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On fuzzy Henstock-Kurzweil-Stieltjes-Diamond-double integral on time scales

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Abstract

We introduce and study some properties of fuzzy Henstock-Kurzweil-Stietljes-$\diamond$-double integral on time scales. Also, we state and prove the uniform convergence theorem, monotone convergence theorem and dominated convergence theorem for the fuzzy Henstock-Kurzweil-Stieltjes-$\diamond$-double integrable functions on time scales.

Keywords: Fuzzy, Convergence theorems, Double integral, Henstock-Kurzweil integral, Time scales.

1. Introduction

In modern day analysis, Henstock [1] and Kurzweil [2] independently introduced the concepts of Henstock and Kurzweil integration. This later developed into Henstock-Kurzweil integration which is the generalization of the two integrals for real-valued functions. The generalization of this concept in the fuzzy setting is a rare case. Wu and Gong [3] introduced Henstock integral of fuzzy-number-valued functions, fuzzy sets and systems and presented some of its basic properties. Gong and Shao [4] gave the controlled convergence theorems for the strong Henstock integrals of $fuzzy$-number-valued functions, fuzzy sets and systems. For other interesting results involving fuzzy Henstock-Kurzweil integral, see e.g., the papers [3, 4, 5] and references cited therein.
In 1988, the theory of time scales was introduced by Hilger in his Ph.D. thesis [6]. The aim is to unify and generalize the concept of discrete and continuous dynamical systems. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [7], and Henstock-Kurzweil integrals on time scales were studied by Thomson [8]. Park et al. [9] studied the convergence results for the Henstock delta integral on time scales. It is clear that most of the properties of a time scale integral can be realized by

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using the techniques tailored to the time scale setting (see [6, 8, 9, 10, 11, 12, 13]) and references cited therein.
In this paper, we introduce fuzzy Henstock-Kurzweil-Stieltjes-◇-double integral on time scales.

2. Preliminaries

A time scale $\mathbb{T}$ is any closed non-empty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$.
Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < d, c < d$, and a rectangle $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) : t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be two non-decreasing functions on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$, respectively. Let $F : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be bounded on $\mathcal{R}$. Let P_1 and P_2 be two partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points from $[a, b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, ..., n$. Similarly, let $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points from $[c, d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, ..., k$.

Definition 2.1. ([5]) Let α be a real axis, a fuzzy subset of $\alpha : \mathbb{R} \to [0, 1]$ is called a fuzzy number if the following conditions occur:
(i) α is normal. That is $x_0 \in \mathbb{R}$ exists with $\alpha(x_0) = 1$;
(ii) α is fuzzy convex, i.e. $\alpha(kx_1 + (1 - k)x_2) \geqslant \min\{\alpha(x_1), \alpha(x_2)\}$ for all $x_1, x_2 \in \mathbb{R}$ and all $k \in (0, 1)$;
(iii) α is upper semi-continuous;
(iv) $[\alpha]^0 = \overline{\{x \in \mathbb{R} : \alpha(x) > 0\}}$ is compact.

We shall denote the space of fuzzy numbers by $f_{\mathbb{R}}$ and define the λ-level set $[\alpha]^\lambda$ by

$$[\alpha]^\lambda = \{x \in \mathbb{R} : \alpha(x) \geqslant \lambda\}, \lambda \in (0, 1].$$

By the conditions (i)-(iv) of Definition 2.1, denote $[\alpha]^\lambda$ by $[\alpha]^\lambda = [\underline{\alpha^\lambda}, \overline{\alpha^\lambda}]$ and for $\alpha_1, \alpha_2 \in f_{\mathbb{R}}$ and $k \in \mathbb{R}$, we define

$$[\alpha_1 + \alpha_2]^\lambda = [\alpha_1]^\lambda + [\alpha_2]^\lambda \text{ and } [k \odot \alpha_1]^\lambda = k[\alpha_1]^\lambda$$

for all $\lambda \in [0, 1]$.

Definition 2.2. ([14]) Let $(f_{\mathbb{R}}, D)$ be a complete metric space. The Hausdorff distance between α_1 and α_2 is defined by

$$D(\alpha_1, \alpha_2) = \sup_{\lambda \in [0,1]} \max\{|\underline{\alpha_1^\lambda} - \underline{\alpha_2^\lambda}|, |\overline{\alpha_1^\lambda} - \overline{\alpha_2^\lambda}|\}.$$

We now introduce Henstock-Kurzweil-Stieltjes-◇-double integral over versions in $\mathbb{T}_1 \times \mathbb{T}_2$. Let $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ be a rectangle, and denote by $f_{\mathbb{R}}$ the space of fuzzy numbers on real line.

Definition 2.3. Let $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to f_{\mathbb{R}}$ be a bounded function on $\mathcal{R}$ and let g_1 and g_2 be increasing functions defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ with partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset$

$[a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, 2, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ for $j = 1, 2, ..., k$. Then

$$S(P_1, P_2, F, g_1, g_2) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$$

is defined as fuzzy Henstock-Kurweil-Stieltjes-◇-double sum of F with respect to functions g_1 and g_2.

Let $P = P_1 \times P_2$ and $\diamond g_{1_i} \diamond g_{2_j} = (g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$, then the Henstock-Kurweil-Stieltjes-◇-double sum of F with respect to functions g_1 and g_2 is denoted by $S(P, F, g)$ is written as

$$S(P, F, g_1, g_2) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)\diamond g_{1_i}\diamond g_{2_j},\ (i = 1, ..., n;\ j = 1, ..., k).$$

3. Main Results

Definition 3.1. Let $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to f_{\mathbb{R}}$ be a fuzzy function on $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} : t \in [a, b)_{\mathbb{T}_1}, s \in [c, d)_{\mathbb{T}_1}$. We say that F is fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrable with respect to non-decreasing functions g_1, g_2 defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ if there is a number $\tilde{L}$, a member of $\mathbb{R}$ such that for every $\varepsilon > 0$, there is a ◇-gauge δ (or γ) such that

$$D\left(S(P, F, g_1, g_2), \tilde{L}\right) < \varepsilon$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ are δ-fine (or γ) partitions of $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$.

A positive function $\delta(t, s), \gamma(t, s) : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to f_{\mathbb{R}}$ such that $\delta(t, s) > 0$ for all t, s in $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ or ($\gamma(t, s) > 0$ for all t, s in $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$) is known as ◇ gauge on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$.

We say that $\tilde{L}$ is the fuzzy Henstock-Kurzweil-Stieltjes-◇-double integral of F with respect to g_1 and g_2 defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}} F(t, s)\diamond_1 g_1(t)\diamond_2 g_2(s) = \tilde{L}.$$

The family of all fuzzy Henstock-Kurzweil-Stieltjes-◇-integrable functions on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ is denoted by $\mathcal{FHKS}_{[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}}$.

Lemma 3.2. *([15]) Suppose that $\alpha \in f_{\mathbb{R}}$. Then,*
(i) the interval $[\alpha]^{\lambda}$ is closed for $\lambda \in [0, 1]$;
(ii) $[\alpha]^{\lambda_1} \supset [\alpha]^{\lambda_2}$ for $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$;
(iii) for any sequence $\{\lambda_n\}$ satisfying $\lambda_n \leqslant \lambda_{n+1}$ and $\lambda_n \to \lambda \in (0, 1]$, we have

$$\bigcap_{n=1}^{\infty} [\alpha]^{\lambda_n} = [\alpha]^{\lambda}.$$

Now we state and provide proofs for our theorems.

Theorem 3.3. *If* $F : (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \to f_{\mathbb{R}}$ *is fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrable with respect to two increasing functions* g_1, g_2 *on* $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$*, then the fuzzy Henstock-Kurzweil-Stieltjes-◇-double integral of* F *is unique.*

Proof. Suppose that $\tilde{L_1}$ and $\tilde{L_2}$ are both fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrals of F on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$. With the assumption that $\tilde{L_1}$ and $\tilde{L_2}$ are not unique, then F is said to be fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrable on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ if it satisfies the following point wise integrability criterion: for every $\varepsilon > 0$ there are ◇-gauges δ_1 and δ_2 (or γ_1 and γ_2) defined on $(a, b]_{\mathbb{T}_1}$ and $(c, d]_{\mathbb{T}_2}$ respectively, such that for every $\varepsilon > 0$, there are ◇-gauges δ_1^1 and δ_2^1 (or γ_1^1 and γ_2^1) for $[a, b)_{\mathbb{T}_1}$ and δ_1^2 and δ_2^2 (or γ_1^2 and γ_2^2) for $[c, d)_{\mathbb{T}_2}$ such that

$$D\left(S(P^1, F, g_1, g_2), \tilde{L_1}\right) < \frac{\varepsilon}{2} \text{ and } D\left(S(P^2, F, g_1, g_2), \tilde{L_2}\right) < \frac{\varepsilon}{2} \text{ for all pairs } P^1 = P_1^1 \times P_2^1 \text{ and } P^2 = P_1^2 \times P_2^2 \text{ of } \delta_1\text{-fine (or } \gamma_1)$$

and for every $\varepsilon > 0$ and $i \in \{1, 2\}$, there are ◇-gauges δ_1^i and δ_2^i (or γ_1^i and γ_2^i) for $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$ respectively such that

$$D(S(P^i, F, g_1, g_2), \tilde{L_i}) < \frac{\varepsilon}{2}$$

provided that $P^i = P_1^i \times P_2^i$ is a pair of δ_1^i-fine (or γ_1^i) and δ_2^i-fine (or γ_2^i) partitions of $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$ respectively.
Let $\delta_1 = \min\{\delta_1^1, \delta_1^2\}$ i.e. $(\delta_1)_L = \min\{(\delta_1^1)_L, (\delta_1^2)_L\}$ and $(\delta_1)_R = \min\{(\delta_1^1)_R, (\delta_1^2)_R\}$ and $\delta_2 = \min\{\delta_2^1, \delta_2^2\}$ i.e. $(\delta_2)_L = \min\{(\delta_2^1)_L, (\delta_2^2)_L\}$ and $(\delta_2)_R = \min\{(\delta_2^1)_R, (\delta_2^2)_R\}$, δ_1 and δ_2 are ◇-gauges for $(a, b]_{\mathbb{T}_1}$ and $(c, d]_{\mathbb{T}_2}$ respectively, and given a pair $P = P_1 \times P_2$ of δ_1-fine and δ_2-fine partitions of $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$, P_1 is a δ_1^1-fine and δ_1^2-fine partition of $(a, b]_{\mathbb{T}_1}$, P_2 is a δ_2^1-fine and δ_2^2-fine partition of $[c, d)_{\mathbb{T}_2}$, hence

$$\begin{aligned} D(\tilde{L_1}, \tilde{L_2}) &\leqslant D((\tilde{L_1}, S(P, F, g_1, g_2) + S(P, F, g_1, g_2), \tilde{L_2})) \\ &\leqslant D(S(P, F, g_1, g_2), \tilde{L_1}) + D(S(P, F, g_1, g_2), \tilde{L_2}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

since for all $\varepsilon > 0$, there are ◇-gauges δ_1 and δ_2 (or γ_1 and γ_2), then it follows that $\tilde{L_1} = \tilde{L_2}$.
Hence, the fuzzy Henstock-Kurzweil-Stieltjes-◇-double integral of F on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ is unique. □

Theorem 3.4. *(Bolzano Cauchy Criterion). Let* $F : (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \to f_{\mathbb{R}}$ *be a fuzzy-valued function over a rectangle* $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ *with respect to* g_1, g_2 *on* $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$*. Then,* F *is fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrable on* $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ *if and only if for each* $\varepsilon > 0$ *there exists a positive function* $\delta : (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2} \to f_{\mathbb{R}}$ *such that* $D(S(P_{\delta_1}, F, g_1, g_2), S(P_{\delta_2}, F, g_1, g_2), F, g)) < \varepsilon$ *for all* $\delta-$ *fine tagged partitions* P_1 *and* P_2 *on* $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$.

Proof. Suppose F is fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrable on $(a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2}$ with respect to g_1 and g_2 , and let

$$\tilde{L} = \int\int_{\mathcal{R}} F(t,s)\diamond g_1(t)\diamond g_2(s).$$

Let $\varepsilon > 0$. There are ◇-gauges δ_1 and δ_2 for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that $D(S(P,F,g_1,g_2),\tilde{L}) < \frac{\varepsilon}{2}$ provided that $P = P_1 \times P_2$ where P_1 is a δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$. Therefore, if $P = P_1 \times P_2$ and $P = P_1^{'} \times P_2^{'}$ are pairs of δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$, then

$$\begin{aligned} D(S(P,F,g_1,g_2),S(P^1,F,g_1,g_2)) &\leqslant D(S(P,F,g_1,g_2),\tilde{L}) \\ &\quad +D(\tilde{L},S(P^1,F,g_1,g_2)) \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, suppose that for all $\varepsilon > 0$ there are ◇-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that
$D(S(P^1,F,g_1,g_2),S(P^2,F,g_1,g_2)) < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of δ_1 (or γ_1)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and δ_2 (or γ_2)-fine partitions of $[c,d)_{\mathbb{T}_2}$.
Let $n \in \mathbb{N}$. Taking $\varepsilon = \frac{1}{n}$, there are ◇-gauges $\delta_{1,n}$ and $\delta_{2,n}$ (or $\gamma_{1,n}$ and $\gamma_{2,n}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that
$D(S(P^{'},F,g_1,g_2),S(P^2,F,g_1,g_2)) < \varepsilon$ for all pairs $P^1 = P_1^1 \times P_2^1$ and $P^2 = P_1^2 \times P_2^2$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$.
By replacing $\delta_{i,n}$ by $\min\{\delta_{i,1},\delta_{i,2},...,\delta_{i,n}\}$ with $i \in \{1,2\}$, we may assume that $\delta_{i,n+1} \leqslant \delta_{i,n}$. Thus, for all $j > n$ $\delta_{i,j} \leqslant \delta_{i,n}$ so any pair $P^n = P_1^n \times P_2^n$ of $\delta_{1,n}$ (or $\gamma_{1,n}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,n}$ (or $\gamma_{2,n}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$ is also a pair of $\delta_{1,j}$ (or $\gamma_{1,j}$)-fine partitions of $[a,b)_{\mathbb{T}_1}$ and $\delta_{2,j}$ (or $\gamma_{2,j}$)-fine partitions of $[c,d)_{\mathbb{T}_2}$, hence

$$D(S(P^n,F,g_1,g_2),S(P^j,F,g_1,g_2)) < \frac{1}{j}.$$

This shows that $\{S(P^n,F,g_1,g_2)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
Let $\tilde{L}$ be the limit of $\{S(P^n,F,g_1,g_2)\}_{n\in\mathbb{N}}$. For all $\varepsilon > 0$, choosing $N > \frac{2}{\varepsilon}$, for ◇-gauges $\delta_{1,N}$ and $\delta_{2,N}$ (or $\gamma_{1,N}$ and $\gamma_{2,N}$) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively,

$$\begin{aligned} D(S(P,F,g_1,g_2),\tilde{L}) &\leqslant D(S(P,F,g_1,g_2),S(P^N,F,g_1,g_2)) \\ &\quad +D(S(P^N,F,g_1,g_2),\tilde{L}) \\ &< \frac{1}{N}+\frac{\varepsilon}{2} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for pair $P = P_1 \times P_2$ such that P_1 is a $\delta_{1,N}$ (or $\gamma_{1,N}$) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a $\delta_{2,N}$ (or $\gamma_{2,N}$) fine partition of $[c,d)_{\mathbb{T}_2}$. □

The following properties are obtained using the definition of fuzzy Henstock-Kurzweil-Stieltjes-◇-double integral of F on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. The proof of the next theorem is straightforward and therefore omitted.

Theorem 3.5. *Let* $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to f_{\mathbb{R}}$ *be a fuzzy-valued function and let* g_1 *and* g_2 *be non-decreasing functions respectively on* $[a,b]_{\mathbb{T}_1}$ *and* $[c,d]_{\mathbb{T}_2}$*. If* F *is fuzzy Henstock-Kurzweil-Stieltjes-*$\diamondsuit$*-double integrable with respect to* g_1 *and* g_2 *on* $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$*. Then,*

i. $\int\int_{\mathcal{R}} \beta \diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s) = \beta(g_1(b) - g_1(a))(g_2(d) - g_2(c))$, β *is a constant;*

ii. $\int\int_{\mathcal{R}} F(t,s) \diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s) = 0$ *when* g_1 *or* g_2 *are constants;*

iii. $\int\int_{\mathcal{R}} F(t,s) \diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s) = f(a,c)(g_1^{\tau_1}(a) - g_1(a))(g_2^{\tau_2}(c) - g_2(c))$
with $b = \tau_1(a)$ *and* $\tau_2(c)$*;*

iv. $\int\int_{\mathcal{R}} \beta F(t,s) \mu[\diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s)] = \int\int_{\mathcal{R}} \beta \mu f(t,s) \diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s)$;
β *and* μ *are constants.*

Theorem 3.6. *Let* F *be a fuzzy-number-valued function, consider a sequence of fuzzy-number-valued function* $F_n : (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \to f_{\mathbb{R}}, n \in \mathbb{N}$ *in* $f_{\mathbb{R}}$ *and increasing functions* $g_1, g_2 : (a,b]_{\mathbb{T}_1} \times (c,d]_{\mathbb{T}_2} \to f_{\mathbb{R}}$*. Assume*
(i) $\lim_{n\to\infty} F_n(t,s) = F(t,s)$ *holds* $\diamondsuit$ *a.e.;*
(ii) $G(t,s) \leqslant F_n(t,s) \leqslant H(t,s)$ *holds* $\diamondsuit$ *a.e.;*
(iii) $F_n(t,s), G(t,s), H(t,s) \in \mathcal{FHKS}_{[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}}$.
Then $F(t,s) \in \mathcal{FHKS}_{[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}}$*. Moreover,*

$$\lim_{n\to\infty} \int\int_{\mathcal{R}} F_n(t,s) \diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s) = \int\int_{\mathcal{R}} F(t,s) \diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s).$$

The proof of Theorem 3.6 is straightforward following the style of proof in [7].

Theorem 3.7. *Let* $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to f_{\mathbb{R}}$ *be a fuzzy-number-valued function and let* g_1 *and* g_2 *be non-decreasing functions respectively on* $[a,b]_{\mathbb{T}_1}$ *and* $[c,d]_{\mathbb{T}_2}$*. Function* $F(t,s) \in \mathcal{FHKS}_{[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}}$ *if and only if* $\underline{F(t,s)^\lambda}, \overline{F(t,s)^\lambda} \in \mathcal{HKS}_{[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}}$ *for all* $\lambda \in [0,1]$ *uniformly.*

Proof. For the necessary condition, let P_1 and P_2 be two partitions of $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c,d]_{\mathbb{T}_2}$. Let $\tilde{L} = \int\int_{\mathcal{R}} F(t,s) \diamondsuit_1 g_1(t) \diamondsuit_2 g_2(s)$. Given $\varepsilon > 0$, there exists a $\delta(t,s)$ such that $D\left(S(P_\delta, F, g_1, g_2), \tilde{L}\right) < \varepsilon$ for any fine tag partition P_1 and P_2. Then,

$$\sup_{\lambda\in[0,1]} \max\left\{\left|[\underline{S(P_\delta, F, g_1, g_2)}]^\lambda - \underline{\tilde{L^\lambda}}\right|, \left|[\overline{S(P_\delta, F, g_1, g_2)}]^\lambda - \overline{\tilde{L^\lambda}}\right|\right\}$$

$$= \sup_{\lambda\in[0,1]} \max\left\{\left|[S(P_\delta, \underline{F^\lambda}, g_1^\lambda, g_2^\lambda)] - \underline{\tilde{L^\lambda}}\right|, \left|[S(P_\delta, \overline{F^\lambda}, g_1^\lambda, g_2^\lambda)] - \overline{\tilde{L^\lambda}}\right|\right\} < \varepsilon$$

and

$$\left|[S(P_\delta, \underline{F^\lambda}, g_1^\lambda, g_2^\lambda)] - \underline{\tilde{L^\lambda}}\right| < \varepsilon, \quad \left|[S(P_\delta, \overline{F^\lambda}, g_1^\lambda, g_2^\lambda)] - \overline{\tilde{L^\lambda}}\right| < \varepsilon$$

for any $\lambda \in [0,1]$ and for partitions $P = P_1 \cup P_2$. Thus, $\underline{F(t,s)}^{\lambda}, \overline{F(t,s)^{\lambda}} \in \mathcal{HKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$ uniformly for any $\lambda \in [0,1]$.

Now for sufficient condition, let $\varepsilon > 0$. By assumption, there exists a $\delta(t,s)$ such that

$$\left|[S(P_\delta, \underline{F^{\lambda}}, g_1^{\lambda}, g_2^{\lambda})] - \underline{\tilde{L^{\lambda}}}\right| < \varepsilon, \quad \left|[S(P_\delta, \overline{F^{\lambda}}, g_1^{\lambda}, g_2^{\lambda})] - \overline{\tilde{L^{\lambda}}}\right| < \varepsilon$$

for any $\lambda \in [0,1]$ and for partitions $P = P_1 \cup P_2$ where

$$\underline{\tilde{L^{\lambda}}} = \int\int_{\mathcal{R}} \underline{F(t,s)^{\lambda}} \diamond_1 g_1(t) \diamond_2 g_2(s), \quad \overline{\tilde{L^{\lambda}}} = \int\int_{\mathcal{R}} \overline{F(t,s)^{\lambda}} \diamond_1 g_1(t) \diamond_2 g_2(s).$$

To prove that $\left\{\left[\underline{\tilde{L^{\lambda}}}, \overline{\tilde{L^{\lambda}}}\right], \lambda \in [0,1]\right\}$ represents a fuzzy number, check that $\left[\underline{\tilde{L^{\lambda}}}, \overline{\tilde{L^{\lambda}}}\right]$ satisfies the conditions (i)-(iii) of Lemma :
(i) for $\lambda \in [0,1]$, if $\underline{F(t,s)^{\lambda}} \leqslant \overline{F(t,s)^{\lambda}}$, then $\underline{\tilde{L^{\lambda}}} \leqslant \overline{\tilde{L^{\lambda}}}$, i.e., the interval $\left[\underline{\tilde{L^{\lambda}}}, \overline{\tilde{L^{\lambda}}}\right]$ is closed.

(ii) $F(t,s)^{\lambda}$ and $\overline{F(t,s)^{\lambda}}$ nondecreasing and nonincreasing functions on $[0,1]$ respectively. For any $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$

$$\begin{aligned}
\int\int_{\mathcal{R}} \underline{F(t,s)^{\lambda_1}} \diamond_1 g_1(t) \diamond_2 g_2(s) &\leqslant \int\int_{\mathcal{R}} \underline{F(t,s)^{\lambda_2}} \diamond_1 g_1(t) \diamond_2 g_2(s) \\
&\leqslant \int\int_{\mathcal{R}} \overline{F(t,s)^{\lambda_2}} \diamond_1 g_1(t) \diamond_2 g_2(s) \\
&\leqslant \int\int_{\mathcal{R}} \overline{F(t,s)^{\lambda_1}} \diamond_1 g_1(t) \diamond_2 g_2(s).
\end{aligned}$$

Thus, $\left[\underline{\tilde{L^{\lambda_1}}}, \overline{\tilde{L^{\lambda_1}}}\right] \supset \left[\underline{\tilde{L^{\lambda_2}}}, \overline{\tilde{L^{\lambda_2}}}\right]$.

(iii) Now, for any $\{\lambda_n\}$ satisfying $\lambda_n \leqslant \lambda_{n+1}$ and $\lambda_n \to \lambda \in (0,1]$, we have

$$\bigcap_{n=1}^{\infty} [F(t,s)]^{\lambda_n} = [F(t,s)]^{\lambda},$$

that is,

$$\bigcap_{n=1}^{\infty} \left[\underline{F(t,s)^{\lambda_n}}, \overline{F(t,s)^{\lambda_n}}\right] = \left[\underline{F(t,s)^{\lambda}}, \overline{F(t,s)^{\lambda}}\right],$$

$\lim_{n\to\infty} \underline{F(t,s)^{\lambda_n}} = \underline{F(t,s)^{\lambda}}$ and $\lim_{n\to\infty} \overline{F(t,s)^{\lambda_n}} = \overline{F(t,s)^{\lambda}}$. Moreover,

$$\underline{F(t,s)^{0}} \leqslant \underline{F(t,s)^{\lambda_n}} \leqslant \underline{F(t,s)^{1}}, \quad \overline{F(t,s)^{1}} \leqslant \overline{F(t,s)^{\lambda_n}} \leqslant \overline{F(t,s)^{0}}.$$

By Theorem 3.6, we have $\underline{F(t,s)^{\lambda}}, \overline{F(t,s)^{\lambda}} \in \mathcal{HKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$ and

$$\lim_{n\to\infty} \int\int_{\mathcal{R}} \underline{F(t,s)^{\lambda_n}} \diamond_1 g_1(t) \diamond_2 g_2(s) = \int\int_{\mathcal{R}} \underline{F(t,s)^{\lambda}} \diamond_1 g_1(t) \diamond_2 g_2(s),$$

$$\lim_{n\to\infty} \int\int_{\mathcal{R}} \overline{F(t,s)^{\lambda_n}} \diamond_1 g_1(t) \diamond_2 g_2(s) = \int\int_{\mathcal{R}} \overline{F(t,s)^{\lambda}} \diamond_1 g_1(t) \diamond_2 g_2(s).$$

Consequently,

$$\bigcap_{n=1}^{\infty}\left[\underline{L^{\tilde{\lambda}_n}},\overline{L^{\tilde{\lambda}_n}}\right]=\left[\underline{L^{\tilde{\lambda}}},\overline{L^{\tilde{\lambda}}}\right].$$

Now, $\tilde{L}=\left\{\left[\underline{L^{\tilde{\lambda}}},\overline{L^{\tilde{\lambda}}}\right],\lambda\in[0,1]\right\}$. Thus,

$$D\left(S(P,F,g_1,g_2),\tilde{L}\right)<\varepsilon$$

for each partitions $P=P_1\cup P_2$ and for any n. This ends the proof. □

Definition 3.8. A sequence $\{F_n(t,s)\}$ of Henstock-Kurzweil-Stieltjes integrable functions is called uniformly fuzzy Henstock-Kurzweil-Stieltjes-◇-integrable on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ with respect to increasing functions g_1 and g_2 if for each $\varepsilon>0$ there exists a $\delta(t,s)$ such that

$$D\left(S(P,F_n(t,s),g_1,g_2),\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)<\varepsilon$$

for any partition P and for any $n\in\mathbb{N}$.

Theorem 3.9. *Let* $F_n(t,s)\in\mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$, $n=1,2,...$, *satisfy:*
(i) $\lim_{n\to\infty}F_n(t,s)=F(t,s)$ *on* $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$;
(ii) $F_n(t,s)$ *are uniformly fuzzy Henstock-Kurzweil-Stieltjes-◇-integrable on* $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$.
Then $F(t,s)\in\mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$ *and*

$$\lim_{n\to\infty}\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)=\int\int_{\mathcal{R}}F(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s).$$

Proof. Let $\varepsilon>0$. There exists a $\delta(t,s)$ such that

$$D\left(S(P,F_n(t,s),g_1,g_2),\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)<\varepsilon$$

for any partition P and for every n. Fix a $P_0\in[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. From above statement of Theorem 3.6, there exists N such that $P_0=P_1\times P_2$ where P_1 is a δ_1 (or γ_1) fine partition of $[a,b)_{\mathbb{T}_1}$ and P_2 is a δ_2 (or γ_2) fine partition of $[c,d)_{\mathbb{T}_2}$ and P_0 is a δ_0 fine partition of $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. Then

$$D\left(S(P_{\delta_0},F_n(t,s),g_1,g_2),S(P_{\delta_0},F_m(t,s),g_1,g_2)\right)<\varepsilon$$

for arbitrary $n,m>N$. Then,

$$\begin{aligned}
& D\left(\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s),\int\int_{\mathcal{R}}F_m(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)\\
\leqslant\ & D\left(S(P_{\delta_0},F_n(t,s),g_1,g_2),\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)\\
+\ & D\left(S(P_{\delta_0},F_n(t,s),g_1,g_2),S(P_{\delta_0},F_m(t,s),g_1,g_2)\right)\\
+\ & D\left(S(P_{\delta_0},F_m(t,s),g_1,g_2),\int\int_{\mathcal{R}}F_m(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)\\
<\ & 3\varepsilon
\end{aligned}$$

for any $n, m > N$ and, hence, $\{\int\int_{\mathcal{R}} F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty}\int\int_{\mathcal{R}} F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s) = \tilde{L}.$$

We now prove that

$$\int\int_{\mathcal{R}} F(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s) = \tilde{L}.$$

Let $\varepsilon > 0$. By hypothesis, there exists a $\delta(t,s)$ such that

$$D\left(S(P, F_n(t,s), g_1, g_2), \int\int_{\mathcal{R}} F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right) < \varepsilon$$

for any partition P and for all n. Choose N satisfying

$$D\left(\int\int_{\mathcal{R}} F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s), \tilde{L}\right) < \varepsilon$$

for all $n > N$. Let there exists $N_0 > N$ satisfying

$$D\left(S(P_\delta, F_{N_0}(t,s), g_1, g_2), S(P_\delta, F(t,s), g_1, g_2)\right) < \varepsilon.$$

Therefore,

$$\begin{aligned}
& D\left(S(P, F(t,s), g_1, g_2), \tilde{L}\right) \\
\leqslant\ & D\left(S(P_\delta, F(t,s), g_1, g_2), S(P_\delta, F_{N_0}(t,s), g_1, g_2)\right) \\
+\ & D\left(S(P_\delta, F_{N_0}(t,s), g_1, g_2), \int\int_{\mathcal{R}} F_{N_0}(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right) \\
+\ & D\left(\int\int_{\mathcal{R}} F_{N_0}(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s), \tilde{L}\right) \\
<\ & 3\varepsilon
\end{aligned}$$

and the result follows. □

Definition 3.10. ([12]) A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is called absolutely continuous on $[a,b]_{\mathbb{T}}$, if for each $\varepsilon > 0$ there exists $\gamma > 0$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \varepsilon$$

whenever $\bigcup_{i=1}^{n}[x_{i-1}, x_i]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$ and $\sum_{i=1}^{n}\Delta x_i < \gamma$

The following theorem is Dominated convergence theorem.

Theorem 3.11. *Let* $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to f_{\mathbb{R}}$ *be a fuzzy-number-valued function and let* g_1 *and* g_2 *be non-decreasing functions respectively on* $[a,b]_{\mathbb{T}_1}$ *and* $[c,d]_{\mathbb{T}_2}$. *If* $F_n(t,s) \in \mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$, $n = 1, 2, \dots$, *satisfy:*

(i) $\lim_{n\to\infty} F_n(t,s) = F(t,s)$ *a.e.;*

(ii) $G(t,s) \leqslant F_n(t,s) \leqslant H(t,s)$ *a.e., and* $G(t,s), H(t,s) \in \mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$*;*
then sequence $\{F_n(t,s)\}$ *is uniformly FHKS-*$\diamond$*-integrable. Thus,* $F(t,s) \in \mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$ *and*

$$\lim_{n\to\infty}\int\int_{\mathcal{R}} F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s) = \int\int_{\mathcal{R}} F(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s).$$

Proof. Let $\varepsilon > 0$ and for arbitrary $p, q > N$, we have

$$\begin{aligned} D\left(F_p(t,s), F_q(t,s)\right) &= \sup_{\lambda\in[0,1]} \max\left\{|\underline{F_p(t,s)^\lambda} - \underline{F_q(t,s)^\lambda}|, |\overline{F_p(t,s)^\lambda} - \overline{F_q(t,s)^\lambda}|\right\} \\ &\leqslant \sup_{\lambda\in[0,1]} \max\left\{|\underline{H(t,s)^\lambda} - \underline{G(t,s)^\lambda}|, |\overline{H(t,s)^\lambda} - \overline{G(t,s)^\lambda}|\right\} \\ &= D\left(H(t,s), G(t,s)\right). \end{aligned}$$

Then, $D\left(H(t,s), G(t,s)\right)$ is Lebesgue-$\diamond$-integrable. Let

$$D(t,s) = \int\int_{\mathcal{R}} D\left(H(x,y), G(x,y)\right)\diamond_1 g_1(x)\diamond_2 g_2(y).$$

From Definition 3.3, $D(t,s)$ is absolutely continuous on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. Let $\varepsilon > 0$. Then there exists $\gamma > 0$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k}|D(\xi_i,\zeta_j) - D(\xi_{i-1},\zeta_{j-1})| < \frac{\varepsilon}{(b-a)(d-c)}$$

whenever $\bigcup_{i=1}^{n}[\xi_{i-1},\xi_i]_{\mathbb{T}_1} \subset [a,b]_{\mathbb{T}_1}$ and $\bigcup_{j=1}^{k}[\zeta_{j-1},\zeta_j]_{\mathbb{T}_2} \subset [c,d]_{\mathbb{T}_2}$ and $\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1})) < \gamma$. The limit

$$\lim_{n\to\infty} F_n(t,s) = F(t,s)$$

holds almost everywhere on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ and $\{D\left(F_n(t,s), F(t,s)\right)\}$ is a sequence of $\diamond$-measurable functions. By Egorov's theorem, there is an open set Ω with $m(\Omega) < \delta$ such that $\lim_{n\to\infty} F_n(t,s) = F(t,s)$ uniformly for $t,s \in [a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\backslash\Omega$. Thus, there exists N such that

$$D\left(F_p(t,s), F_q(t,s)\right) < \frac{\varepsilon}{(b-a)(d-c)}$$

for any $p, q > N$ and for any $t,s \in [a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\backslash\Omega$.
Suppose that δ_1 is in the set of $\diamond$-gauge on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ such that

$$\left|S(D(H(t,s), G(t,s), P_{\delta_1}, g_1, g_2) - \int\int_{\mathcal{R}} D\left(H(t,s), G(t,s)\right)\diamond_1 g_1(t)\diamond_2 g_2(s)\right| < \varepsilon$$

and

$$D\left(S(P_{\delta_1}, F_n(t,s), g_1, g_2), \int\int_{\mathcal{R}} F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right) < \varepsilon$$

for $1 \leqslant n \leqslant N$ and for any partition P. Define $\delta(\xi,\zeta)$ in the set of $\diamond$-gauge on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ as

$$\delta(\xi,\zeta) = \begin{cases} \delta_1(\xi,\zeta), & \text{if } \xi,\zeta \in [a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\backslash\Omega \\ \min\{\delta_1(\xi,\zeta),\ \rho((\xi,\zeta),\Omega)\}, & \text{if } (\xi,\zeta) \in \Omega, \end{cases}$$

where $\rho((\xi,\zeta),\Omega)=\inf\{|(\xi,\zeta)-(\xi',\zeta')|:(\xi',\zeta')\in\Omega\}$. Fix $n>N$. We have

$$\begin{aligned}
& D\left(S(P_\delta,F_n(t,s),g_1,g_2),S(P_\delta,F_N(t,s),g_1,g_2)\right)\\
= \; & D\left(\sum_{i=1}^{n}\sum_{j=1}^{k}F_n(\xi_i,\zeta_j)\diamond g_1(t)g_2(s),\sum_{i=1}^{n}\sum_{j=1}^{k}F_N(\xi_i,\zeta_j)\diamond g_1(t)g_2(s))\right)\\
+ \; & D\left(\sum_{\xi_i\in\Omega}\sum_{\zeta_j\in\Omega}F_n(\xi_i,\zeta_j)\diamond g_1(t)g_2(s),\sum_{\xi_i\in\Omega}\sum_{\zeta_j\in\Omega}F_N(\xi_i,\zeta_j)\diamond g_1(t)g_2(s))\right)\\
\leqslant \; & \varepsilon+\sum_{\xi_i\in\Omega}\sum_{\zeta_j\in\Omega}D\left(F_n(\xi_i,\zeta_j),F_N(\xi_i,\zeta_j)\right)\diamond g_1(t)g_2(s)\\
\leqslant \; & \varepsilon+|\sum_{\xi_i\in\Omega}\sum_{\zeta_j\in\Omega}D(H(\xi_i,\zeta_j),G(\xi_i,\zeta_j)\diamond g_1(t)g_2(s)\\
& -\int\int_{\Omega}D(H(t,s),G(t,s)\diamond g_1(t)g_2(s)|\\
+ \; & \left|\int\int_{\Omega}D(H(t,s),G(t,s)\diamond g_1(t)g_2(s)\right|\\
\leqslant \; & 3\varepsilon
\end{aligned}$$

for any partition P. Hence,

$$\begin{aligned}
& D\left(S(P,F_n(t,s),g_1,g_2),\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)\\
\leqslant \; & D\left(S(P,F_n(t,s),g_1,g_2),S(P,F_N(t,s),g_1,g_2)\right)\\
+ \; & D\left(S(P,F_N(t,s),g_1,g_2),\int\int_{\mathcal{R}}F_N(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)\\
+ \; & D\left(\int\int_{\mathcal{R}}F_N(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s),\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)\right)\\
\leqslant \; & 5\varepsilon,
\end{aligned}$$

which ends the proof of dominated convergence theorem for fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrable functions on time scales. □

We obtain the following monotone convergence theorem as a consequence of dominated convergence theorem. The monotone convergence theorem is stated as follows:

Theorem 3.12. *Let* $F:[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\to f_{\mathbb{R}}$ *be a fuzzy-number-valued function and let* g_1 *and* g_2 *be non-decreasing functions respectively on* $[a,b]_{\mathbb{T}_1}$ *and* $[c,d]_{\mathbb{T}_2}$. *If* $F_n(t,s)\in\mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$, $n=1,2,...$, *satisfy:*

(i) $\lim_{n\to\infty}F_n(t,s)=F(t,s)\diamond$ *a.e.;*
(ii) $\{F_n(t,s)\}$ *is a monotone sequence and* $F_n(t,s)\in\mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$*;*
then sequence $\{F_n(t,s)\}$ *is uniformly FHKS-◇-integrable. Consequently,* $F(t,s)\in\mathcal{FHKS}_{[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}}$ *and*

$$\lim_{n\to\infty}\int\int_{\mathcal{R}}F_n(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s)=\int\int_{\mathcal{R}}F(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s).$$

4. Conclusion

This paper exposed some of the properties of fuzzy Henstock-Kurzweil-Stietljes-◇-double integral on time scales. Uniform convergence theorem and dominated convergence theorem are proved for the fuzzy Henstock-Kurzweil-Stieltjes-◇-double integrable functions on time scales. We also obtained the monotone convergence theorem as a consequence of dominated convergence theorem.

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