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Enhancement of heat and mass transfer of a physical model using Generalized Caputo fractional derivative of variable order and modified Laplace transform method

Muhammad Imran Asjad [a], Waqas Ali Faridi[a], Muhammad Abubakar[a], Maryam Aleem[a], Fahd Jarad [b]

[a] Department of Mathematics, University of Management and Technology, Lahore, 54770, Pakistan

[b] Department of Mathematics, Çankaya University, 06790 Etimesgut, Ankara, Turkey

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Abstract

In this paper, we use a model of non-Newtonian second grade fluid which having three partial differential equations of momentum, heat and mass transfer with initial condition and boundary condition. We develop the modified Laplace transform of this model with fractional order generalized Caputo fractional operator. We find out the solutions for temperature, concentration and velocity fields by using modified Laplace transform and investigated the impact of the order α and ρ on temperature, concentration and velocity fields respectively. From the graphical results, we have seen that both the α and ρ have reverse effect on the fluid flow properties. In consequence, it is observed that flow properties of present model can be enhanced near the plate for smaller and larger values of ρ. Furthermore, we have compared the present results with the existing literature for the validation and found that they are in good agreement.

Keywords: Modified Laplace transform, Generalized Caputo, Application to heat and mass transfer problem.
2010 MSC: 35A22, 26A33, 34A08..

1. Introduction

The fractional calculus and its applications in assorted fields of science and engineering is considered. Now, it is an important sub-field of mathematics which has an ability to assist the dynamics of non-local complex systems in new directions. The fractional differential operators describe the better memory effect caused by the non-locality of these operators [1, 2, 3, 4, 5], there has been an interest in generalizing these operators in system to better understanding the impact of non-locality [6, 7, 8, 9, 10, 11, 12, 13].

*Corresponding author: imran.asjad@umt.edu.pk

In the present era, fractional calculus is more attractive as compared to other fields due to the diversity of the fractional derivatives operators. We can study the Caputo fractional operator and the R–L fractional operator by [14, 15], the AB-fractional operator [16, 17, 18, 19], the CF-fractional operator by [20], and many others. The AB-fractional operator and the CF-fractional operator are newly implemented in the construction of physical situations [21, 22, 23, 24]. The CF-fractional operator and the CF-fractional operators are great agreement in construction of real world phenomena. The technique of fractional calculus is an elderly and has been enlarged in different domains such as non-integer-order multiples in electromagnetism, electrochemistry, tracer in fluid flows, in soliton theory, neurons model in biology, finance, signal processing, applied mathematics, bio-engineering, viscoelasticity, fluid mechanics, and fluid dynamics [25, 26, 27, 28, 29].

In recent time, Amir Khan and Gul Zaman find out the analytic solutions of unsteady magnetohydrodynamic (MHD) flows of a generalized second-grade fluid [30]. Amer Rashid and Abdul Wahab studied about the unsteady flow of an anomalous Oldroyd-B fluid [31]. Qi Haitao and Xu Mingyu discussed about the fractional derivative Maxwell model (FDMM) which contained viscoelastic fluid with unsteady flow [32]. Shaowei Wang and Moli Zhao gave analytical solutions of generalized fractional Maxwell fluid which has transient electro-osmotic flow with help of fractional derivative [33]. Amir Mahmood and Saima Parveen also discussed generalized fractional Maxwell fluid which has torsional oscillatory flow and find its exact analytic solutions [34]. Also, Jarad and Jawad [35] defined a modified Laplace transform for particulargeneralized fractional operators namely, Riemann and generalized Caputo fractional operators. For the moment, a modified Laplace transform for Caputo-Fabrizio and Atangana-Baleanu is not defined yet in the existing literature. Therefore, we have applied the modified Laplace transform for generalized Caputo fractional derivative operator to some fluid flow problem in transport phenomena. Some relevant studies about non-Newtonian fluids from the literature can be seen in references in [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. Through the present study, we have investigated the influence of parameters α and ρ on the flow properties of non-Newtonian fluid and found that fluid properties can be enhanced by increasing the value of ρ. We have also compared the present study with the recent literature and found that they are in good agreement.

In below study, we have written some fundamental definitions and lemmas. After it, we have discussed about the physical model and their obtained graphs. And at the end, we have mentioned the conclusion.

2. Preliminaries

This section is about the discussion of the modified Laplace transform, some lemmas.

Definition 2.1. Let $h : [0, \infty) \to \mathbb{R}$ be a real valued function. The modified Laplace transform of h is defined by [26],

$$_{\rho}\{h(t)\}(s) = \int_0^{\infty} e^{-s\frac{t^{\rho}}{\rho}} h(t) \frac{dt}{t^{1-\rho}}, \quad \rho > o, \quad \forall\, s. \tag{2.1}$$

(For all values of s, the integral is valid.)

Lemma 2.2. *As in reference [26], consider* h *and* g *be two piecewise continuous functions at each interval* $[0,\ T]$ *and of exponential order. Then define the modified convolution of* h *and* g *by,*

$$(g *_\rho h)(t) = \int_0^t g((t^\rho - \tau^\rho)^{\frac{1}{\rho}}) g(\tau) \frac{d\tau}{\tau^{1-\rho}}. \tag{2.2}$$

Then, the commutative of the modified convolution of two functions is,

$$h \ *_b \ g = g \ *_b \ h. \tag{2.3}$$

Modified Laplace transform of some functions can be defined in following lemma.

Lemma 2.3. *Modified laplace transform of some basic functions [26]:*
(1) $_\rho\{1\}(s) = \frac{1}{s}, \quad s > 0,$
(2) $_\rho\{t^p\}(s) = \rho^{\frac{p}{\rho}} \frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}},\ p \in \mathbb{R},\ s > 0,$
(3) $_\rho\{e^{\lambda \frac{t^\rho}{\rho}}\}(s) = \frac{1}{s-\lambda},\ s > \lambda.$

3. Modified Laplace of Function and Fractional Operators:

3.1. *The Mittag-Leffler Function*

There is a vital contribution of Mittag-Leffler functions in the field of Fractional Calculus. Since we expect solutions of the problems in the frame generalized fractional operators, we have to set the relation between these functions and modified Laplace transform respectively. We denote the Mittag-Leffler function by [26],

$$E_\alpha(z) = \sum_{j=0}^{n} \left(\frac{z^j}{\Gamma(j\alpha + 1)} \right), z \in C, Re(\alpha) > 0, \tag{3.1}$$

A more generalization of Mittag-Leffler function with the two constants is defined as [21],

$$E_{\alpha,\ \beta}(Z) = \sum_{j=0}^{n} \left(\frac{z^j}{\Gamma(j\alpha\ +\ \beta)} \right),\ z \in\ C,\ Re\ (\alpha)\ >\ 0, \tag{3.2}$$

and it can be observed clearly by equations (2.4) and (2.5) that,

$$E_{\alpha,1}(z) = E_\alpha(z). \tag{3.3}$$

In the lemma below, we perceived the modified Laplace transform of Mittag-Leffler functions [26],

Lemma 3.1. *Consider* $Re(\alpha) > 0$ *and* $|\left(\frac{\lambda}{s^\alpha}\right)| < 1$, *we get,*

$$_\rho\left\{ E_\alpha(\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha \right\} = \left(\frac{s^{\alpha-1}}{s^\alpha - \lambda} \right), \tag{3.4}$$

and

$$_\rho\left\{ \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}(\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha \right\} = \left(\frac{1}{s^\alpha - \lambda} \right). \tag{3.5}$$

3.2. *The Generalized Right and Left Fractional Integrals*

The generalized right and left fractional integrals are defined respectively [26],

$$({}_aI^{\alpha,\rho}h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - u^\rho}{\rho}\right)^{\alpha-1} h(u)\frac{du}{u^{1-\rho}}, \tag{3.6}$$

and

$$(I_b^{\alpha,\rho}h)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{u^\rho - t^\rho}{\rho}\right)^{\alpha-1} h(u)\frac{du}{u^{1-\rho}}. \tag{3.7}$$

It should be mentioned that once $\rho = 1$, the integrals in Eq. (3.6) and (3.7) become the Riemann-Liouville fractional integrals.

3.3. *Right and Left Riemann-Liouville Fractional Integral*

Let $\alpha \in C$, Re $(\alpha) > 0$, the left Riemann-Liouville fractional integral, beginning from "a" with order α has the following form [26],

$$({}_aI^{\alpha}h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1} h(u)du, \tag{3.8}$$

while the right R-L fractional integral, ending at $b > a$, with order $\alpha > 0$ is given by

$$(I_b^{\alpha}h)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (u-t)^{\alpha-1} h(u)du. \tag{3.9}$$

3.4. *Hadamard Fractional Integral*

By taking limit as $\rho \to 0$ into Eq. (3.6) and (3.7),we have the Hadamard fractional integral, starting from a, having order $\alpha \in C$, $\mathrm{Re}(\alpha) > 0$ takes the form [26],

$$({}_aJ^{\alpha}h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (\ln t - \ln u)^{\alpha-1} \frac{h(u)}{u}du, \tag{3.10}$$

and the right Hadamard fractional integral, ending at $b > a$, having order α is given as,

$$(J_b^{\alpha}h)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\ln u - \ln t)^{\alpha-1} \frac{h(u)}{u}du. \tag{3.11}$$

3.5. *The Right and Left Generalized form of Fractional Derivatives*

The right and left generalized form of fractional derivatives with order $\alpha > 0$, are defined by [26],

$$({}_aD^{\alpha,\rho}h)(t) = \frac{\gamma^n}{\Gamma(n-\alpha)}\int_a^t \left(\frac{t^\rho - u^\rho}{\rho}\right)^{n-\alpha-1} h(u)\frac{du}{u^{1-\rho}}, \tag{3.12}$$

and

$$(D_b^{\alpha,\rho}h)(t) = \frac{-\gamma^n}{\Gamma(n-\alpha)}\int_t^b \left(\frac{u^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} h(u)\frac{du}{u^{1-\rho}}, \tag{3.13}$$

where $\rho > 0$ and $\Gamma = t^{1-\rho}\frac{d}{dt}$.

3.6. Caputo Modification of the Right and Left Generalized Fractional Derivatives

The right and left generalized fractional derivatives with Caputo modification is presented as [50],

$$({}_{a}^{c}D^{\alpha,\rho}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\left(\frac{t^{\rho}-u^{\rho}}{\rho}\right)^{n-\alpha-1}(\gamma^{n})h(u)\frac{du}{u^{1-\rho}}\,, \tag{3.14}$$

and

$$({}^{c}D_{b}^{\alpha,\rho}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}\left(\frac{u^{\rho}-t^{\rho}}{\rho}\right)^{n-\alpha-1}(-\gamma^{n})h(u)\frac{du}{u^{1-\rho}}\,. \tag{3.15}$$

3.7. Generalized Caputo Fractional Derivatives With Modified Laplace Transform

The Caputo generalized fractional derivative by the modified Laplace transform is presented as follow [48],

$$_{\rho}\{(D_{c}^{\alpha,\rho}h)(t)\} = s^{\alpha}{}_{\rho}\{h(t)\} - \sum_{j=0}^{n-1} s^{\alpha-k-1}(\gamma^{j}g)(0). \tag{3.16}$$

4. Application to transport phenomena of non-Newtonian fluid:

We analyze the unsteady MHD flow of non-Newtonian fluid over an exponentially swift isothermal vertical plate having infinite dimensions, with temperature and mass diffusion changes, keeping the heat absorption under consideration. We administer the conducting property of liquid to be minor. Consequently, the magnetic Reynolds number appears smaller than one. So, we observe that the transverse magnetic field is larger as compared to the induced magnetic field. Let the supposition, there is absence of applied voltage, since no electric field is present. We ignore the Joule heating and Viscous dissipation in the equation of energy. The corresponding generalized Caputo fractional model [47]:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial^2 y\partial t} + g\beta_T(T-T_\infty) + g\beta_c(C-C_\infty) - \frac{\sigma\beta_o^2}{\rho}u, \tag{4.1}$$

$$\rho C_p\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial^2 y} - Q(T-T_\infty), \tag{4.2}$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial^2 y} - D(C-C_\infty), \tag{4.3}$$

with the following boundary and initial conditions,

$$u(y,0)=0, \quad T(y,0)=0, \quad C(y,0)=0, \quad y>0, \tag{4.4}$$

$$u(0,t) = U_o H(t)e^{\lambda\frac{t^{\rho}}{\rho}}, \;\; T(0,t) = T_\infty + a_0(T_w - T_\infty)t^{\rho},$$
$$C(0,t) = C_\infty + b_0(C_w - C_\infty)t^{\rho}\;, 0\leqslant\rho\leqslant 1,\; t>0, \tag{4.5}$$

$$u \to 0,\ T \longrightarrow 0,\ C \to 0,\ \text{as}\ \ y \to \infty. \tag{4.6}$$

We introduce the dimensionless variables and parameters as follow,

$$y^* = \frac{U_o y}{\nu},\ t^* = \frac{U_o^2 t}{\nu},\ u^* = \frac{u}{U_o}, T^* = \frac{T - T_\infty}{T_w - T_\infty},\ C^* = \frac{C - C_\infty}{C_w - C_\infty}, \tag{4.7}$$

$$Gr = \frac{g\beta_T \nu (T_w - T_\infty)}{U_o^3},\ Gm = \frac{g\beta_c \nu (C_w - C_\infty)}{U_o^3},\ M = \frac{\sigma\beta_o^2 \nu}{\rho U_o^2}, \tag{4.8}$$

$$Pr = \frac{\mu C_p}{k},\ Sc = \frac{\nu}{D},\ \alpha_2 = \frac{\alpha_1 U_o^2}{\mu\nu}. \tag{4.9}$$

In the Eqs. (4.1)- (4.6), drop the Asterisk documentation, we obtain the accompanying introductory limit issue:

$$D_t^{\alpha,\rho} u(y,t) = \frac{\partial^2 (y,t)}{\partial y^2} + \alpha_2 D_t^{\alpha,\rho} \frac{\partial^2 u(y,t)}{\partial y^2} + GrT(y,t) + GmC(y,t) - Mu(y,t), \tag{4.10}$$

$$D_t^{\alpha,\rho} T(y,t) = \frac{1}{Pr} \frac{\partial^2 T(y,t)}{\partial y^2}, \tag{4.11}$$

$$D_t^{\alpha,\rho} C(y,t) = \frac{1}{Sc} \frac{\partial^2 C(y,t)}{\partial y^2}, \tag{4.12}$$

with dimensionless initial and boundary condition,

$$u(y,0) = 0,\ T(y,0) = 0,\ C(y,0) = 0,\ \ y > 0, \tag{4.13}$$

$$u(0,t) = H(t) e^{a \frac{t^\rho}{\rho}},\ T(0,t) = a_0 t^\rho,\ C(0,t) = b_0 t^\rho\ t > 0, \tag{4.14}$$

$$u \to 0,\ T \longrightarrow 0,\ C \to 0,\ \text{as}\ \ y \to \infty, \tag{4.15}$$

where,

$$Gr = \frac{g\beta_T \nu (T_w - T_\infty)}{U_o^3},\ \ Gm = \frac{g\beta_c \nu (C_w - C_\infty)}{U_o^3},\ \ M = \frac{\sigma\beta_o^2 \nu}{\rho U_o^2}, \tag{4.16}$$

$$Pr = \frac{\mu C_p}{k},\ \ Sc = \frac{\nu}{D},\ \ \alpha_2 = \frac{\alpha_1 U_o^2}{\mu\nu}, \tag{4.17}$$

here, Gr, Gm, M, Pr, Sc represent the Grashof number of heat transfer, Grashof number of mass transfer, Mach number, Prandtl number and Schmidt number, respectively. Also a_0 and b_0 are coefficients with dimension of $\frac{1}{t^\rho}$. $D_t^{\alpha,\rho} u(y,t)$ represents the generalized Caputo fractional derivative of $u(y,t)$ given as [26],

$$D_t^{\alpha,\rho} u(y,t) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x^\rho - t^\rho)^{n-\alpha-1} \frac{\gamma^n h(t)(dt)}{t^{1-\rho}}. \tag{4.18}$$

5. Solution of the problem:

5.1. Temperature Calculation

Using modified Laplace transform in the Equations. (4.11), (4.14), (4.15), and take the initial condition, we get:

$$s^{\alpha}T(y,s) = \frac{1}{Pr}\frac{\partial^2 T(y,s)}{\partial y^2}, \tag{5.1}$$

$$T(0,s) = a_0 \rho^{\frac{p}{\rho}} \frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}}, \quad as \quad T(y,s) \to 0, \quad as\ y \to \infty. \tag{5.2}$$

The working of the partial differential Equation. (5.1), with the application of conditions in Equation. (5.2),

$$T(y,s) = a_0 \rho^{\frac{p}{\rho}} \frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}} exp(-y\sqrt{Pr}\sqrt{s^{\alpha}}). \tag{5.3}$$

We take the inverse modified Laplace transform of Eq. (5.3) apply convolution theorem as follow

$$T(y,t) = a_0 \int_0^t (t-\tau)^{\rho} f(\tau) d(\tau), \tag{5.4}$$

and

$$f(\tau) = \tfrac{1}{t}\psi\left(0, -\tfrac{\alpha}{2}; -y\sqrt{Pr}t^{-\frac{\alpha}{2}}\right),$$

where ψ is a Wright's function[51, 52, 53, 54].

5.2. Concentration Calculation

We investigate that the concentration C(y, t) and temperature field have similar structure in the initial-boundary value problem, i.e.

$$C(y,s) = b_0\ \rho^{\frac{p}{\rho}} \frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}} exp(-y\sqrt{Sc}\sqrt{s^{\alpha}}). \tag{5.5}$$

We apply the inverse modified Laplace transformation on Eq.(3.23) by convolution theorem and get,

$$C(y,t) = b_0 \int_0^t (t-\tau)^{\rho} g(\tau) d(\tau), \tag{5.6}$$

and

$$g(\tau) = \tfrac{1}{t}\psi\left(0, -\tfrac{\alpha}{2}; -y\sqrt{Sc}t^{-\frac{\alpha}{2}}\right),$$

where ψ is a Wright's function[51, 52, 53, 54].

5.3. Velocity Calculation

Using modified Laplace transformation in the Eqs. (4.18), (4.14) , (4.15) and also the initial condition in the Eqs. (5.3) and (5.5) as:

$$(s^{\alpha}+M)\tilde{u}(y,s) = \frac{\partial^2\tilde{u}(y,s)}{\partial y^2} + \alpha_2 s^{\alpha}\frac{\partial^2\tilde{u}(y,s)}{\partial y^2} + G_r\ a_0\ \rho^{\frac{p}{\rho}}\frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}}\exp(-y\sqrt{P_r}\sqrt{s^{\alpha}}) + G_m\ b_0\ \rho^{\frac{p}{\rho}}\frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}}\exp(-y\sqrt{S_c}\sqrt{s^{\alpha}}), \tag{5.7}$$

where,

$$\tilde{u}(0,s) = \frac{1}{s-\lambda}, \quad \tilde{u}(y,s)\to 0, \quad \text{if} \quad y\to\infty, \tag{5.8}$$

where, $\tilde{u}(y,s)$ denotes the modified Laplace transformation with regards to the given function $u(y,s)$. and the s is transform variable. The solution of the partial differential equation (5.7), under conditions (5.8) obtained as:

$$\tilde{u}(y,s) = \frac{1}{s^{\alpha}-\lambda}\exp\left(-y\sqrt{\frac{(s^{\alpha}+M)}{1+\alpha_2 s^{\alpha}}}\right) + Gr\ a_0\rho^{\frac{p}{\rho}}\frac{\Gamma(1+\frac{p}{\rho})}{s^{1+\frac{p}{\rho}}}\frac{(1+\alpha_2 s^{\alpha})}{(s^{\alpha}+M)-(Prs^{\alpha})(1+\alpha_2 s^{\alpha})} \left[exp\left(-y\sqrt{Pr}\sqrt{s^{\alpha}}\right) - exp\left(-y\sqrt{\frac{(s^{\alpha}+M)}{1+\alpha_2 s^{\alpha}}}\right)\right] + Gm\ b_0\rho^{\frac{p}{\rho}}\frac{\Gamma(1+\frac{p}{\rho})}{s^1} + \frac{p}{\rho}\frac{(1+\alpha_2 s^{\alpha})}{(s^{\alpha}+M)-(Scs^{\alpha})(1+\alpha_2 s^{\alpha})}\left[exp\left(-y\sqrt{Sc}\sqrt{s^{\alpha}}\right) - exp\left(-y\sqrt{\frac{(s^{\alpha}+M)}{1+\alpha_2 s^{\alpha}}}\right)\right]. \tag{5.9}$$

The Eq. (5.9) is complex and difficult to obtain the analytical inverse modified Laplace transform. So, the following numerical inverse modified Laplace transform algorithms will be employed to obtain solution for velocity field. Algorithms to find inverse Laplace transform are given as:

$$^{-1}[\bar{F}(y,s,\alpha)] = F(y,t,\alpha) = \frac{e^{4.70}}{t}\left[\frac{1}{2}\bar{F}(y,\frac{4.70}{t},\alpha) + Re\left[\sum_{m=1}^{n}(-1)\bar{F}\left(y,\frac{4.70+k\pi i}{t},\alpha\right)\right]\right]. \tag{5.10}$$

Where $Re(.)$ is the real part, i is the imaginary part and $n\in N$. Stehfest's Algorithm is:

$$^{-1}[\bar{F}(y,s,\alpha)] = F(y,t,\alpha) = \frac{\ln 2}{t}\left[\sum_{k=1}^{2n} d_k\bar{F}\left(y,\frac{k\ln(2)}{t},\alpha\right)\right], \tag{5.11}$$

where

$$d_k = (-1)^{k+n}\sum_{[l=floor\frac{k+1}{2}]}^{\frac{(k+n)-|k-n|}{2}}\left[\frac{i^n(2l)}{(n-1)(l-1)(k-1)(2l-k)}\right]. \tag{5.12}$$

6. Graphical results and discussion:

This part deals with graphical discussion of the results obtained of the present intricate study. We have applied the modified Laplace transformation technique defined to support some generalized forms of fractional operators to transport phenomena of the second grade fluid in mass and heat transmission problem. We observe the impact of the variables α and ρ on the flowing properties of non-Newtonian fluids presented in Figures (6.1)-(6.2) and table 1 and 2.

Figures-(6.1)(a) is presented to observe the impact of 'α' on the concentration field in presence of the other parameters as constant. From the figure we have perceived that as we increase the value of α, the concentration level shows decreasing trend, means concentration level decays near the plate, while increasing farther away from the plate in the integrated domain. Finally, it asymptotically comes close to zero as y increases.

On the other hand, in Figure-(6.1)(b) we have investigated the impact of 'ρ' on concentration field in presence of all other parameters as constant. When we increase the value of $0 < \rho < 1$ then the concentration level shows an increasing trend against to the effect of α. Also we have seen that the distance between the thermal and boundary layers increases which lead to enhance the flow properties of the fluid flow.

Figure-(6.2)(a),which shows the same trend as in figure-(6.1)(a). If we fix the $\rho=1$ and increase the value of α then concentration level decreases as well as the boundary layer.

Same as in figure-(6.2)(b), in this figure we see the same trend to Figure-(6.1)(b). But one more thing is observed that if we fix the value of $\alpha=1$ and increase the value of ρ then we obtained the maximum value of concentration level near the plate.which shows the same trend as in figure-(6.1). If we fix the $\rho=1$ and increase the value of α then concentration level decreases as well as the boundary layer.

Figure-(6.3)(a)is presented to see the influence of α on the temperature field in the presence of the other parameters as constant. From the figure, we have observed that as we increase the value of α then temperature level shows decreasing trend, means temperature level decay near the plate, while increasing farther away from the plate in the integrated domain. Finally, it asymptotically comes close to zero as y increases.

On the other hand, in figure-(6.3)(b) we have investigated the impact of ρ on temperature field in presence of all other parameters as constant. When we increase the value of $0 < \rho < 1$ then the temperature level shows an increasing trend against to the effect of α. Also we have seen that the distance between the thermal boundary layers increases which lead to enhance the flow properties of the fluid flow.

Figure-(6.4)(a) is presented to see the influence of α on the temperature field in the presence of the other parameters as constant. From the figure, we have observed that as we increase the value of α then temperature level shows decreasing trend, means temperature level decay near the plate but this trend is not same as in figure-(6.3)(a). In the start, temperature decrease slowly and for large values of α it becomes close to y.

Figure-(6.4)(b), in this figure we see the same trend to figure-(6.3)(b) But one more thing is observed that if we fix the value of '$\alpha=1$' and increase the value of 'ρ ' then we obtained the maximum value of temperature level.

Figure-(6.5)(a),is presented to see the influence of 'α' on the velocity distribution in the presence of the other parameters as constant. From the figure we have perceived that as we increase the value of 'α' then the velocity distribution level shows decreasing trend, means velocity distribution level decay near the plate, while increasing farther away from the plate in the integrated domain. Finally, it asymptotically comes close to zero as y increases.

Figure-(6.5)(b), we have investigated the impact of 'ρ ' on velocity distribution in presence of all other parameters as constant. When we increase the value of $0 < \rho < 1$ then the velocity level shows an increasing trend against to the effect of α. Also we have seen that the distance between the thermal and boundary layers increases which lead to enhance the flow properties of the fluid flow.

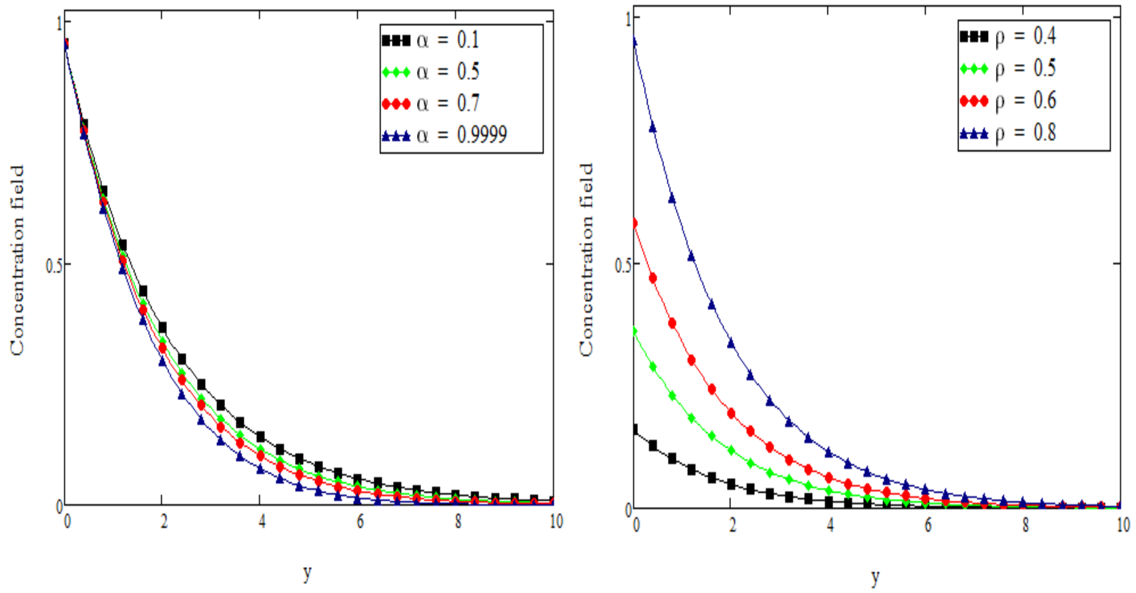
Figure-(6.6)(a) is presented to analyze the effect of 'ρ' on the concentration field in the presence of the other parameters as constant. From the figure we have investigated that as we increase the value of 'ρ' where $\rho > 1$, then the concentration level shows increasing trend as in figure-(6.1)(a).

Figure-(6.6)(b) is presented to see the influence of 'ρ' on the temperature field in the presence of the other parameters as constant. From the figure we have perceived that as we increase the value of 'ρ' where $\rho > 1$, then the temperature level shows increasing trend as in figure-(6.3)(b).

Figure-(6.7) is presented to see the influence of 'ρ' on the velocity distribution in the presence of the other parameters as constant. From the figure we have perceived that as we increase the value of 'ρ' where $\rho > 1$, then the velocity distribution level shows increasing trend as in figure-(6.5)(a).

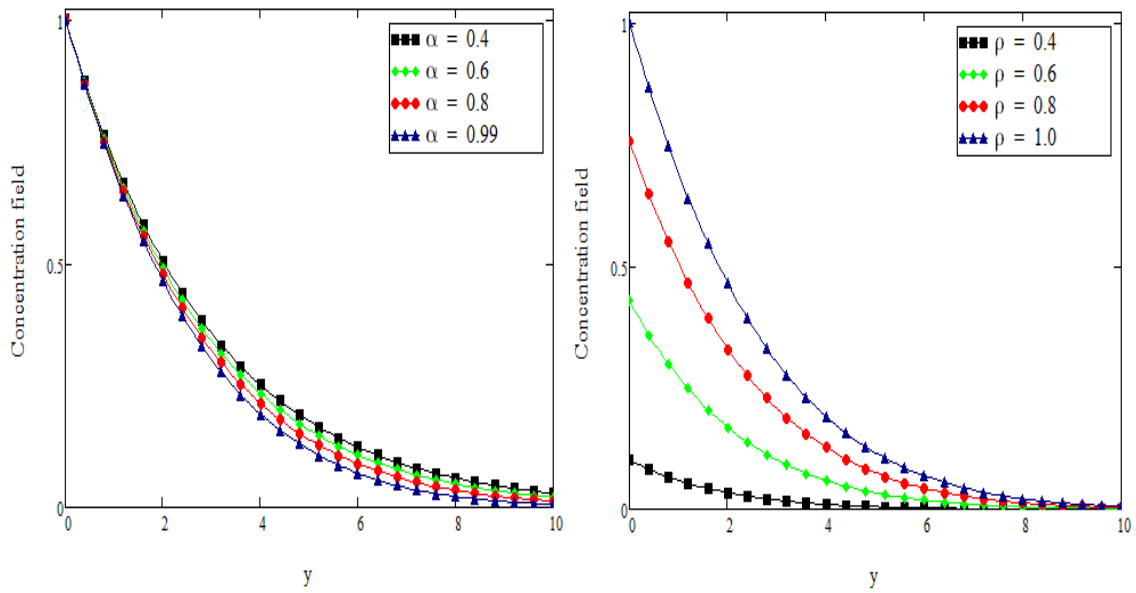
Figure-(6.8) is presented to see the effect of 'M' on the velocity distribution in the presence of the all other parameters as constant. From the figure-(6.8), we have perceived that as we increase the value of 'M' where $M > 0$, then the velocity distribution level shows decreasing trend. Physically, it is because of the fact that increased value of magnetic field parameter produce Lorentz force which is resistive force acting in opposite direction of fluid motion and slower down the motion of fluid.

Figure-(6.9), is presented to see the influence of 'Pr' on the velocity distribution in the presence of the other parameters as constant. From the figure we have perceived that as we increase the value of 'Pr' where $Pr > 0$, then the velocity distribution level shows decreasing trend. Physically, this is mainly due to the enhanced value of Prandtl number that decreases the amount of thermal conductivity and in turn magnifies the fluid viscosity that causes the thermal boundary layer thickness to be reduced. For the validation of the present study we have plotted the Figures (6.10), (6.11), (6.12) and table 1 and 2.



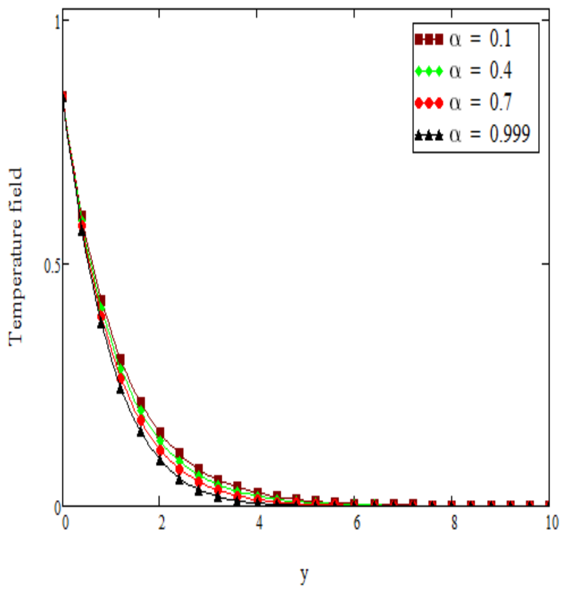
(a) Graphical results of α fixing by $\rho = 0.5$ (b) Graphical results of ρ by fixing $\alpha = 0.5$

Figure 1: Concentration profile of α and ρ on values $Sc = 0.22$, $t = 1.2$.

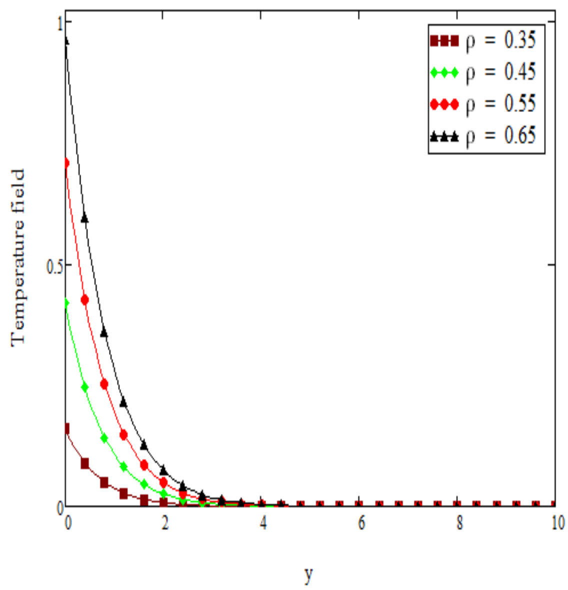


(a) Graphical results of α by fixing $\rho = 1$ (b) Graphical results of ρ by fixing $\alpha = 1$

Figure 2: Concentration profile of α and ρ on values $Sc = 0.1$, $t = 1.0$.

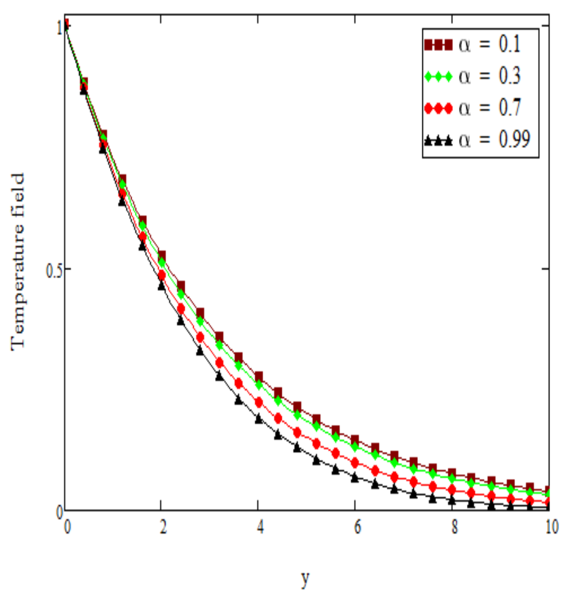


(a) Graphical results of α fixing by $\rho = 0.6$, and $Pr = 0.17$

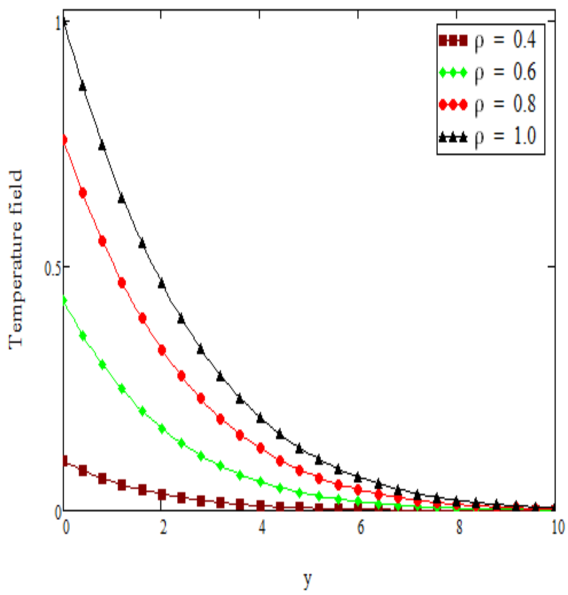


(b) Graphical results of ρ by fixing $\alpha = 0.7$, and $Pr = 1.2$

Figure 3: Temperature profile of α and ρ on value $t = 1.5$.

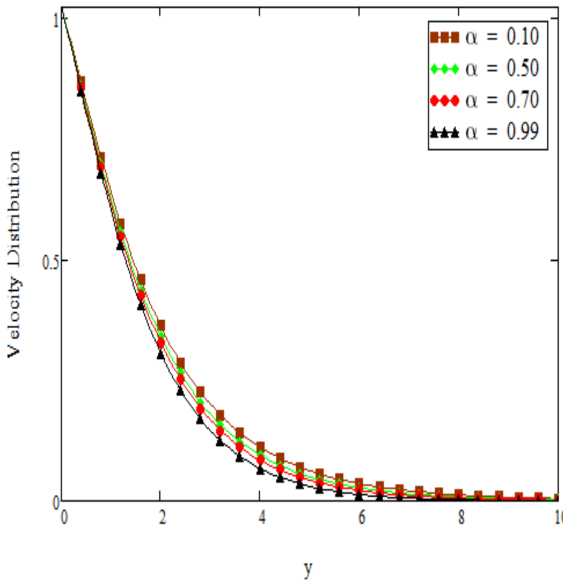


(a) Graphical results of α by fixing $\rho = 1$ and $Pr = 0.1$

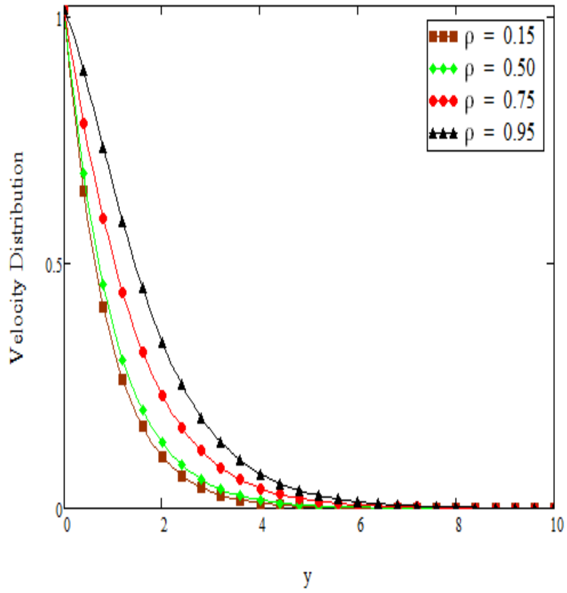


(b) Graphical results of ρ by fixing $\alpha = 1$ and $Pr = 0.1$

Figure 4: Temperature profile of α and ρ on value $t = 1$.

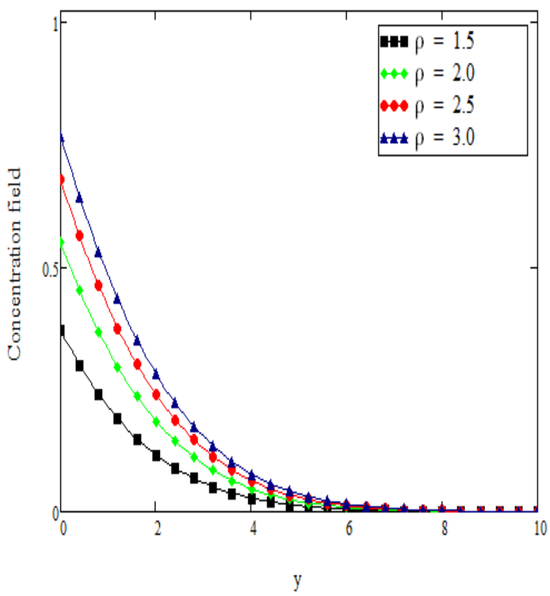


(a) Graphical results of α by fixing $\rho = 1$, $\lambda = 0.02,\ M = 1.8,\ Pr = 0.7$.

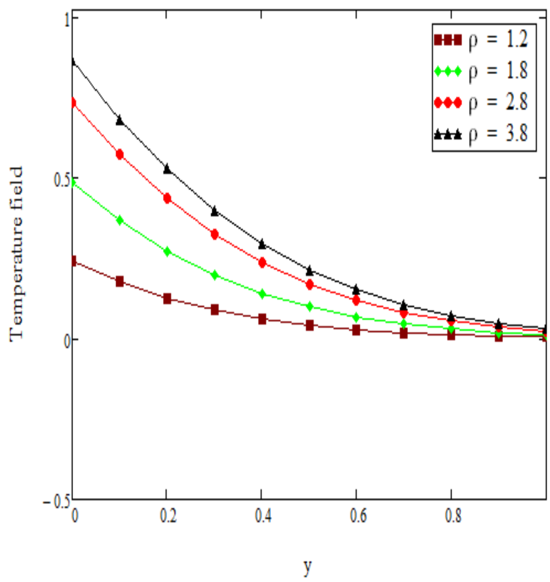


(b) Graphical results of ρ by fixing $\rho = 1$, $Pr = 0.75$, $\lambda = 0.01,\ M = 1.2$.

Figure 5: Velocity profile of α and ρ on values $Sc = 0.22$, $t = 1.2$, $\alpha_2 = 0.6$, $Gr = 1.2$ and $Gm = 1.8$



(a) Concentration profile



(b) Temperature profile

Figure 6: Concentration profile for large numeric values of ρ when $\alpha = 0.9, t = 0.15,\ SC = 0.22$ and , $Pr = 1.5$

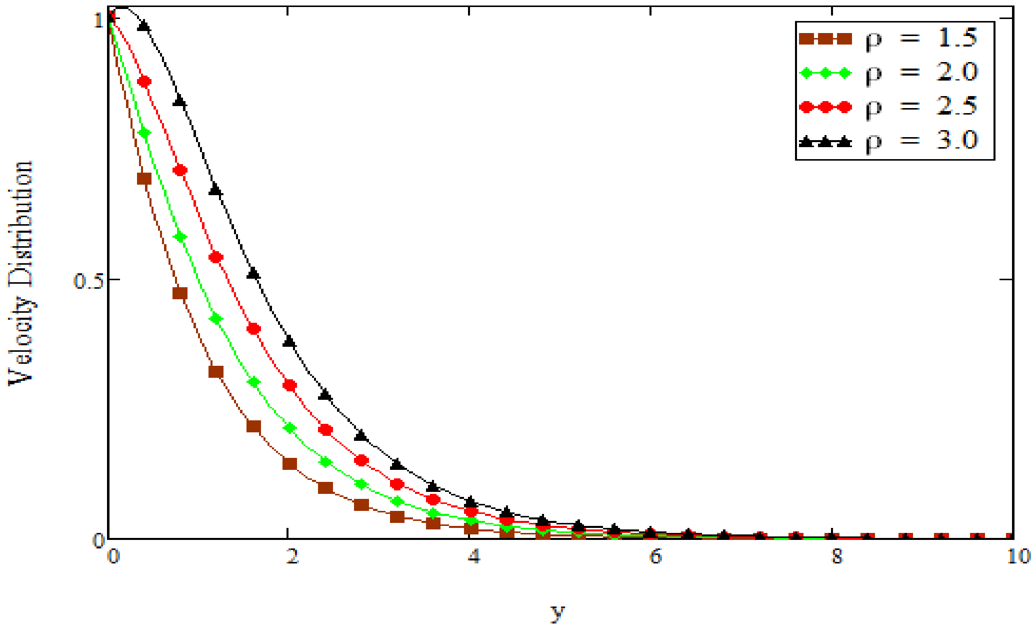


Figure 7: Velocity in case of large numeric values of α at
$Sc = 0.22,\ \alpha = 0.4,\ t = 0.1,\ \alpha_2 = 0.6,\ \lambda = 0.01,\ M = 1.2,\ Gr = 1.2,\ Gm = 0.5,$
$Pr = 0.75.$

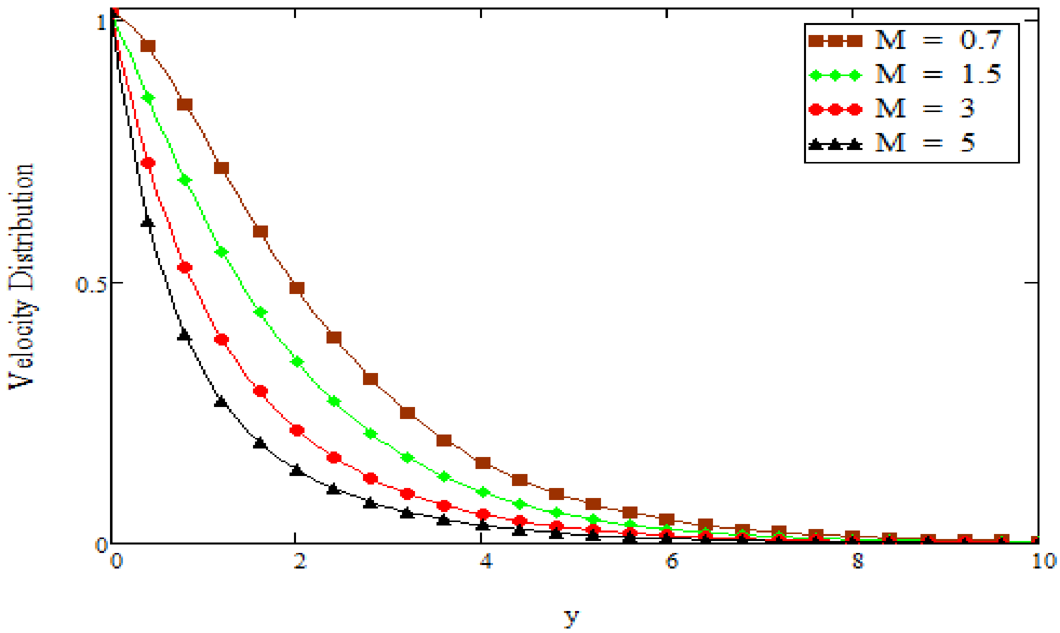


Figure 8: Velocity profile with M when
$Sc = 0.22,\ \rho = 0.9,\ t = 1.2,\ \alpha_2 = 0.6,\ \lambda = 0.01,\ Gr = 1.5,\ Gm = 0.2,\ Pr = 0.5,\ \alpha = 0.1.$

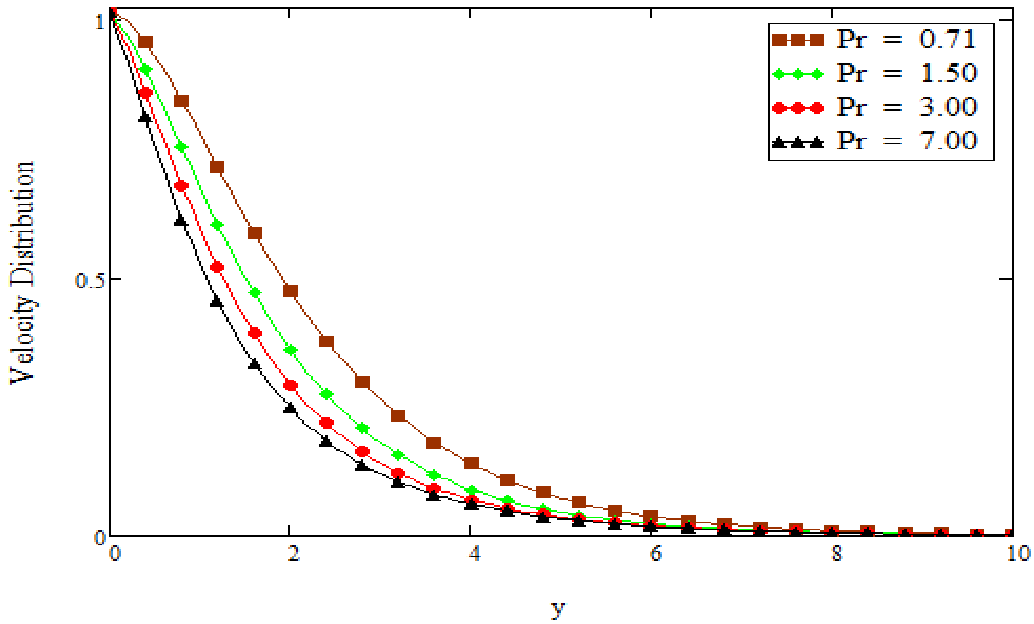


Figure 9: Velocity profile with Pr when
$Sc = 0.22,\ \rho = 0.9,\ t = 1.2,\ \alpha_2 = 0.6\ ,\ \lambda = 0.01,\ Gr = 1.5,\ Gm = 0.2,\ M = 0.5,\ \alpha = 0.1.$

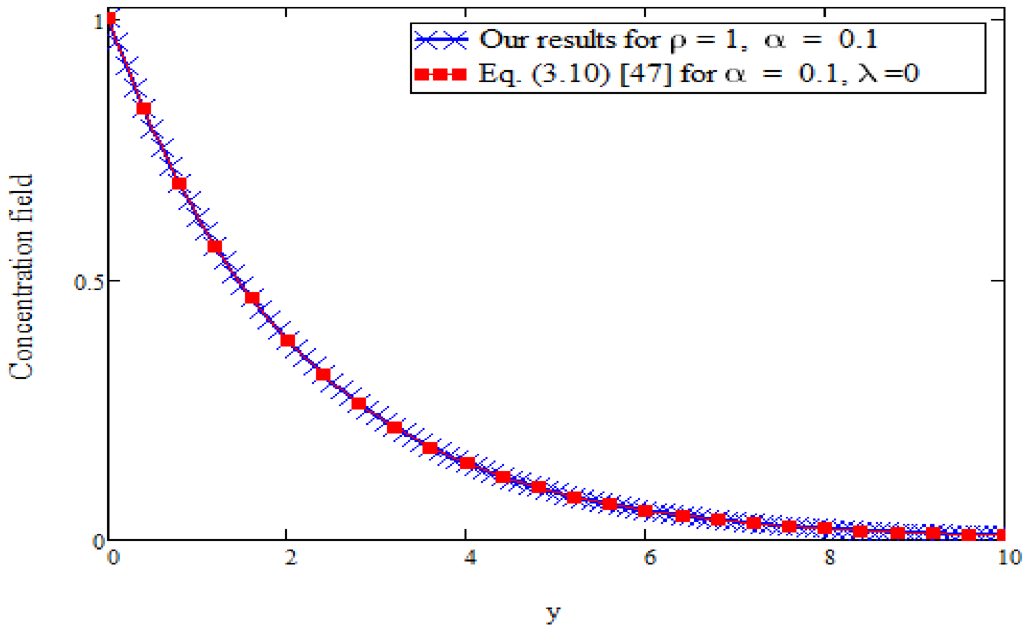


Figure 10: Concentration comparison with [47] (no presence of chemical reaction) when
$Sc = 0.22,\ \rho = 1,\ t = 1.$

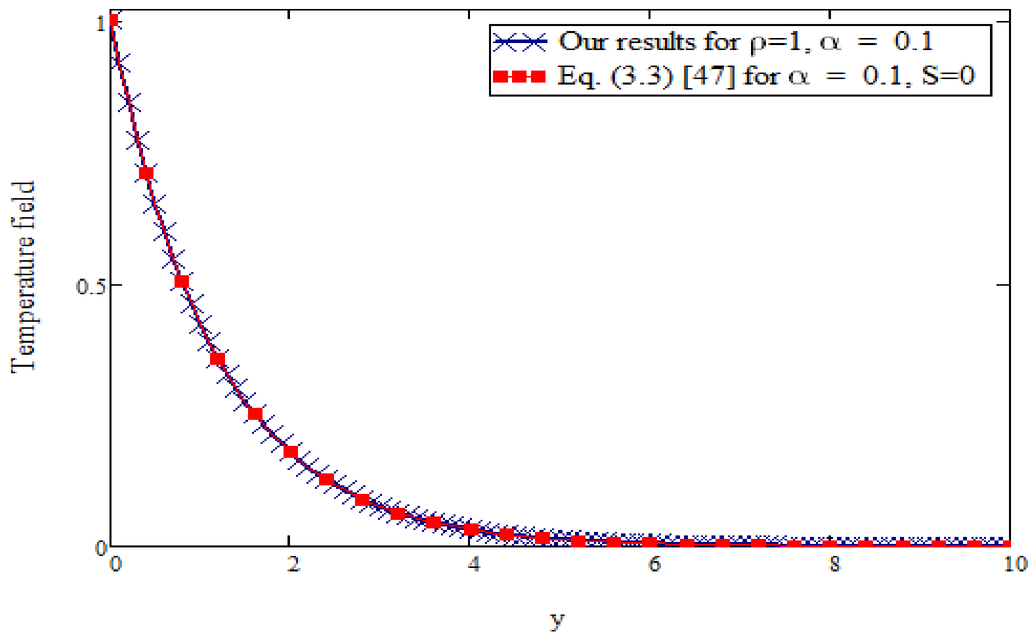


Figure 11: Temperature comparison with [47] (no presence of heat source) at $Pr = 0.71,\ \rho = 1,\ t = 1$.

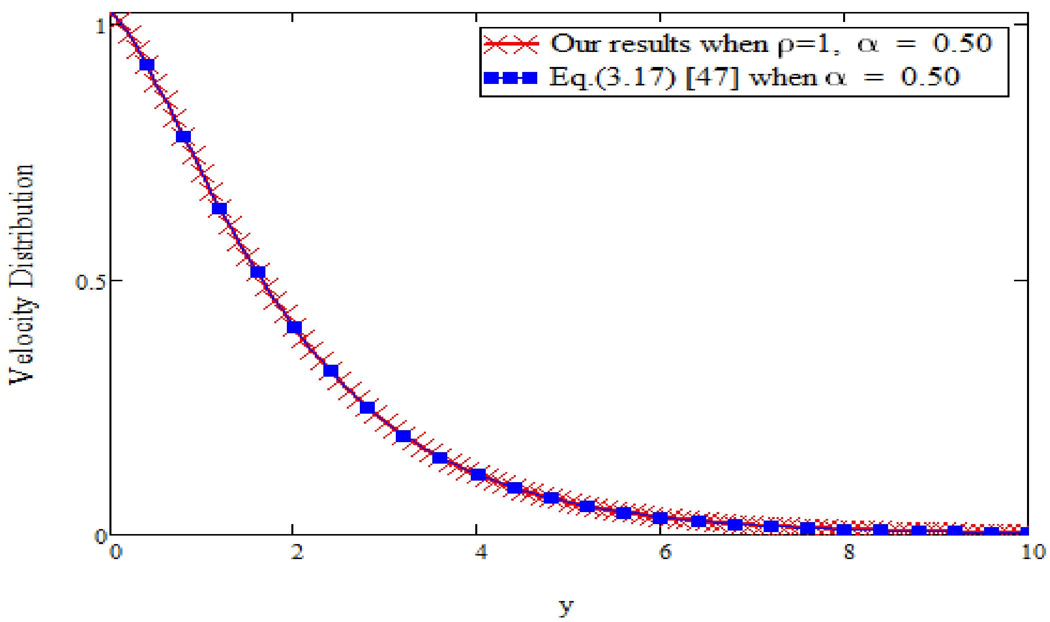


Figure 12: Velocity comparison with [47] (no presence of source of heat and chemical reaction) at $Sc = 0.22,\ \rho = 1,\ t = 1.2,\ \alpha_2 = 0.6\ ,\ \lambda = 0.02,\ Gr = 1.2,\ Gm = 1.8,\ M = 1.8, \alpha = 0.5,\ Pr = 0.71$.

Table 1

Table 1 Statistical analysis of our problem for ρ and α by keeping others parameters fixed:

y	T(y,t)			C(y,t)			u(y,t)		
	$\rho<1$ $\rho=0.6$	$\rho>1$ $\rho=2$	$\alpha=0.7$	$\rho<1$ $\rho=0.6$	$\rho>1$ $\rho=2$	$\alpha=0.7$	$\rho<1$ $\rho=0.6$	$\rho>1$ $\rho=2$	$\alpha=0.7$
0	0.839	0.548	0.839	0.578	548	0.95	1.012	1.001	1.024
0.1	0.743	0.417	0.764	0.548	0.522	0.902	0.928	0.944	0.982
0.2	0.657	0.312	0.695	0.519	0.497	0.856	0.852	0.887	0.940
0.3	0.580	0.228	0.632	0.492	0.473	0.813	0.782	0.831	0.897
0.4	0.512	0.164	0.575	0.466	0.450	0.771	0.717	0.777	0.854
0.5	0.452	0.115	0.522	0.418	0.428	0.731	0.659	0.724	0.811
0.6	0.398	0.079	0.474	0.396	0.406	0.694	0.605	0.673	0.769
0.7	0.351	0.053	0.431	0.375	0.386	0.658	0.556	0.625	0.729
0.8	0.308	0.035	0.391	0.355	0.366	0.623	0.511	0.579	0.689
0.9	0.271	0.022	0.354	0.337	0.347	0.591	0.47	0.536	0.651
1.0	0.238	0.014	0.321	0.319	0.329	0.560	0.433	0.495	0.614

Table 2

Table 2 Comparison of attained our outcomes of velocity, concentration and temperature to [47]:

y	T(y,t)		C(y,t)		u(y,t)	
	our value $\rho=1$, $\alpha=0.1$	Eq.(3.3)[10] $S=0$, $\alpha=0.1$	our value $\rho=1$, $\alpha=0.1$	Eq.(3.10)[10] $\lambda=0$, $\alpha=0.1$	our value $\rho=1$, $\alpha=0.1$	Eq.(3.13)[10] $\lambda=0$, $\alpha=0.1$
0	1	1	1	1	1.024	1.024
0.1	0.918	0.918	0.953	0.954	1.006	1.006
0.2	0.842	0.842	0.909	0.869	0.982	0.982
0.3	0.773	0.773	0.866	0.829	0.955	0.955
0.4	0.709	0.709	0.826	0.791	0.925	0.925
0.5	0.650	0.650	0.787	0.755	0.893	0.893
0.6	0.597	0.597	0.750	0.720	0.859	0.859
0.7	0.548	0.548	0.715	0.687	0.825	0.825
0.8	0.502	0.502	0.682	0.565	0.791	0.791
0.9	0.461	0.461	0.650	0.626	0.756	0.756
1.0	0.423	0.423	0.619	0.597	0.722	0.722
1.1	0.388	0.388	0.591	0.570	0.688	0.688

7. Conclusions

The present analysis studied the application of modified Laplace transform to mass and heat transmission of the second grade fluid flow using generalized Caputo fractional derivative. Our interest is to see the impact of two parameters α and ρ on the fluid properties like those of concentration, temperature and velocity field distributively. A few key points in the analysis can be observed as given below.

• For larger values of alpha, the fluid properties (temperature, concentration and velocity) observed decay near the plate.

• On the other hand, fluid properties can be enhanced for the values of $\rho \leqslant 1$ and moreover $\rho > 1$ for also increases behavior observed.

• The results obtained with modified Laplace transform are more efficient and can be used to enhance the fluid flow properties.

• Maximum enhancement of fluid properties can be obtained by increasing the value of ρ for fixed value of $\alpha = 1$.

• By limiting the parameter ρ, we obtained the outcomes from the existing literature and they are in good premises.

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