}{l} x dx.$$

Therefore, $T_n(t), n = 1, 2, \ldots$, the solutions of equation (9) that satisfy the initial conditions are

$$T_n(0)=\varphi_{0n} ,\tag{13}$$

$$T_n'(0)=\varphi_{1n}\tag{14}$$

and impulse conditions

$$\Delta T_n'(t)\Big|_{t=t_k}=I_{kn} .\tag{15}$$

The notations are introduced here:

$$\varphi_{0n}=\frac{2}{l}\int_0^l \varphi_0(x)\sin\frac{n\pi}{l}xdx\tag{16}$$

$$\varphi_{1n}=\frac{2}{l}\int_0^l \varphi_1(x)\sin\frac{n\pi}{l}xdx\tag{17}$$

$$I_{kn}=\frac{2}{l}\int_0^l I_k(x)\sin\frac{n\pi}{l}xdx\tag{18}$$

where $n=1,2,\ldots;\quad k=1,2,\ldots.$

The problem of constructing a solution of equation (9) satisfying conditions (16)-(18) is solved as follows.

First, construct the solution of equation $T_n^1(\mathrm{t})$ (9) under initial conditions (13) and (14), we have

$$T_n^1(\mathrm{t})=\varphi_{0n}\cos\omega_n t+\frac{1}{\omega_n}\varphi_{1n}\sin\omega_n t+\frac{1}{\omega_n}\int_0^t f_n(\tau)\sin\omega_n(t-\tau)d\tau\tag{19}$$

Next, we continue this solution on a half-interval $[t_1,t_2)$ as follows: we present the general solution of equation (9)

$$T_n^2(\mathrm{t})=C_1\cos\omega_n t+C_2\sin\omega_n t+\frac{1}{\omega_n}\int_{t_1}^t f_n(\tau)\sin\omega_n(t-\tau)d\tau,\tag{20}$$

where C_1 and C_2 are arbitrary constants. To determine them, we use the conditions of impulse actions (15) at the point $t=t_1$, we have

$$\begin{cases}T_n^2(\mathrm{t}_1)=T_n^1(\mathrm{t}_1)\\ (T_n^2(\mathrm{t}_1))'=(T_n^1(\mathrm{t}_1))'+I_{n1}\end{cases}\tag{21}$$

The solution of system (21) is relative C_1 and C_2 has the form:

$$C_1=\varphi_{0n}-\frac{1}{\omega_n}\int_0^{t_1} f_n(\tau)\sin\omega_n\tau d\tau-\frac{1}{\omega_n}I_{1n}\sin\omega_n t_1\tag{22}$$

$$C_2=\frac{\varphi_{1n}}{\omega_n}+\frac{1}{\omega_n}\int_0^{t_1} f_n(\tau)\cos\omega_n\tau d\tau+\frac{1}{\omega_n}I_{1n}\cos\omega_n t_1 .$$

Substituting the values C_1 and C_2 from (21) to (20), we write down the solution of the task (9), the (13)-(15) for $t\in[t_1,t_2)$ in the form

$$T_n^2(t)=\varphi_{0n}\cos\omega_n t+\frac{\varphi_{1n}}{\omega_n}\sin\omega_n t+\frac{1}{\omega_n}\int_0^t f_n(\tau)\sin\omega_n(t-\tau)d\tau+\frac{I_{1n}}{\omega_n}\sin\omega_n(t-t_1).\tag{23}$$

Further continuing in the same way the process of sequentially continuing the solution of the problem (9), (13)-(15) for $t\in[t_m,t_{m+1}),\, m\in N$, we have

$$T_n^m(t)=\varphi_{01}\cos\omega_n t+\frac{\varphi_{1n}}{\omega_n}\sin\omega_n t+\frac{1}{\omega_n}\int_{t_0}^t f_n(\tau)\sin\omega_n(t-\tau)d\tau+$$
$$+\sum_{i=1}^{m-1}\frac{I_{in}}{\omega_n}\sin\omega_n(t-t_i).\tag{24}$$

It follows that when in equation (9) the functions $f_n(t),\, n\in N$ are defined and continuous for all $t\in R$, the solution of the Cauchy problem with initial conditions (13), (14) for differential equation (9) with impulse action (15) exists and is unique. This solution is defined for everyone $t \in R$ and is represent able using the formula

$$T_n(t) = \varphi_{01}\cos\omega_n t + \frac{\varphi_{1n}}{\omega_n}\sin\omega_n t + \frac{1}{\omega_n}\int_{t_0}^{t} f_n(\tau)\sin\omega_n(t-\tau)d\tau + \sum_{0\le t_i<t}\frac{I_{in}}{\omega_n}\sin\omega_n(t-t_i).$$

Substituting the found expressions for $T_n(t)$ in the series (5), we obtain a solution to the problem (1)-(4) if the series (5) and the series obtained from it by direct differentiation x and t to two times inclusive, uniformly converge. It can be shown that such convergence of the series will be ensured if we require a uniform n boundedness of the partial sums of the series $\sum_{k=1}^{\infty} I_{kn}$ and the continuous function $f(x,t)$ had continuous partial derivatives up x to the second order and that the conditions $f(0,t) = f(l,t) = 0, I_k(0) = I_k(l) = 0, \forall k \in N$ were met for all values t.

From the above it follows that the solution of the problem (1)-(4) is expressed as a series

$$u(x,t) = \sum_{k=1}^{\infty}[(\varphi_{01} - \frac{I_{in}}{\omega_n}\sum_{0\le t_i<t}\sin\omega_n t_i)\cos\omega_n t + (\frac{\varphi_{1n}}{\omega_n} + \frac{I_{in}}{\omega_n}\sum_{0\le t_i<t}\cos\omega_n t_i)\sin\omega_n t]\sin\frac{k\pi x}{l} +$$
$$+\frac{1}{\omega_n}\sum_{k=1}^{\infty}\int_{t_0}^{t} f_n(\tau)\sin\omega_n(t-\tau)d\tau\sin\frac{k\pi x}{l}.$$

Note that the formulas obtained completely coincide with similar formulas of classical mathematical physics obtained by studing the string oscillations of finite length with fixed ends.

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