SABA Publishing

Journal of Mathematical Analysis and Modeling
jmam.sabapub.com
ISSN 2709-5924

J Math Anal & Model
(2022)3(1) : 15-29
doi:10.48185/jmam.v3i1.441

Completely positive map from $\mathbb{M}_4(\mathbb{C})$ to $\mathbb{M}_5(\mathbb{C})$ on positive semidefinite Matrices

C. A. Winda [a], N. B. Okelo [b,*], Omolo Ong'ati [c]

[a,b,c] Department of Pure and Applied Mathematics,
Jaramogi Oginga Odinga University of Science and Technology, Kenya

• Received: 08 January 2022 • Accepted: 13 March 2022 • Published Online: 29 March 2022

Abstract

Positive maps are essential in the description of quantum systems. However, characterization of the structure of the set of all positive maps is a challenge in mathematics and mathematical physics. We construct a linear positive map from $\mathbb{M}_4$ to $\mathbb{M}_5$ and state the conditions under which they are positive and completely positive (copositivity of positive).

Keywords: Positive maps, 2-positivity, Choi matrix, completely positivity, decomposable maps.
2010 MSC: put your Mathematics Subject Classification 2010 (MSC) 47B65, 15A60, 15A63, 15B48.

Choi [1] noted that the positive map ϕ is congruence if and only if ϕ is of the form $\phi(X) = V^*XV$ for all $A \in \mathbb{M}_n$, with V being an $m \times n$ matrix. Though it was conjectured that extreme rays of positive maps from $\mathbb{M}_n$ to $\mathbb{M}_m$ are all congruence maps, choi established by a counterexample in biquadratic forms to disapprove this conjecture. The famous Choi result in [1] affirms that a map ϕ is completely positive if and only if it's Choi matrix C_ϕ is positive definite. The positive map ϕ is completely positive if and only if the block matrix $[\phi(A)]$ is positive , otherwise it is not completely positive. It is more convenient to express n-positivity by using a block matrix notation. Since (X_{ij}) is positive semidefinite matrix, then $(\mathcal{I}_n \otimes \phi)(X_{ij})$ is the induced map, represented by the block matrix $[\phi(X_{ij})]$.

The construction of Choi's map [1], [2], [3] and Robertson′s map [4], [5] among other indecomposable maps have been used to justify the importance of these maps in their application in quantum mechanics. On the other hand, indecomposable maps may be considered as a huge obstacle in getting a canonical form for a positive map. The first example of indecomposable map was given by Choi [6] and [7] for a $\mathbb{M}_3$ commonly refereed to as the Choi map.

A family of indecomposable maps for an arbitrary finite dimension $n = 3$ was constructed by Kossakowski [8]. Several methods of construction of indecomposable maps

*Corresponding author: windac758@gmail.com

have been proposed by Kim and Kye [9] , Osaka [10], [11] and Tang [12] most of which are in the context of quantum entanglement. In this paper we have constructed a map from $\mathbb{M}_4$ to $\mathbb{M}_5$ like the two maps [7] and [13] with the off diagonal entries being a product of a negative parameter.

The rest of the paper is organized as follows. In Section 2, we introduce some notations, definitions and the construction of our map. Section 3 concept of biquadratic polynomial is used to state the positivity of the map. In Section 4, we establish the conditions under which the map is completely positive and completely copositive in Proposition 2.4 and Proposition 2.5. Finally we gives a concrete example in Example 2.6

By $\mathbb{M}_n$ we denote the set of positive semidefinite matrices of order n, that is $A \in \mathbb{M}_n$. The identity map on $\mathbb{M}_n(\mathbb{C})$ and the transpose map on $\mathbb{M}_n(\mathbb{C})$ are denoted by $\mathcal{I}_n$ and τ_n respectively. Let A be a $n \times n$ square matrix, A is positive semidefinite if, for any vector v with real components, $\langle x, Ax\rangle \geqslant 0$ for all $x \in \mathbb{R}^n$ or equivalently A is Hermitian and all its eigenvalues are non negative and positive definite if, in addition, $\langle x, Ax\rangle > 0$ for all $x \neq 0$. A linear map ϕ is from $\mathbb{M}_n(\mathbb{C})$ to $\mathbb{M}_m(\mathbb{C})$ is called positive if $\phi(\mathbb{M}_n(\mathbb{C}))^+ \subseteq \mathbb{M}_m(\mathbb{C})^+$. A map $\phi : \mathbb{M}_n(\mathbb{C}) \longrightarrow \mathbb{M}_m(\mathbb{C})$ is $n-$*positive* if $\mathcal{I} \otimes \phi : \mathbb{M}_n \otimes \mathbb{M}_n \longrightarrow \mathbb{M}_n \otimes \mathbb{M}_m$ is positive. On the other hand, $\phi : \mathbb{M}_n(\mathbb{C}) \longrightarrow \mathbb{M}_m(\mathbb{C})$ is $n-$*copositive* if the map $\tau_n \otimes \phi : \mathbb{M}_n \otimes \mathbb{M}_n \longrightarrow \mathbb{M}_n \otimes \mathbb{M}_n$ is positive.

We construct a linear map $\phi_{(\mu,c_1,c_2,c_3)}$ from $\mathbb{M}_4$ to $\mathbb{M}_5$ and study its properties of positivity, completely positivity and decomposability. The values of parameters $\mu, c_1, c_2, c_3 \in \mathbb{R}^+$.

Let $X \in \mathbb{M}_n(\mathbb{C})$ be a positive semidefinite matrix written, $X = \left(x_i x_j^*\right)$, where $x_i = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ is a column vector and x_j^* is the transpose conjugate(row vector) of x_i. The diagonal elements of the positive semidefinite matrix X given by $x_n^* x_n = |x_n|$ are positive real numbers. We denote the diagonal entries $x_n x_n^* \in \mathbb{R}$ by α_n.

Definition 0.1. Let X be a 4×4 a positive semidefinite matrix with complex entries. Let $c_1, c_2, c_3 \in \mathbb{R}^+$, $0 < \mu \leqslant 1$ and $r \in \mathbb{N}$. Then we define the family of positive maps $\phi_{(\mu,c_1,c_2,c_3)}$ as follows:

$$\phi_{(\mu,c_1,c_2,c_3)} : \mathbb{M}_4(\mathbb{C}) \longrightarrow \mathbb{M}_5(\mathbb{C}).$$

$$X \mapsto \begin{pmatrix} P_1 & -c_1x_1x_2^* & -c_2x_1x_3^* & 0 & -\mu x_1x_4^* \\ -c_1x_2x_1^* & P_2 & -c_2x_2x_3^* & -c_3x_2x_4^* & 0 \\ -c_2x_3x_1^* & -c_2x_3x_2^* & P_3 & -c_3x_3x_4^* & 0 \\ 0 & -c_nx_4x_2^* & -c_nx_4x_3^* & P_4 & 0 \\ -\mu x_4x_1^* & 0 & 0 & 0 & P_5 \end{pmatrix},$$

where

$$\begin{aligned} P_1 &= \mu^{-r}(\alpha_1 + c_1\alpha_2\mu^r + c_2\alpha_3\mu^r + c_3\alpha_4\mu^r) \\ P_2 &= \mu^{-r}(\alpha_2 + c_1\alpha_3\mu^r + \alpha_4c_2\mu^r + \alpha_1c_3\mu^r) \\ P_3 &= \mu^{-r}(\alpha_3 + c_1\alpha_1\mu^r + \alpha_2c_2\mu^r + \alpha_3c_3\mu^r) \\ P_4 &= \mu^{-r}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\ P_5 &= \mu^r(\alpha_4 + c_1\alpha_1\mu^r + c_2\alpha_2\mu^r + c_3\alpha_4\mu^r) \end{aligned}$$

1. Positivity

A crucial problem in applications of positive maps is checking whether or not they are positive. It is well-known that determining positivity of linear maps is equivalent to detecting nonnegativity of biquadratic forms. it is known that there is a positive semidefinite biquadratic form that is not the sum of squares of bilinear forms [7], Theorem 1.

A linear map ϕ from $\mathbb{M}_n(\mathbb{C})$ to $\mathbb{M}_m(\mathbb{C})$ preserving symmetry is said to be positive if the matrices $\phi(X)$ are positive semidefinite for all symmetric positive semidefinite matrices $X \in \mathbb{M}_n(\mathbb{C})$. The linear map ϕ is the image of positive semidefinite matrices of rank 1. That is , if the matrix $x_i x_j^*$ has rank 1 where x is a column vector. Then, from the definition of positive semidefinite matrices positivity of the map ϕ give by the biquadratic polynomials of $\phi(X)$. The linear map ϕ is uniquely determined by the polynomial function;

$$F(z, x) := z\phi_{(c_1,\dots,c_{n-1})}(x_i x_j^*)z^T$$

as a biquadratic function in $x := (x_1, \dots, x_n)$ and $z := (z_1, \dots, z_{n+1})$. The map $\phi_{(c_1,\dots,c_{n-1})}$ is positive if and only if the biquadratic form $F(z, x)$ is positive semi-definite (the biquadratic function is a sum of squares).

Lemma 1.1. *Then the function*

$$\begin{aligned} F(z_1, z_2, z_3, z_4, z_5, t) &= c_3|t|z_1^2 + (c_3 + c_2|t| - 2\mu^r c_2^2)z_2^2 + (c_1|t| + c_3)z_3^2 \\ &+(3\mu^{-r} + \mu^{-r}|t| - 3\mu^r c_3 \mathrm{Re}(t)^2)z_4^2 + (c_1 + c_2 + c_3 + |t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \\ &+c_1(z_1 - z_2)^2 + c_2(z_1 - z_3)^2 \\ &+\frac{\mu^{-r}}{2}(z_3 - 2\mu^r c_2 z_2)^2 + \mu^{-r}(z_2 - 2\mu^r c_3 \mathrm{Re}(t)z_4)^2 \\ &+\mu^{-r}(z_3 - 2\mu^r c_3 \mathrm{Re}(t)z_4)^2 + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 \end{aligned}$$

is positive semidefinite for every z_1, z_2, z_3, z_4, z_5 and $t \in \mathbb{C}$ whenever it satisfy the inequalities,

$$\mu^{-r} \geqslant 2c_3, \tag{1.1}$$
$$\mu^{-r} \geqslant 2c_1, \tag{1.2}$$
$$\mu^{-r} \geqslant c_2, \tag{1.3}$$
$$c_1\mu^{-r} \geqslant c_2^2. \tag{1.4}$$

Proof. If $z_1 = 0$. Then,
$F(0, z_2, z_3, z_4, z_5, t)$

$$\begin{aligned} &= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)z_3^2 \\ &\quad +\mu^{-r}(3 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 \\ &\quad -2c_2z_2z_3 - 2c_3\mathrm{Re}(t)z_2z_4 - 2c_3\mathrm{Re}(t)z_3z_4 \\ &= \mu^{-r}(1 + c_1\mu^r)z_2^2 + c_2(|t| - 1)z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r)z_3^2 + 3\mu^{-r}z_4^2 \\ &\quad +\mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 + c_2(z_3 - c_2)^2 + c_3(z_2 - \mathrm{Re}(t)z_4)^2 \\ &\quad +(\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2 + c_3(z_3 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2. \end{aligned}$$

From the coefficients of z_2^2 and z_4^2 we have,

$$\mu^{-r} + c_1 + c_2|t| - c_2 = (\mu^{-r} - c_2) + c_1 + c_2|t|$$

and

$$3\mu^{-r} + \mu^{-r}|t| - 2c_3\mathrm{Re}(t)^2 = 3\mu^{-r} + \mu^{-r}(|x|^2 + |y|^2) - 2c_3|x|^2$$

respectively. The function $F(0, z_2, z_3, z_4, z_5, t)$ is positive whenever it satisfy the inequalities, $\mu^{-r} \geqslant c_2$ and $\mu^{-r} \geqslant 2c_3$.

If $z_2 = 0$. Then ,

$$\begin{aligned} F(z_1, 0, z_3, z_4, z_5, t) &\\ &= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)z_3^2 \\ &\quad + \mu^{-r}(3 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 \\ &\quad - 2c_2z_1z_3 - 2c_3\mathrm{Re}(t)z_3z_4 - 2\mu\mathrm{Re}(t)z_1z_5 \\ &= (c_1 + c_3|t|)z_1^2 + (c_1|t| + c_3)z_3^2 + 3\mu^{-r}z_4^2 + (c_1 + c_2 + c_3)z_5^2 + c_2(z_1 - z_3)^2 \\ &\quad + \mu^{-r}(z_3 - \mu^r c_3\mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2 \\ &\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \\ &\geqslant 0 \end{aligned}$$

whenever the coefficients of z_4^2 satisfy the inequality

$$\begin{aligned} \mu^{-2r}(3 + |t|) - c_3^2\mathrm{Re}(t)^2 &= 3\mu^{-2r} + \mu^{-2r}(|x|^2 + |y|^2) - c_3^2|x|^2 \\ &\geqslant 0 \end{aligned} \tag{1.5}$$

whenever (1.1) hold.

If $z_3 = 0$. Then ,

$$\begin{aligned} F(z_1, z_2, 0, z_4, z_5, t) &\\ &= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2 \\ &\quad + \mu^{-r}(3 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 \\ &\quad - 2c_1z_1z_2 - 2c_3\mathrm{Re}(t)z_2z_4 - 2\mu\mathrm{Re}(t)z_1z_5 \\ &= (c_2 + c_3|t|)z_1^2 + (c_1 + c_2|t|)z_2^2 + 3\mu^{-r}z_4^2 + (c_1 + c_2 + c_3)z_5^2 + c_1(z_1 - z_2)^2 \\ &\quad + (c_3 - c_1)z_2^2 + \mu^{-r}(z_2 - \mu^r c_3\mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2 \\ &\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \\ &\geqslant 0 \end{aligned}$$

with the coefficients of z_4^2 satisfying the inequality (1.1).

If $z_4 = 0$. Then,

$$\begin{aligned} F(z_1, z_2, z_3, 0, z_5, t) &\\ &= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2 \\ &\quad + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)z_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 \\ &\quad - 2c_1z_1z_2 - 2c_2z_1z_3 - 2c_2z_2z_3 - 2\mu\mathrm{Re}(t)z_1z_5 \\ &= c_3|t|z_1^2 + c_1z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_3\mu^r)z_3^2 + \mu^{-r}(|t| + c_1\mu^r \\ &\quad + c_2\mu^r + c_3\mu^r)z_5^2 + c_1(z_1 - z_2)^2 + c_2(z_1 - z_3)^2 + \mu^{-r}(z_3 - \mu^r c_2z_2)^2 \\ &\quad + (c_2|t| - \mu^r c_2^2)z_2^2 + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \\ &\geqslant 0 \end{aligned}$$

The function $F(z_1,z_2,z_3,0,z_5,t)$ is positive if the coefficients of z_2^2 satisfy the inequality (1.4).

If $z_5=0$. Then,

$$
\begin{aligned}
F(z_1,z_2,z_3,z_4,0,t)&\\
=\ & \mu^{-r}(1+c_1\mu^r+c_2\mu^r+c_3|t|\mu^r)z_1^2+\mu^{-r}(1+c_1\mu^r+c_2|t|\mu^r+c_3\mu^r)z_2^2\\
&+\mu^{-r}(1+c_1|t|\mu^r+c_2\mu^r+c_3\mu^r)z_3^2+\mu^{-r}(3+|t|)z_4^2\\
&-2c_1z_1z_2-2c_2z_1z_3-2c_2z_2z_3-2c_3Re(t)z_2z_4-2c_3Re(t)z_3z_4\\
=\ & \mu^{-r}(1+c_3|t|\mu^r)z_1^2+c_3z_2^2+\mu^{-r}z_3^2+3\mu^{-r}z_4^2+c_1(z_1-z_2)^2+c_2(z_1-z_3)^2\\
&+c_2(|t|z_2-z_3)^2+(c_1|t|-c_2)z_3^2+\mu^{-r}(z_2-c_1\mu^rRe(t)z_4)^2\\
&+(\mu^{-r}\frac{|t|}{2}-\mu^rc_1^2Re(t)^2)z_4^2+c_3(z_3-Re(t)z_4)^2+(\mu^{-r}\frac{|t|}{2}-c_3Re(t)^2)z_4^2
\end{aligned}
$$

The function $F(z_1,z_2,z_3,z_4,0,t)$ is positive whenever the coefficients of z_3^2 satisfy (1.3) while the coefficients z_4^2 satisfy the inequalities (1.1) and (1.2).

Let $z_i\neq 0, i=1,2,3,4,5$ and assume that there exist $z_1,z_2,z_3,z_4,z_5\in \mathrm{Real}$ and $t\in\mathbb{C}$ such that $z_1\neq 0$ and $F(z_1,z_2,z_3,z_4,z_5,t)<0$. Since $0<\mu<1$ and $c_1,c_2\geqslant 0$. Then,

$$
\begin{aligned}
F(z_1,z_2,z_3,z_4,z_5,t)&\\
=\ & \mu^{-r}(1+c_1\mu^r+c_2\mu^r+c_3|t|\mu^r)z_1^2+\mu^{-r}(1+c_1\mu^r\\
&+c_2|t|\mu^r+c_3\mu^r)z_2^2+\mu^{-r}(1+c_1|t|\mu^r+c_2\mu^r+c_3\mu^r)z_3^2\\
&+\mu^{-r}(3+|t|)z_4^2+\mu^{-r}(|t|+c_1\mu^r+c_2\mu^r+c_3\mu^r)z_5^2\\
&-2c_1z_1z_2-2c_2z_1z_3-2c_2z_2z_3-2c_3Re(t)z_2z_4-2c_3Re(t)z_3z_4-2\mu Re(t)z_1z_5\\
=\ & c_3|t|z_1^2+\mu^{-r}z_2^2+\mu^{-r}z_3^2+3\mu^{-r}z_4^2+(c_1+c_2+c_3)z_5^2+c_1(z_1-z_2)^2\\
&+c_2(c_1-c_3)^2+c_2(|t|z_2-z_3)^2+(c_1|t|-c_2)z_3^2+c_3(z_2-Re(t)z_4)^2\\
&+(\mu^{-r}\frac{|t|}{2}-c_3Re(t)^2)z_4^2+c_3(z_3-Re(t)z_4)^2+(\mu^{-r}\frac{|t|}{2}-c_3Re(t)^2)z_4^2\\
&+\mu^{-r}(z_1-\mu^{1+r}Re(t)z_5)^2+(|t|\mu^{-r}-\mu^{2+r}Re(t)^2)z_5^2\\
<\ & 0
\end{aligned}
$$

is a contradiction when the inequalities (1.1) and (1.3) hold . Thus
$F(z_1,z_2,z_3,z_4,z_5,t)\geqslant 0$ for every $z_1,z_2,z_3,z_4,z_5\in \mathrm{Real}$ and $t\in\mathbb{C}$ □

Proposition 1.2. *The linear map $\phi_{(\mu,c_1,c_2,c_3)}:\mathbb{M}_4\longrightarrow\mathbb{M}_5$ is positive provided Lemma 1.1 is satisfied.*

Proof. We need to show that,

$$\phi_{(\mu,c_1,c_2,c_3)}\left(\begin{pmatrix} q\\ s\\ u\\ t\end{pmatrix}\begin{pmatrix}\bar{q} & \bar{s} & \bar{u} & \bar{t}\end{pmatrix}\right)\in\mathbb{M}_5^+$$

for every $q,s,u,t\in\mathbb{C}$.

That is,

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}^T \begin{pmatrix} p_1 & -c_1 q\bar{s} & -c_2 q\bar{u} & 0 & -\mu q\bar{t} \\ -c_1 s\bar{q} & p_2 & -c_2 s\bar{u} & -c_3 s\bar{t} & 0 \\ -c_2 u\bar{q} & -c_2 u\bar{s} & p_3 & -c_3 u\bar{t} & 0 \\ 0 & -c_3 t\bar{s} & -c_3 t\bar{u} & p_4 & 0 \\ -\mu t\bar{q} & 0 & 0 & 0 & p_5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} \geqslant 0 \qquad (1.6)$$

where,

$$\begin{aligned} p_1 &= \mu^{-r}(|q|^2 + |s|^2 c_1 \mu^r + |u|^2 c_2 \mu^r + c_3 |t| \mu^r) \\ p_2 &= \mu^{-r}(|s|^2 + |u|^2 c_1 \mu^r + c_2 |t| \mu^r + |q|^2 c_3 \mu^r) \\ p_3 &= \mu^{-r}(|u|^2 + c_1 |t| \mu^r + |q|^2 c_2 \mu^r + |s|^2 c_3 \mu^r) \\ p_4 &= \mu^{-r}(|q|^2 + |s|^2 + |u|^2 + |t|) \\ p_5 &= \mu^{-r}(|t| + |q|^2 c_1 \mu^r + |s|^2 c_2 \mu^r + |u|^2 c_3 \mu^r) \end{aligned}$$

for every $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$ and $q, s, u, t \in \mathbb{C}$.
Taking $q = s = u = 0$,

$$c_3|t|z_1^2 + c_2|t|z_2^2 + c_1|t|z_3^2 + \mu^{-r}|t|z_4^2 + \mu^{-r}|t|z_5^2 \geqslant 0.$$

If $q = 0$, given that $0 < \mu < 1$. Then,

$$\begin{aligned} & (c_1 + c_2 + c_3|t|)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)z_2^2 + \mu^{-r}(1 + c_1\mu^r + c_3\mu^r)z_3^2 \\ & +\mu^{-r}(2 + |t|)z_4^2 + \mu^{-r}(|t| + c_2\mu^r + c_3\mu^r)z_5^2 \\ & -2c_2z_2z_3 - 2c_3\mathrm{Re}(t)z_2z_4 - 2c_3\mathrm{Re}(t)z_3z_4 \\ = \; & (c_1 + c_2 + c_3|t|)z_1^2 + c_1z_2^2 + \mu^{-r}z_3^2 + 2\mu^{-r}z_4^2 + \mu^{-r}(|t| + c_2\mu^r \\ & +c_3\mu^r)z_5^2 + c_2(|t|z_2 - z_3)^2 + (\frac{c_1}{c_2} - 1)z_2^2 + \mu^{-r}(z_2 - \mu^r c_3\mathrm{Re}(t)z_4)^2 \\ & +(\mu^{-r}\frac{|t|}{2} - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2 + c_3(z_3 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2 \end{aligned}$$

is positive by inequality (1.1) and (1.3).
If $s = 0$. Since $0 < \mu < 1$. Then,

$$\begin{aligned} & \mu^{-r}(1 + c_2\mu^r + c_3|t|\mu^r)z_1^2 + (c_1 + c_2|t| + c_3)z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)z_3^2 \\ & +\mu^{-r}(2 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_3\mu^r)z_5^2 \\ & -2z_2z_3c_2 - 2z_3z_4c_3\mathrm{Re}(t) - 2z_1z_5\mu\mathrm{Re}(t) \\ = \; & c_3|t|z_1^2 + (c_1 + c_2|t| + c_3)z_2^2 + c_1|t|z_3^2 + 2\mu^{-r}z_4^2 + (c_1 + c_3)z_5^2 + c_2(z_1 - z_3)^2 \\ & +\mu^{-r}(z_3 - \mu^r c_3\mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2 \\ & +\mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \end{aligned}$$

is positive when the inequality (1.3) hold.

If $u = 0$ and $0 < \mu < 1$. Then,

$$\begin{aligned}
&\mu^{-r}(1 + c_1\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_2|t|\mu^r + c_3\mu^r)z_2^2\\
&+(c_1|t| + c_2 + c_3)z_3^2 + \mu^{-r}(2 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r\\
&+c_2\mu^r)z_5^2 - 2c_1z_1z_2 - 2c_3\mathrm{Re}(t)z_2z_4 - 2\mu\mathrm{Re}(t)z_1z_5\\
= \;& c_3|t|z_1^2 + \mu^{-r}(1 + c_2|t|)z_2^2 + (c_1|t| + c_2 + c_3)z_3^2 + 2\mu^{-r}z_4^2 + (c_1 + c_2)z_5^2\\
&+c_1(z_1 - z_2)^2 + (\mu^{-r} - c_1)z_2^2 + c_3(z_2 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - c_3\mathrm{Re}(t)^2)z_4^2\\
&+\mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2
\end{aligned}$$

is positive when the inequalities (1.1) and (1.2) are satisfied.
Now if q, s and u are not equal to zero. Assume that $q = s = u = 1$. Then, by Lemma 1.1

$$z^T\phi_{(\mu,c_1,c_2,c_3)}\left(\begin{pmatrix}1\\1\\1\\t\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & \bar{t}\end{pmatrix}\right)z$$

is positive for every $z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{R}^5$ and $t \in \mathbb{C}$ □

2. Completely positivity

The tensor product of positive semidefinite matrices $\mathbb{M}_n$ and $\mathbb{M}_{n+1}$ is isomorphic to the block matrices $\mathbb{M}_n(\mathbb{M}_{n+1})$. That is, $\mathbb{M}_n(\mathbb{M}_{n+1}) \cong \mathbb{M}_n \otimes \mathbb{M}_{n+1}$.

$$\mathbb{M}_n \otimes \mathbb{M}_{n+1} \cong \mathbb{M}_n(\mathbb{M}_{n+1}) \cong \mathbb{M}_2(\mathbb{M}_q) \text{ for some } k \in \mathbb{N}.$$

This gives the Choi matrix described in [14] which we write as,

$$C_\phi = \left(\begin{array}{cc|cc} a & C_{1\times m} & 0 & Y_{2\times m}\\ C^*_{m\times 1} & B_{m\times m} & Z^*_{m\times 1} & T_{m\times m}\\ \hline 0 & Z_{1\times m} & d & 0_{1\times m}\\ Y^*_{m\times 1} & T^*_{m\times m} & 0_{m\times 1} & U_{m\times m}\end{array}\right) \tag{2.1}$$

where $A, D \in \mathbb{M}_2$ are positive diagonal matrices. B and U are positive semidefinite matrices in $\mathbb{M}_{n+1}$, $T \in \mathbb{M}_{n+1}$ not necessarily positive and $C, Y, Z \in \mathbb{M}_{2\times(n+1)}$. The map $\phi \mapsto C_\phi$ is linear, injective and is surjective, and given an operator $\sum_{i,j=1}^{n} E_{ij} \otimes \phi(E_{ij}) \in \mathbb{M}_n \otimes \mathbb{M}_{n+1}$. By and canonical shuffling the Choi matrix of the linear map $\phi_{(\mu,c_1,\ldots,c_n)}$ is such that $C_\phi \in \mathbb{M}_{2q}$.

The Choi result in [1] affirms that a map ϕ is completely positive if and only if the Choi matrix C_ϕ is positive definite. For convenience we express n-positivity by using a block matrix notation. Since (X_{ij}) is positive semidefinite matrix, then $(\mathcal{I}_n \otimes \phi)(X_{ij})$ is the induced map, represented by the block matrix $[\phi(X_{ij})]^n$. We need to note that the positivity of the Choi matrix depends on the choice of matrix units (E_{ij}).

Note that the positive map ϕ is completely positive if and only if it is k-positive. Since our map ϕ from $\mathbb{M}_n$ to $\mathbb{M}_{n+1}$ is 2-positive, we look at the conditions for complete positivity and complete copositivity of this map by applying the next propositions in [14].

Proposition 2.1. *([14], Proposition 3.1) Let $\phi : \mathbb{M}_n \longrightarrow \mathbb{M}_{n+1}$ be a 2-positive map with the Choi matrix of the form, 2.1. ϕ is completely positive if the following conditions hold.*

(i). $Z = 0$.

(ii). $C^*C \leqslant B$.

(iii). $dU \geqslant 0$.

(iv) $Y^*Y \leqslant U$.

(v) if B is invertible, then $T^*B^{-1}T \leqslant U$.

Remark 2.2. The transposition in this case imply the Partial Positive transpose of the Choi matrix $C_\phi \in \mathbb{M}_n(\mathbb{M}_{n+1})$. The transposition is operated with respect to the blocks $\mathbb{M}_n$. This leads to the Partial Positive transpose Choi matrix $C_\phi^\Gamma \in \mathbb{M}_n(\mathbb{M}_{n+1})$ with the structure;

$$C_\phi^\Gamma = \left(\begin{array}{cc|cc} a & C^*_{1\times m} & 0 & Z^*_{1\times m} \\ C_{m\times 1} & B_{m\times m} & Y_{m\times 1} & T^*_{m\times m} \\ \hline 0 & Y^*_{1\times m} & d & 0_{2\times m} \\ Z_{m\times 1} & T_{m\times m} & 0_{m\times 1} & U_{m\times m} \end{array}\right).$$

We show the proof of (i)since the other parts of the proof follows from the proof of Theorem 2.1

Proposition 2.3. *([14], Proposition 3.2) Let* $\phi : \mathbb{M}_n \longrightarrow \mathbb{M}_{n+1}$ *be a 2-positive map with the Choi matrix of the form, 2.1.* ϕ *is completely copositive if the following conditions hold.*

(i). $Y = 0$.

(ii). $C^*C \leqslant B$.

(iii). $dU \geqslant 0$.

(iv) $Z^*Z \leqslant U$.

(v) if B is invertible, then $T^*B^{-1}T = U$.

2.1. *Completely (co)positivity of* $\phi_{((\mu,c_1,c_2,c_3)}$

Proposition 2.4. *Let* $\phi_{((\mu,c_1,c_2,c_3)}$ *be a positive map given by (1.6). Then following conditions are equivalent:*

(i) $\phi_{(\mu,c_1,c_2,c_3)}$ *is completely positive.*

(ii) $\phi_{(\mu,c_1,c_2,c_3)}$ *is 2-positive.*

Proof. (ii) $\Rightarrow$ (iii).
Assume $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive. Consider a rank one matrix $P = [x_i x_j]$ a positive element in $\mathbb{M}_2(\mathbb{M}_5(\mathbb{C}))$ where $x_i = (1,1,0,0,1,1,0,0,1,1)^T$, we have that,

$$\mathcal{I}_2 \otimes \phi_{(\mu,c_1,c_2,c_3)}(P) = \left(\begin{array}{ccccc|ccccc} \mu^{-r} & . & . & . & . & . & -c_1 & -c_2 & . & -\mu \\ . & c_3 & . & . & . & . & . & . & . & . \\ . & . & c_2 & . & . & . & . & . & . & . \\ . & . & . & \mu^{-r} & . & . & . & . & . & . \\ . & . & . & . & c_1 & . & . & . & . & . \\ \hline . & . & . & . & . & c_1 & . & . & . & . \\ -c_1 & . & . & . & . & . & \mu^{-r} & -c_2 & -c_3 & . \\ -c_2 & . & . & . & . & . & -c_2 & \mu^{-r} & -c_3 & . \\ . & . & . & . & . & . & -c_3 & -c_3 & \mu^{-r} & . \\ -\mu & . & . & . & . & . & . & . & . & \mu^{-r} \end{array}\right) \quad (2.2)$$

in $\mathbb{M}_2(\mathbb{M}_5(\mathbb{C}))$ where zeros are replaced by dots. Since $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive, the above matrix is positive definite. Therefore,

$$\begin{vmatrix} \mu^{-r} & -c_1 & -c_2 & . & \mu \\ -c_1 & \mu^{-r} & -c_2 & -c_3 & . \\ -c_2 & -c_2 & \mu^{-r} & -c_3 & . \\ . & -c_3 & -c_3 & \mu^{-r} & . \\ \mu & . & . & . & \mu \end{vmatrix} \geqslant 0 \tag{2.3}$$

provided

$$\mu^{-r} > c_1, \quad \mu^{-r} > c_2 \quad \text{and} \quad \mu^{-r} \geqslant 2c_3 \tag{2.4}$$

. The Choi matrix $C_{\phi_{(\mu,c_1,c_2,c_3)}}$ is;

$$\left(\begin{array}{c|ccccccccc||c|ccccccccc}
\mu^{-r} & . & . & . & . & . & -c_1 & . & . & . & . & . & -c_2 & . & . & . & . & . & . & -\mu \\ \hline
. & c_3 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & c_2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & \mu^{-r} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & c_1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & c_1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
-c_1 & . & . & . & . & . & \mu^{-r} & . & . & . & . & . & -c_2 & . & . & . & . & . & -c_3 & . \\
. & . & . & . & . & . & . & c_3 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & \mu^{-r} & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & c_2 & . & . & . & . & . & . & . & . & . & . \\ \hline\hline
. & . & . & . & . & . & . & . & . & . & c_2 & . & . & . & . & . & . & . & . & . \\ \hline
. & . & . & . & . & . & . & . & . & . & . & c_1 & . & . & . & . & . & . & . & . \\
-c_2 & . & . & . & . & . & -c_2 & . & . & . & . & . & \mu^{-r} & . & . & . & . & . & -c_3 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & \mu^{-r} & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & c_3 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & c_3 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & c_2 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & c_1 & . & . \\
. & . & . & . & . & . & -c_3 & . & . & . & . & . & -c_3 & . & . & . & . & . & \mu^{-r} & . \\
-\mu & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & \mu^{-r}
\end{array}\right).$$

Since $a \geqslant 0$,

$$aB - C^*C = \begin{pmatrix} c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-2r} - c_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2\mu^{-r} \end{pmatrix}.$$

The inequality hold when $\mu^{-r} > c_1$.

$$dU = c_2 \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

The inequality $dU \geqslant 0$ holds when $\mu^{-r} > c_3$.

$$aU - Y^*Y = \begin{pmatrix} c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-2r} - c_2^2 & 0 & 0 & 0 & 0 & 0 & -c_3\mu^{-r} & 0 \\ 0 & 0 & \mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 \\ 0 & -c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} - \mu^2 \end{pmatrix}$$

is positive since $\mu^{-2r} > c_2^2 + c_3^2$ provided $\mu^{-r} \geqslant c_2$ and $\mu^{-r} \geqslant c_3$.

$U - T^*BT$

$$= \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{c_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c_2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 - c_2^2\mu^{r} - c_3^2\mu^{r} & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

All the principal minors of $U - TB^{-1}T$ are positive when $c_1\mu^{-r} - (c_2^2 + c_3^2) > 0$ and $\mu^{-r} > c_3$.

Hence the set of inequalities (2.4) are satisfied, consequently $C_{\phi_{((\mu,c_1,c_2,c_3)}}$ is positive semidefinite. Hence, complete positivity of $\phi_{((\mu,c_1,c_2,c_3)}$ follows. □

Proposition 2.5. *Let $\phi_{((\mu,c_1,c_2,c_3)}$ be a positive map given by (1.6) . The positive map $\phi_{((\mu,c_1,c_2,c_3)}$ is completely copositive if the following conditions holds.*

(i) *$\phi_{(\mu,c_1,c_2,c_3)}$ is 2-copositive.*

(ii) *$\phi_{(\mu,c_1,c_2,c_3)}$ is completely copositive.*

Proof. Assume $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-copositive. Consider a rank one matrix P an element in $\mathbb{M}_2(\mathbb{M}_5(\mathbb{C}))$ where $x_i = (1,1,0,0,1,1,0,0,1,1)^T$, we have that,

$$\tau_2 \otimes \phi_{(\mu,c_1,c_2,c_3)}(P) = \left(\begin{array}{ccccc|ccccc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_3 & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_2 & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_1 & -\mu & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & -c_1 & -c_2 & \cdot & -\mu & c_1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & -c_2 & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \mu^{-r} & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & -c_3 & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \tag{2.5}$$

in $\mathbb{M}_2(\mathbb{M}_5(\mathbb{C}))$. By computation of the minors, $\mathcal{J}_2 \otimes \phi_{(\mu,c_1,c_2,c_3)}(P)$ is positive semidefinite on condition that;

$$\mu^{-r} > c_1, \quad \mu^{-r} > c_2 \quad \mu^{-r} \geqslant 2c_3 \quad c_3 \geqslant c_1 \quad \text{and} \quad c_1 \geqslant c_2 \tag{2.6}$$

hold.

The choi matrix ,

$$C^{\Gamma}_{\phi_{(\mu,c_1,c_2,c_3)}} = \left(\begin{array}{c|ccccccccc||c|ccccccccc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & c_3 & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot \\
\cdot & -c_1 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right).$$

Since $a \geqslant 0$ and $C = 0$.

$$aB - CC^* = \mu^{-r}\begin{pmatrix}
c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 \\
0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2
\end{pmatrix}$$

The inequality hold when $c_3 \geqslant c_1$.

Since F is a zero matrix, $dU - FF^*$ is positive when the inequality $c_1\mu^{-r} > c_3^2$ hold.

$$aU - ZZ^* = \begin{pmatrix} \mu^{-r}c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-2r} - c_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-2r} & 0 & 0 & 0 & -c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & \mu^{-r}c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^{-r}c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-r}c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3\mu^{-r} & 0 & 0 & 0 & \mu^{-r}c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} \end{pmatrix}.$$

The matrix is positive when the inequality $c_1\mu^{-r} > c_2^2$ holds.
Finally,

$$U - TB^{-1}T^*$$

$$= \begin{pmatrix} c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{T}$$

$$= \begin{pmatrix} c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & \frac{-c_2\mu}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & \frac{c_1^2+c_1c_3-c_2^2}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & -c_3\mu^{1+r} & 0 & 0 & 0 & \mu^{-r} - c_3^2\mu^{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

The matrix $U - TB^{-1}T^*$ is positive provided the inequalities,

$$\mu^{-r} > c_3, \quad c_1 \geqslant c_2 \quad \text{and} \quad c_1\mu^{-r} > c_3^2$$

hold. □

Example 2.6. Let $r = 3$ The map $\phi_{(\frac{1}{2},\frac{2}{3},\frac{1}{5},\frac{3}{4})}$ is both completely positive and completely copositive but is not easy to find the values of $p, t_1, t_2, t_3 \in [0, 1]$ for which it is decomposable. B computation the 2-positive map $\phi_{(\frac{1}{2},\frac{2}{3},\frac{1}{5},\frac{3}{4})}$ yields the Choi matrix C_ϕ as,

$$
\left(\begin{array}{c|ccccccccc||c|ccccccccc}
8 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ \hline
0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{array}\right)
$$

with eigenvalues

$$9.18471, 8.53052, 8.13487, 8., 8., 8., 7.51328, 6.63662, 0.75, 0.75,$$

$$0.75, 0.75, 0.666667, 0.666667, 0.666667, 0.666667, 0.2, 0.2, 0.2, 0.2.$$

The 2-copositive map $\phi_{(\frac{1}{2},\frac{2}{3},\frac{1}{5},\frac{3}{4})}$ yields the Choi matrixC_ϕ^Γ as,

$$
\left(\begin{array}{c|ccccccccc||c|ccccccccc}
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \frac{3}{4} & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\hline
0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{array}\right)^{\Gamma}
$$

with eigenvalues

$$8.07592, 8.07146, 8., 8., 8., 8., 8., 8., 1.3763, 1.21007, 0.912628, 0.75,$$

$$0.590748, 0.504039, 0.4, 0.2066, 0.2, 0.128539, 0.0403659, 0.$$

Acknowledgement

We wish to thank the anonymous referees for their insightful suggestions that helped to improve the content of the paper.

References

[1] Choi M-D. (1975). *Completely positive maps on complex matries*. Linear Algebra and its Applications **10**: 285-290. https://doi.org/10.1016/0024-3795(75)90075-0

[2] Hall W.(2006). *Constructions of indecomposable positive maps based on a new criterion for indecomposability*. https://arXiv:quant-ph/0607035v1: 1-11.

[3] Hall W.(2006). *A new criterion for indecomposability of positive maps,*. J. Phys. A. **39**: 14119-14131. https://doi.org/10.1088/0305-4470/39/45/020

[4] Robertson A. G.(1985). *Positive prjection on* C^*-*algebra and extremal positive maps,*. J. London Math. Soc. **2 32**: 133-140. http://dx.doi.org/10.1112/jlms/s2-32.1.133

[5] Robertson A. G.(1983). *Automorphisms of spin factors and the decomposition of positive maps*. Quart. J.Math. Oxford. **34**: 87-96. http://dx.doi.org/10.1093/qmath/34.1.87

[6] Choi M-D. (1988). *Some assorted inequalities for positive linear maps on* C^**-algebras*. Linear Operator Theory **4**: 271-285. https://www.researchgate.net/publication/266539707

[7] Choi, M.-D. (1975). *Positive semidefinite biquadratic Forms*. Linear Algebra and its Applications **12**: 95-100. https://core.ac.uk/download/pdf/82119059.pdf

[8] Kossakowski A.(2003). *A class of linear positive maps in matrix algebras,*. Open Sys. and Information Dyn. **10**: 213-220. https://doi.org/10.1023/A:1025101606680

[9] Kim H-J. and Kye S-H.(1994). *Indecomposable extreme positive linear maps in matrix algebras,*. London Math. Soc. **26**: 575-581. https://doi.org/10.1112/blms/26.6.575

[10] Osaka H.(1991). *Indecomposable Positive Maps in Low Dimensional Matrix Algebras*. Linear Aligebra and its Applications **153**: 73-83. https://doi.org/10.1016/0024-3795(91)90211-E

[11] Osaka H.(1993). *A series of absolutely indecomposable positive maps in matrix algebras,*. Linear Algebra and its Applications **186**: 45-53. https://doi.org/10.1016/0024-3795(93)90283-T

[12] Tang' W.(1986). *On Positive Linear Maps between Matrix Algebras*. Linear Algebra and its Applicatlons **79**: 33-44. https://doi.org/10.1016/0024-3795(86)90290-9

[13] Choi, M.-D. and Lam, T. Y.(1977). *Extremal positive semidefinite forms*. Math. Ann. **231**: 1-18. http://dx.doi.org/10.1007/BF01360024

[14] Winda C. A., Okelo N. B. and Omolo Ongati.(2020). *Choi Matrices of 2-positive Maps on Positive Semidefinite Matrices*. Asian Research Journal of Mathematics **16(4)**: 60-71. http://dx.doi.org/10.9734/ARJOM/2020/16i556092