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The solution of fifth and sixth order linear and non linear boundary value problems by the Improved Residual Power Series Method

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Abstract

In this paper, we solve some fifth and sixth-order boundary value problems (BVPs) by the improved residual power series method (IRPSM). IRPSM is a method that extends the residual power series method (RPSM) to BVPs without requiring an exact solution. The presented method is capable of handling both linear and nonlinear BVPs effectively. The solutions provided by IRPSM are compared with the exact solutions and with the existing approximate solutions. The results demonstrate that the approach is extremely accurate and dependable.

Keywords: Improved residual power series method, Fifth and sixth order BVPs, Comparative analysis, Different techniques.

1. Introduction

In engineering and science, real-world problems can be modelled mathematically by differential equations (DEs). Only a limited class of them can be solved exactly, and for the rest of the equations, numerical methods are used to provide approximate solutions of acceptable accuracy. Recent numerical methods include the Adomian decomposition method [1, 2, 3, 4, 5], variational iteration method [6], homotopy perturbation method [7, 8], homotopy analysis method [9] and the differential transform method [10]. Traditional perturbation methods use large or small parameters and are unable to produce a general form of approximate solutions, especially in nonlinear problems. Non-perturbation techniques like DTM and ADM can handle highly nonlinear problems, but their series solution convergence zone is often limited. The HPM, which is an excellent integration of homotopy and perturbation methods, overcomes the limitations of problems with small or large parameters. It successfully solves a wide range of nonlinear problems. Vasile Marinca et al. [11, 12, 13, 14] recently proposed OHAM for the approximate solution of nonlinear problems of fourth-grade fluid thin film flow down a vertical cylinder. They

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employed OHAM to better understand the behavior of non-linear mechanical vibration in electrical machines in their research. In the mathematical modeling of viscoelastic flows [15], fifth-order BVPs arise. The thin convicting layers are bordered by stable layers that are thought to surround A-type stars and may be represented by sixth-order BVPs that are known to arise in astrophysics [16]. Glatzmaier [17] also looked at how such equations may be used to predict dynamo activity in some stars. Wazwaz [4, 5] used the decomposition technique to solve fifth and sixth-order linear and non-linear problems. Noor et al. [18, 19, 20] used the VIM with He's Polynomials, the HPM, and the VIM to study these types of problems. Javed Ali et al. [21, 22] used OHAM and VDM to solve fifth and sixth-order BVPs. The residual power series method (RPSM) was recently established for solving reduced-order initial value problems (IVPs). RPSM creates the approximate solution of linear and nonlinear order initial value problems (IVPs) in the form of a polynomial. Numerous mathematicians have used RPSM to solve different problems. Some of them are given below. Arqub and his co-author implemented the RPSM for the linear and nonlinear Lane Emden equation [23]. Al-Smadi applied the RPSM to solve first-order linear and nonlinear IVPs in [24]. These problems are initial values and low orders. We extend this method to boundary value problems by solving fifth and sixth-order boundary value problems. To find the missing initial conditions, a truncated series is developed, and then the residual of the solution is forced to zero at the given boundary conditions. With this extension, we name this method the Improved Residual Power Series Method (IRPSM). This approach is easy to understand, dependable, and well defined. The results obtained by using our approach are validated against OHAM, MDM, HPM, VAM, VDM, and exact solutions.

The following is a breakdown of the structure of the paper. The basic idea of the proposed method is covered in Section 2. In Section 3, you will find several numerical examples. In Section 4, we discussed the outcomes of the numerical simulation performed with Mathematica 11.3.

2. Basic idea of IRPSM

In this section, we explain the IRPSM for n^{th} order IVPs and BVPs along with the conditions. The IRPSM comprises the expansion of the power series about the initial point $x = x_0$ for finding the solution of BVPs. For unknown initial conditions, we use assumed values and later use boundary values to find them. Consider n^{th} order BVP:

$$f^{(n)}(x) = \psi(x, f^{(m)}(x)) , \qquad 0 \leqslant x \leqslant t \qquad m = 0, 1, 2, 3, ..., n-1 , \tag{2.1}$$

with boundary conditions:

$$f^{(m)}(x) = \alpha_m , \qquad m = 0, 1, 2, 3, ..., n-1$$

assume initial conditions:

$$f^{(m)}(0) = \beta_m , \qquad m = 0, 1, 2, 3, ..., n-1 , \tag{2.2}$$

In Eq. (2.1) and Eq. (2.2) ψ and β_m are known functions or constants while f is unknown. Now we assume solution of the given problem by k^{th} truncated power series as

$$f(x) = \sum_{i=0}^{k} A_i x^i , \qquad k = 0, 1, 2, 3, ..., n-1 , \tag{2.3}$$

where A_i are unknown to be calculated. Since the differential equation is of n^{th} order so we have to calculate the constant A_i for $i = 0, 1, 2, ..., n-1$. For $k = 0$ Eq. (2.3) reduces to

$$f(x) = \sum_{i=0}^{0} A_i x^i \,. \tag{2.4}$$

Putting $x = 0$ in Eq. (2.4) and comparing with first initial conditions, we have

$$A_0 = \beta_0 \,. \tag{2.5}$$

For the value of A_1 taking $k = 1$ then find out the 1st derivative of Eq. (2.3) and using $x = 0$. Afterward comparing with second initial condition leads to

$$A_1 = \beta_1 \,. \tag{2.6}$$

Repeating the same procedure for $k = 2$, find out the second derivative of Eq. (2.3) and using $x = 0$. Afterward comparing with third initial condition leads to

$$A_2 = \frac{\beta_2}{2!} \,. \tag{2.7}$$

Similarly for $i = n-1$, find out the $(n-1)^{th}$ derivative of Eq. (2.3) and using $x = 0$, then comparing it with the $(n-1)^{th}$ initial condition leads to

$$A_{n-1} = \frac{\beta_{n-1}}{(n-1)!} \,. \tag{2.8}$$

In case of boundary value problems we assume the values of unknown initial conditions, then latter on we find it by using boundary conditions.
For greater values of the constant A_i, $k = n, n+1, n+2....$, can be calculated by the following method. We consider the k^{th} truncated series

$$f(x) = f_{initial}(x) + \sum_{(i=n)}^{k} A_i \, x^i \tag{2.9}$$

$f_{initial}(x)$ is the k^{th} truncated series, for $k = 0, 1, 2, ..., n-1$, $f_{initial}(x)$ can obtained as:

$$f_{initial}(x) = f^{(n-1)}(x) = A_0 + A_1 \, x + A_2 \, x^2 + A_3 \, x^3 ... + A_{(n-1)} x^{(n-1)} \tag{2.10}$$

Now by putting the values of A_i in Eq. (2.10) we get:

$$f_{initial}(x) = f^{(n-1)}(x) = \beta_0 + \beta_2 \, x + \frac{\beta_2}{2!} \, x^2 + \frac{\beta_3}{3!} \, x^3 ... + \frac{\beta_{n-1}}{(n-1)!} x^{(n-1)}. \tag{2.11}$$

Furthermore, to find the residuals, Eq (2.1) can be written as follows:

$$f^{(n)}(x) - \psi(x, f^{(m)}) = 0 \tag{2.12}$$

Using Eq. (2.3) in Eq. (2.12) gives the definition of the k^{th} residual function as follows:

$$\begin{aligned}\mathrm{Res}^k(x) &= \sum_{i=n}^{k} i(i-1)(i-2)(i-3)...(i-n+1)\ A_i\ x^{i-n} \\ &- f(x, \sum_{i=m}^{k} i(i-1)(i-2)...(i-m+1)A_i x^{i-m}, \quad m = 0, 1, ..., n-1. \end{aligned} \tag{2.13}$$

Now by taking derivative with respect to x and then putting $x = 0$ on both sides of Eq. (2.13) to get:

$$\begin{aligned}\frac{d^{k-n}}{d\,x^{k-n}} \mathrm{Res}^k(x=0) &= \frac{d^{k-n}}{d\,x^{k-n}} \sum_{i=n}^{k} i(i-1)(i-2)(i-3)...(i-n+1)\ A_i\ x^{i-n} \\ &- \frac{d^{k-n}}{d\,x^{k-n}} f\Big(x, \sum_{i=m}^{k} i(i-1)(i-2)...(i-m+1\Big)A_i x^{i-m}, \end{aligned} \tag{2.14}$$

for $m = 0, 1, ..., n-1$. From Eq. (2.14), we can find out the values of A_n, so the n^{th} truncated series will be shown as below:

$$f^n(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 .... + A_n x^n\ . \tag{2.15}$$

This process can be repeated until the problem solution does not achieve the required accuracy. By obtaining more solution coefficients, higher accuracy will be obtained, or by selecting a larger value of k in the truncation series (2.3), better results will be obtained. After measuring all the coefficients, the equation (2.15) can be used to calculate an approximate solution. From here, we find all the assumed initial conditions β_m by using all the given boundary conditions.

$$f^{(m)}(x) = \alpha_m\ , \qquad 0 \leqslant x \leqslant t \qquad m = 0, 1, 2, 3, ..., n-1\ .$$

From here, we get the $n-1$ equation, and then the corresponding equations are solved with the help of NSolve Mathematica's built-in code for unknown.

3. Numerical Illustration

Example 3.1. Fifth order non-linear BVP ([18, 19])

$$f^{(5)}(t) = f^2(t)\ e^{-t}\ , \qquad 0 < t < 1, \tag{3.1}$$

with BCs,

$$f(0) = 1\ , \quad f'(0) = 1\ , \quad f''(0) = 1\ , \quad f(1) = e\ , \quad f'(1) = e\ .$$

The exact solution of this problem is

$$f(t) = e^t.$$

$$f(t) = \sum_{i=0}^{m} A_i\, t^i. \tag{3.2}$$

Eq. (3.2) can be used to calculate the values of unknown A_i. Initially for $m = 0$ we have

$$f(t) = \sum_{i=0}^{0} A_i t^i = A_0.$$

Using $t = 0 \quad f(0) = A_0$. From initial condition $f(0) = 1$

$$A_0 = 1.$$

Now for $m = 1$ taking first derivative of Eq. (3.2), then using $t = 0$. Comparing with initial conditions the obtained result is

$$f'(t) = \sum_{i=0}^{1} i\, A_i\, t^{i-1} = A_1.$$

Using initial condition we get $A_1 = 1$.
In the same way, we calculated $A_2 = \frac{1}{2}, \quad A_3 = \frac{a}{6} \quad A_4 = \frac{b}{24}$. Rewriting the given BVP Eq. (3.1) in the form:

$$f^{(5)}(t) - e^{-t}\, f^2(t) = 0\,, \qquad 0 < t < 1\,, \tag{3.3}$$

Plugging the m^{th} truncated series in Eq. (3.3) leads to

$$Res^m(t) = \sum_{i=5}^{m} i(i-1)(i-2)(i-3)(i-4)A_i\, t^{i-5} - e^{-t}\Big(\sum_{i=0}^{k} A_i\, t^i\Big)^2. \tag{3.4}$$

To obtain 5^{th} order approximate solution put $m = 5$ and $t = 0$ in Eq. (3.4) we get

$$Res^5(0) = -1 + 120\, A_5 = 0\,.$$

From above equation it follows that $A_5 = \frac{1}{120}$. Continuing in this way we obtain 14^{th} order solution. Using procedure in section 2, we obtain the following values by using boundary conditions

$$\frac{d^{k-5}}{d\,x^{k-5}}\, Res^5(t) = \frac{d^{k-5}}{d\,x^{k-5}}\Big(\sum_{i=5}^{5} i(i-1)(i-2)(i-3)(i-4)A_i\, t^{i-5} - e^{-t}\Big(\sum_{i=0}^{5} A_i\, t^i\Big)^2\Big),$$

where k=6,7,8,...

$$\begin{aligned} f(t) =& 1 + t + t^2/2 + (at^3)/6 + (bt^4)/24 + t^5/120 + t^6/720 + t^7/5040 - t^8/40320 \\ &+ (at^8)/20160 - t^9/362880 + (bt^9)/181440 + t^{10}/3628800 + t^{11}/1900800 \\ &- (at^{11})/997920 + (a^2t^{11})/1995840 - (23t^{12})/159667200 + (at^{12})/2280960 \\ &- (a^2t^{12})/3421440 - (bt^{12})/6842880 + (abt^{12})/6842880 + (67t^{13})/6227020800 \\ &- (139at^{13})/1556755200 + (a^2t^{13})/11119680 + (bt^{13})/14826240 - (abt^{13})/11119680 \\ &+ (b^2t^{13})/88957440 + (23t^{14})/9686476800 + (at^{14})/103783680 - (a^2t^{14})/51891840 \\ &- (157bt^{14})/10897286400 + (abt^{14})/34594560 - (b^2t^{14})/138378240 \end{aligned}$$

Here we are using the boundary conditions on above equation to find out the assume initial condition using NSolve Mathematica inbuilt code to get:
$a = 0.9999999999456852, \ b = 1.00000000023766.$
As a result, the solution became

$$\begin{aligned} f(t) =& 1 + t + t^2/2 + 0.166667t^3 + 0.0416667t^4 + t^5/120 + t^6/720 + t^7/5040 + 0.0000248016t^8 \\ & + 2.75573 \times 10^{-6}t^9 + 2.75573 \times 10^{-7}t^{10} + 2.50521 \times 10^{-8}t^{11} + 2.08768 \times 10^{-9}t^{12} \\ & + 1.6059 \times 10^{-10}t^{13} + 1.14707 \times 10^{-11}t^{14} \end{aligned} \tag{3.5}$$

Table 1: The IRPSM solution (3.5) is compared to the exact solution as well as the error estimations. Second Last column of table 1, are the errors in the solutions of OHAM, for the same problem [21]. Last column of table 1, are the errors in the solutions of VIM, for the same problem [18].

x	**Exact solution**	**IRPSM solution**	**E*(IRPSM)**	**E (OHAM)[21]**	**E (VIM)[18]**
0.	1.	1.	0.000	0.000	0.000
0.1	1.10517	1.10517	8.1×10^{-15}	1.9×10^{-10}	1.0×10^{-9}
0.2	1.2214	1.2214	5.6×10^{-14}	1.2×10^{-9}	2.0×10^{-9}
0.3	1.34986	1.34986	1.6×10^{-13}	3.3×10^{-9}	1.0×10^{-8}
0.4	1.49182	1.49182	3.2×10^{-13}	6.3×10^{-9}	2.0×10^{-8}
0.5	1.64872	1.64872	5.1×10^{-13}	9.3×10^{-9}	3.1×10^{-8}
0.6	1.82212	1.82212	6.7×10^{-13}	1.1×10^{-8}	3.7×10^{-8}
0.7	2.01375	2.01375	7.3×10^{-13}	1.1×10^{-8}	4.1×10^{-8}
0.8	2.22554	2.22554	6.0×10^{-13}	8.2×10^{-9}	3.1×10^{-8}
0.9	2.4596	2.4596	2.6×10^{-13}	1.9×10^{-9}	1.4×10^{-8}
1.	2.71828	2.71828	1.3×10^{-14}	0.000	0.000

Table 1: E*(IRPSM)=Actual-Approx

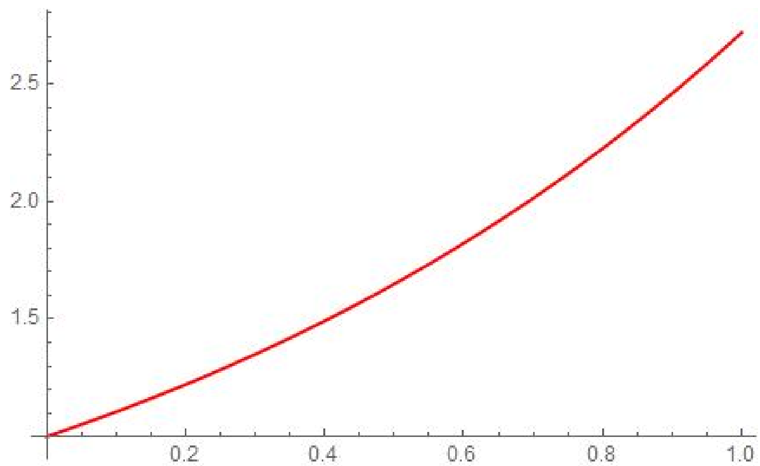

Figure 1: Solution graph of the Eq. (3.5).

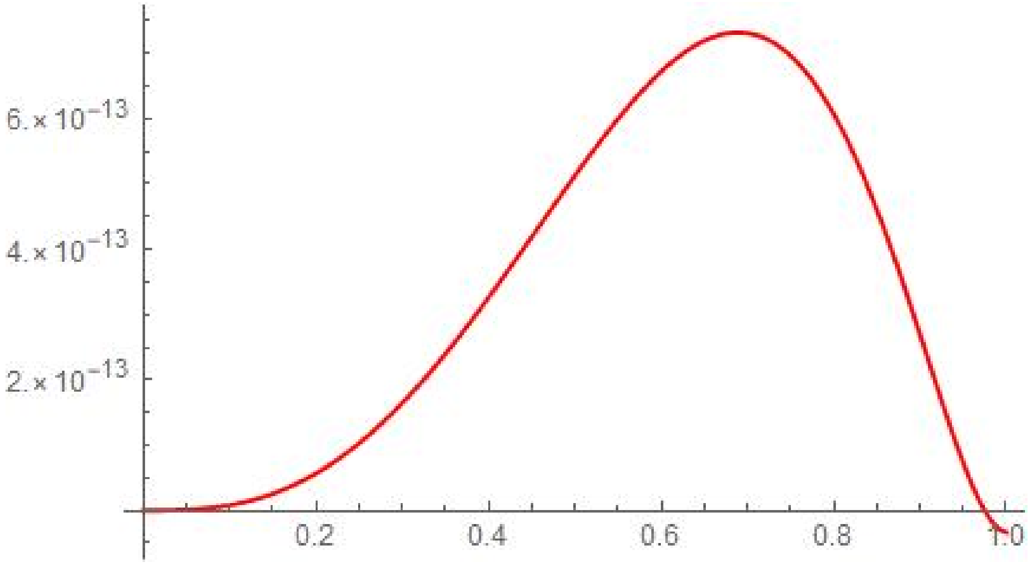

Figure 2: Residual Error graph of IRPSM.

Example 3.2. Consider the following special fifth order linear boundary value problem ([18, 21]):

$$g^{(5)}(t) = g(t) - 15\, e^{t} - 10\, t\, e^{t}, \qquad 0 < t < 1 \tag{3.6}$$

$$g(0) = 0\,, \quad g'(0) = 1\,, \quad g''(0) = 0\,, \quad g(1) = 0\,, \quad g'(1) = -e\,.$$

The actual solution of this problem is $g(t) = t(1-t)\, e^{t}$.

$$g(t) = \sum_{i=0}^{m} A_i t^i. \tag{3.7}$$

We get the following 14^{th} order solution using the IRPSM technique described in section 2.
$a = -2.999999989324808, \quad b = -8.000000046558213.$

$$\begin{aligned} g(t) =& t - 0.5t^3 - 0.333333t^4 - t^5/8 - t^6/30 - t^7/144 - 0.00119048t^8 - 0.000173611t^9 \\ & - t^{10}/45360 - t^{11}/403200 - t^{12}/3991680 - 2.2964 \times 10^{-8}t^{13} - 1.92709 \times 10^{-9}t^{14} \end{aligned} \tag{3.8}$$

Table 2: The IRPSM solution (3.8) is compared to the actual solution as well as the error estimations. Second last column of table 2, are the errors in the solutions of OHAM, for the same problem [21]. Last column of table 2, are the errors in the solutions of VIM, for the same problem [18].

x	Exact solution	IRPSM solution	E*(IRPSM)	E (OHAM)[21]	E (VIM)[18]
0.	1.	1.	0.000	0.000	0.000
0.1	0.0994654	0.0994654	1.5×10^{-12}	-9×10^{-11}	-3×10^{-11}
0.2	0.195424	0.195424	1.1×10^{-11}	-4×10^{-10}	-2×10^{-10}
0.3	0.28347	0.28347	3.2×10^{-11}	-5×10^{-10}	-4×10^{-10}
0.4	0.358038	0.358038	6.4×10^{-11}	-2×10^{-11}	-8×10^{-10}
0.5	0.41218	0.41218	1.0×10^{-10}	1×10^{-9}	-1×10^{-9}
0.6	0.437309	0.437309	1.3×10^{-10}	2×10^{-9}	-2×10^{-9}
0.7	0.422888	0.422888	1.4×10^{-10}	2×10^{-9}	-2×10^{-9}
0.8	0.356087	0.356087	1.2×10^{-10}	1×10^{-9}	-2×10^{-9}
0.9	0.221364	0.221364	5.7×10^{-11}	4×10^{-10}	-1×10^{-9}
1.	0.000	4.8×10^{-17}	-6.6×10^{-18}	0.000	0.000

Table 2: E*(IRPSM)=Actual-Approx

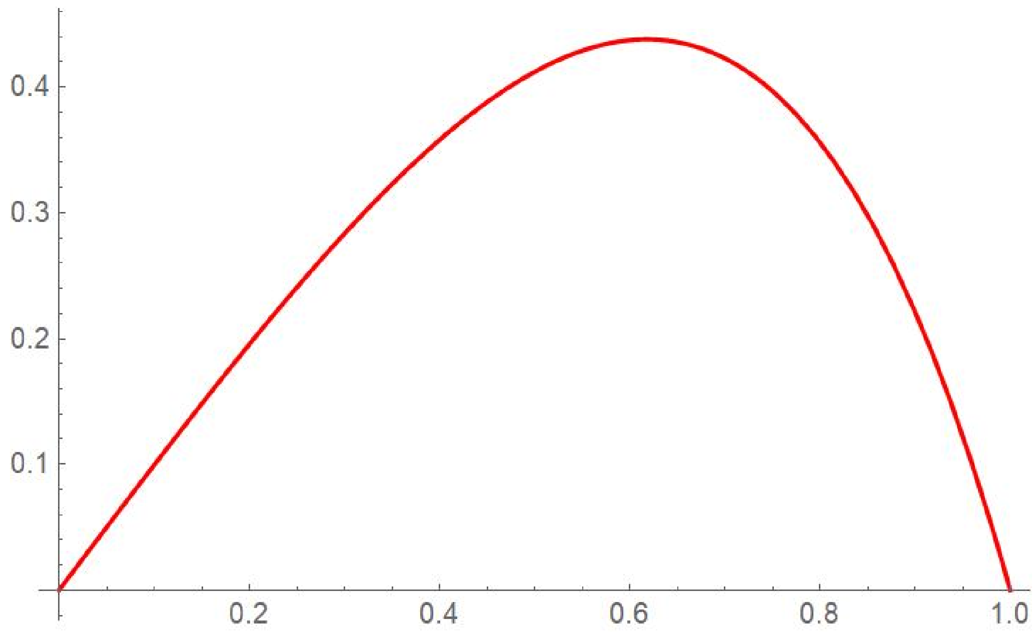

Figure 3: Solution graph of the Eq. (3.8).

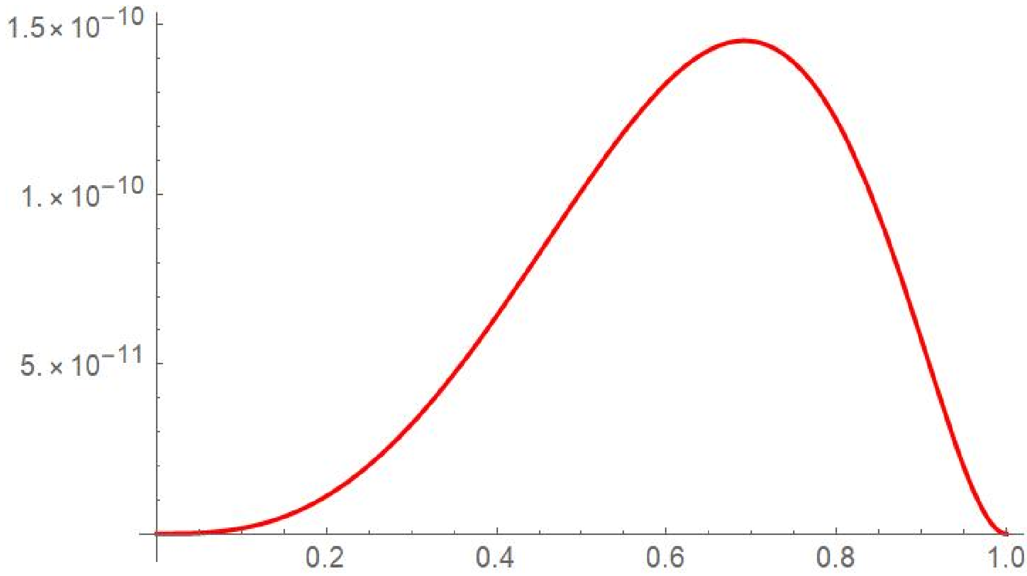

Figure 4: Residual Error graph of IRPSM.

Example 3.3. Sixth order non-linear boundary value problem of ([5, 19]):

$$g^{(6)}(t) = g^2(t)e^t \qquad 0 < t < 1 \tag{3.9}$$

$$g(0) = 1\,, \quad g'(0) = -1\,, \quad g''(0) = 1\,, \quad g(1) = e^{-1}\,, \quad g'(1) = -e^{-1}\,, \quad g''(1) = e^{-1}\,.$$

The actual solution for this problem is $g(t) = e^{-t}$.

$$g(t) = \sum_{i=0}^{m} A_i\, t^i. \tag{3.10}$$

We get the following 12^{th} order solution using the IRPSM technique described in section 2. We obtain the following values for a, b and c using boundary conditions.
$a = -1.000000031823586$, $b = 1.0000002831121835$ and $c = -1.0000007970627998$

$$\begin{aligned} g(t) =& 1 - t + t^2/2 - 0.166667t^3 + 0.0416667t^4 - 0.00833334t^5 + t^6/720 - t^7/5040 + t^8/40320 \\ & - 2.75573 \times 10^{-6}t^9 + 2.75573 \times 10^{-7}t^{10} - 2.50521 \times 10^{-8}t^{11} + 2.08768 \times 10^{-9}t^{12} \end{aligned} \tag{3.11}$$

Table 3: The IRPSM solution (3.11) is compared to the actual solution as well as the error estimations. Second last column of table 3, are the errors in the solutions of MDM, for the same problem [5]. Last column of table 3, are the errors in the solutions of HPM, for the same problem [19].

x	Exact solution	IRPSM solution	E* (IRPSM)	E (MDM) [5]	E (HPM) [19]
0.	1.	1.	0.000	0.000	0.000
0.1	0.904837	0.904837	4.1×10^{-12}	-2.3×10^{-7}	-1.2×10^{-4}
0.2	0.818731	0.818731	2.5×10^{-11}	-1.3×10^{-6}	-2.3×10^{-4}
0.3	0.740818	0.740818	6.3×10^{-11}	-3.3×10^{-6}	-3.2×10^{-4}
0.4	0.67032	0.67032	1.0×10^{-10}	-5.2×10^{-6}	-3.8×10^{-4}
0.5	0.606531	0.606531	1.3×10^{-10}	-6.1×10^{-6}	-4.0×10^{-4}
0.6	0.548812	0.548812	1.3×10^{-10}	-5.7×10^{-6}	-3.9×10^{-4}
0.7	0.496585	0.496585	1.0×10^{-10}	-4.0×10^{-6}	-3.3×10^{-4}
0.8	0.449329	0.449329	5.2×10^{-11}	-1.9×10^{-6}	-2.4×10^{-4}
0.9	0.40657	0.40657	1.0×10^{-11}	-3.5×10^{-7}	-1.2×10^{-4}
1.	0.367879	0.367879	2.1×10^{-17}	-5.0×10^{-10}	2.0×10^{-9}

Table 3: E* (IRPSM) = Actual-Approx

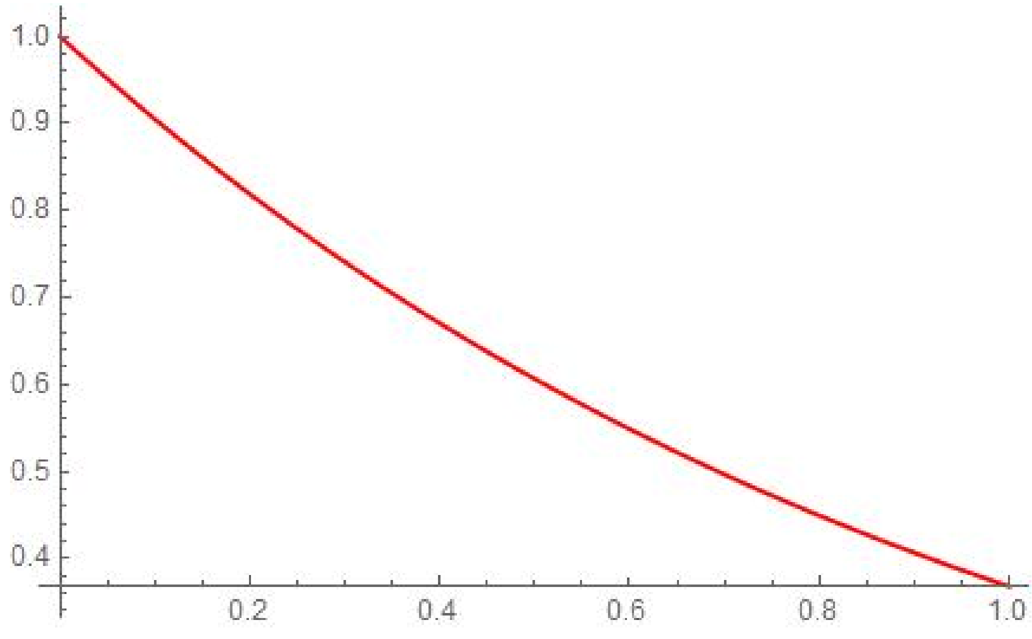

Figure 5: Solution graph of the Eq. (3.11).

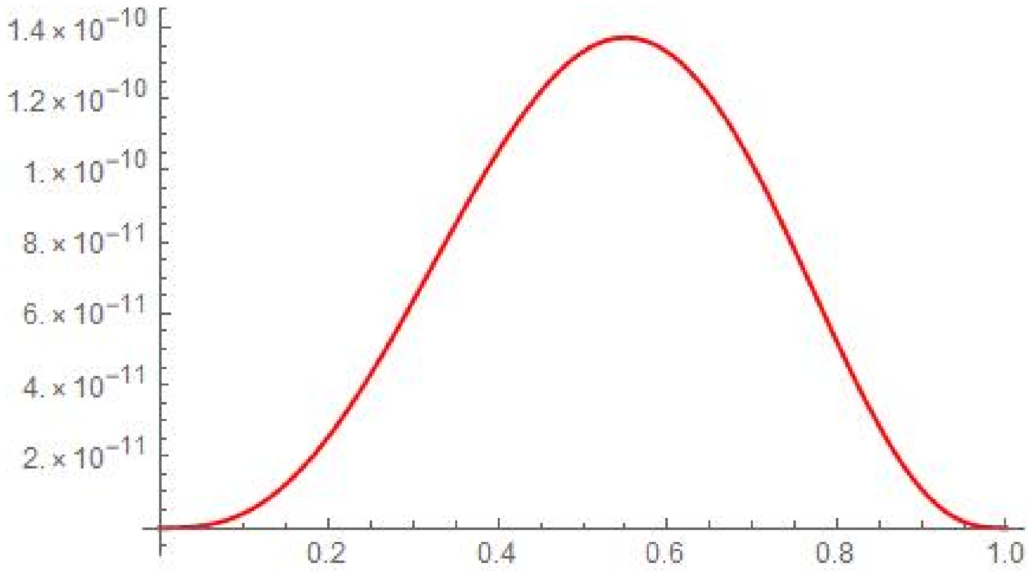

Figure 6: Residual Error graph of IRPSM.

Example 3.4. Sixth order linear BVP involving a parameter d ([19, 20, 22]):

$$g^{(6)}(t) = (1+d)g^{(4)}(t) - d\, g^{(2)}(t) + d\, t, \qquad 0 < t < 1 \tag{3.12}$$

$$g(0) = 1\,, \quad g'(0) = 1\,, \quad g''(0) = 0\,, \quad g(1) = \frac{7}{6} + \sinh(1)$$

$$g'(1) = \frac{1}{2} + \cosh(1)\,, \quad g''(1) = 1 + \sinh(1)\,.$$

The actual solution of this problem is $g(t) = 1 + \frac{1}{6}t^3 + \sinh(t)$.

$$g(t) = \sum_{i=0}^{m} A_i t^i. \tag{3.13}$$

We consider the 11^{th} order solution. Using procedure in section 2, we obtain the following values for a, b and c. $a = 2.0000000299279628$, $b = -2.889814925018749 \times 10^{-7}$ and $c = 1.0000011603456829$

$$g(t) = 1 + t + 0.333333t^3 - 1.20409 \times 10^{-8}t^4 + 0.00833334t^5 - 4.415 \times 10^{-9}t^6 + 0.000198415t^7 - 7.95559 \times 10^{-10}t^8 + 2.75608 \times 10^{-6}t^9 - 8.84751 \times 10^{-11}t^{10} + 2.50836 \times 10^{-8}t^{11} \tag{3.14}$$

Table 4: The IRPSM solution (3.14) is compared to the actual solution as well as the error estimations. Third last column of table 4, are the errors in the solutions of HPM, for the same problem [19]. Second last column of table 4, are the errors in the solutions of VIM, for the same problem [20]. Last column of table 4, are the errors in the solutions of VDM, for the same problem [22].

x	Exact solution	IRPSM solution	E*(IRPSM)	E (HPM)[19]	E (VIM)[20]	E (VDM)[22]
0.	1.	1.	0.000	0.000	0.000	0.000
0.1	1.10033	1.10033	-3.8×10^{-12}	1.2×10^{-6}	1.2×10^{-6}	2.9×10^{-6}
0.2	1.20267	1.20267	-2.3×10^{-11}	7.2×10^{-6}	1.2×10^{-6}	1.6×10^{-6}
0.3	1.30902	1.30902	-5.7×10^{-11}	1.7×10^{-5}	1.2×10^{-5}	3.6×10^{-5}
0.4	1.42142	1.42142	-9.5×10^{-11}	2.7×10^{-5}	1.2×10^{-5}	5.3×10^{-5}
0.5	1.54193	1.54193	-1.2×10^{-10}	3.4×10^{-5}	1.2×10^{-5}	6.0×10^{-5}
0.6	1.67265	1.67265	-1.2×10^{-10}	3.2×10^{-5}	1.2×10^{-5}	5.3×10^{-5}
0.7	1.81575	1.81575	-9.3×10^{-11}	2.3×10^{-5}	1.2×10^{-5}	3.5×10^{-5}
0.8	1.97344	1.97344	-4.8×10^{-11}	1.1×10^{-5}	1.2×10^{-5}	1.5×10^{-5}
0.9	2.14802	2.14802	-1.0×10^{-11}	2.2×10^{-6}	1.2×10^{-6}	2.7×10^{-6}
1.	2.34187	2.34187	0.000	0.000	0.000	0.000

Table 4: E*(IRPSM)=Actual-Approx

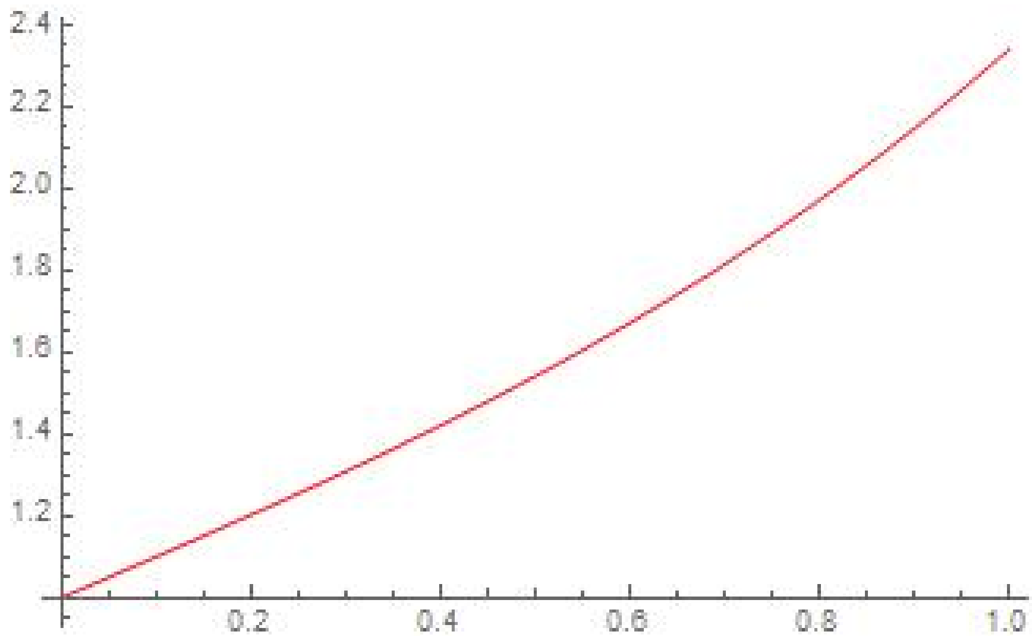

Figure 7: Solution graph of the Eq. (3.14).

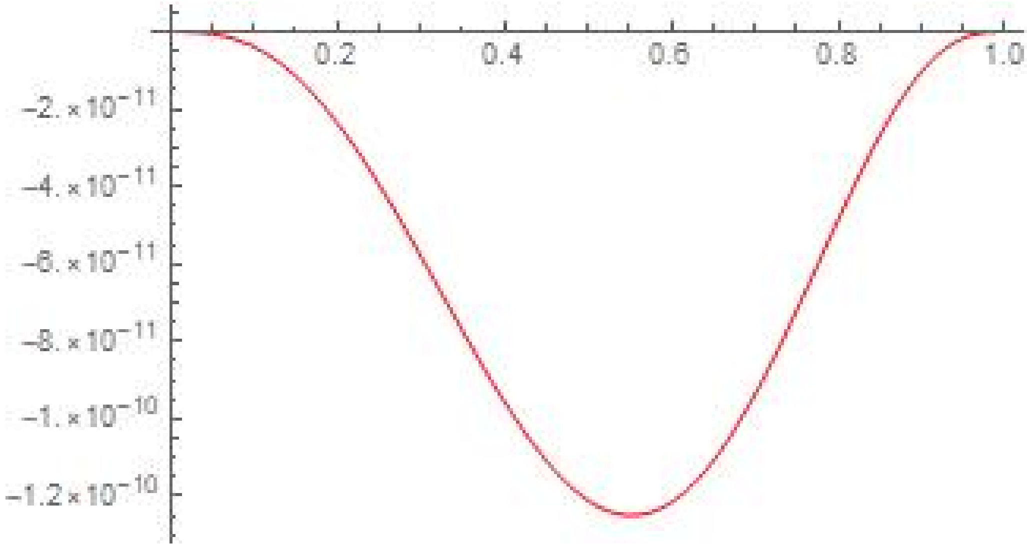

Figure 8: Residual Error graph of IRPSM.

4. Conclusion

When the exact solution does not exist, then the residual power series method (RPSM) cannot solve BVPs. The improved residual power series method (IRPSM) can solve BVPs without an existing exact solution. In this study, we employed IRPSM to solve fifth and sixth-order linear and nonlinear BVPs. The simulations associated with the four examples discussed above were performed using Mathematica. 11.3. The proposed algorithm produced a rapidly convergent series. When the obtained results are compared to other work, the IRPSM technique is found to be more reliable and efficient than other techniques. No restrictive assumptions are needed and one feels very comfortable as the convergence of the method is not dependent on the initial guess. The low order solutions show excellent agreement with the exact solution, and the remarkable low error is notable. The solution curve is remarkably smooth, and it can be investigated and interpreted in any way. Furthermore, the method's results are extremely close to the exact solution. This strategy offers a lot of potential in terms of attracting researchers, scientists, and engineers from many fields.

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