Implicit fractional differential equation with nonlocal integral-multipoint boundary conditions in the frame of Hilfer fractional derivative

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Abstract

This article deals with a nonlinear implicit fractional differential equation with nonlocal integral-multipoint boundary conditions in the frame of Hilfer fractional derivative. The existence and uniqueness results are obtained by using the fixed point theorems of Krasnoselskii and Banach. Further, to demonstrate the effectiveness of the main results, suitable examples are granted.

Keywords: fractional differential equations, Hilfer fractional derivative, fixed point theorems.
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1. Introduction:

Fractional differential equations (FDEs) with initial/boundary conditions emerge from a variety of applications inclusive in diverse fields of science and engineering, e.g., practical problems concerning mechanics, conservative systems, economy, control systems, chemistry, physics, harmonic oscillator, biology, atomic energy, medicine, information theory, nonlinear oscillations, the engineering technique fields, dynamics in Hamiltonian systems, stability and instability of geodesic on Riemannian manifolds, etc. This is because FDEs characterize many real-world processes linked to memory and hereditary properties of different materials more carefully as compared to classical order differential equations. For further details [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

There are sundry definitions of fractional calculus (FC), from the most common of them Riemann-Liouville (RL) (and Caputo) fractional derivatives (FDs) to other less-known definitions such as Erdelyi-Kober (and Hadamard) FDs and so on. A generalization

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of FDs of RL and Caputo was given by R. Hilfer in [13], which so-called the Hilfer FD of order σ_1 and a type $\sigma_2 \in [0,1]$. RL and Caputo FDs can get by giving $\sigma_2 = 0$ and $\sigma_2 = 1$ respectively in the formula of Hilfer FD. Such a derivative interpolates between the RL and Caputo FDs. More details on this FD mentioned above can be found in [14, 15] and references cited therein.

Besides the extensive development of FDs, several articles have been concerned with the existence and uniqueness of solutions for FDEs [16, 17, 18, 19, 20, 21, 22] and the references contained therein.

Hilfer FD of initial value problems (IVPs) were studied by different authors, see [23, 24]. However, there are some papers on boundary value problems (BVPs) of Hilfer FD. In [25] the authors initiate the study of nonlinear BVPs of Hilfer FD. For some more new works on BVPs with Hilfer FD can be seen in [26, 27, 28].

Motivated by the above works, in this article, we investigate the existence and uniqueness of solutions for an implicit FDE with nonlocal integral-multipoint boundary conditions in the frame of Hilfer FD of the form:

$$\begin{cases} {}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(s) = g(s, \varkappa(s), {}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(s)), & s \in \mathfrak{J} := [a, b], \\ \varkappa(a) = 0, \ \int_a^b \varkappa(\tau)d\tau + \lambda = \sum_{j=1}^{m-2} \xi_j \varkappa(\vartheta_j), \end{cases} \tag{1.1}$$

where ${}_H\mathbb{D}^{\sigma_1,\sigma_2}$ is the Hilfer FD of order σ_1 $(1 < \sigma_1 < 2)$, and parameter σ_2 $(0 \leqslant \sigma_2 \leqslant 1)$, $g : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a < \vartheta_1 < \vartheta_2 < \cdots < \vartheta_{m-2} < b$, $a \geqslant 0$, and $\xi_j, \vartheta_j \in \mathbb{R}$, $j = 1, 2, \cdots, m-2$.

We give attention to the subject of nonlocal problems, because in many cases a nonlocal condition in this type of problem reflects physical phenomena more exactly than classical boundary conditions. We prove the existence and uniqueness results by applying classical fixed point techniques. Here, we use Banach's fixed point approach to get the uniqueness result. Whereas Krasnosel'skii's fixed point theorem [29] is used to get the existence results for the problem (1.1). The work completed in this article fresh and enriches the literature on BVPs of Hilfer-type FDEs.

The present article orderly as follows: Sect. 2 some notations are presented and we give some concepts of preliminaries about Hilfer FD. Our main results for the problem (1.1) are given in Sect. 3. At the final, some examples are constructed to explain the applicability of the proved results.

2. Preliminaries:

In this portion, we provide some preliminary facts of FC which will be used throughout this article, see [11, 12].

Definition 2.1. The left sided RL fractional integral of order σ_1 is given by

$$\mathbb{I}^{\sigma_1}\nu(s) = \frac{1}{\Gamma(\sigma_1)} \int_a^s (s-\tau)^{\sigma_1-1}\nu(\tau)d\tau,$$

where $\Gamma(\cdot)$ denotes the Gamma function, $\sigma_1 > 0$ and ν be a locally integrable function on $(a, +\infty)$.

Definition 2.2. The left sided RL FD of order σ_1 is defined by

$$^{RL}\mathbb{D}^{\sigma_1}v(s) := \mathbb{D}^n \, \mathbb{I}^{n-\sigma_1}v(s),$$

where $n = [\sigma_1] + 1$, $[\sigma_1]$ symbolize the integer part of the real number σ_1, v is a continuous function, and $\mathbb{D}^n = \left(\frac{d}{ds}\right)^n$.

Definition 2.3. The Caputo FD of order $\sigma_1 > 0$ of a continuous function v is defined as

$$^{C}\mathbb{D}^{\sigma_1}v(s) := \mathbb{I}^{n-\sigma_1}\mathbb{D}^n v(s).$$

where $n = [\sigma_1] + 1$.

Definition 2.4. The Hilfer FD of order σ_1 and parameter σ_2 is defined by

$$_H\mathbb{D}^{\sigma_1,\sigma_2}v(s) = \mathbb{I}^{\sigma_2(n-\sigma_1)}\mathbb{D}^n \, \mathbb{I}^{(1-\sigma_2)(n-\sigma_1)}v(s),$$

where $n - 1 < \sigma_1 < n$, $0 \leqslant \sigma_2 \leqslant 1$, $s > a$.

Remark 2.5. In Definition 2.4, type σ_2 allows $\mathbb{D}^{\sigma_1,\sigma_2}$ to interpolate continuously between the classical RL FD and Caputo FD. When $\sigma_2 = 0$ the Hilfer FD corresponds to the RL FD, i.e. $_H\mathbb{D}^{\sigma_1,0}v(s) = \mathbb{D}^n \, \mathbb{I}^{n-\sigma_1}v(s)$, whereas when $\sigma_2 = 1$ the Hilfer FD corresponds to the Caputo FD, i.e., $_H\mathbb{D}^{\sigma_1,1}v(s) = \mathbb{I}^{n-\sigma_1}\mathbb{D}^n v(s)$.

Lemma 2.6. *[15] Let* $g \in L(a,b)$, $\sigma_1 \in (n-1,n]$ $(n \in \mathbb{N})$, $\sigma_2 \in [0,1]$. *If* $\mathbb{I}^{(1-\sigma_2)(n-\sigma_1)}g \in AC^k(\mathfrak{J})$, *then*

$$\left(\mathbb{I}^{\sigma_1} \, _H\mathbb{D}^{\sigma_1,\sigma_2}g\right)(s) = g(s) - \sum_{k=0}^{n} \frac{(s-a)^{k-(n-\sigma_1)(1-\sigma_2)}}{\Gamma(k-(n-\sigma_1)(1-\sigma_2)+1)} \lim_{s \to a^+} \left(\mathbb{I}^{(1-\sigma_2)(n-\sigma_1)}g\right)(s).$$

Let $C(\mathfrak{J},\mathbb{R})$ and $L(\mathfrak{J},\mathbb{R})$ are the Banach space of continuous functions and Lebesgue integrable functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norms

$$\|g\| = \sup\{|g| : s \in \mathfrak{J}\}, \text{ and}$$

$$\|g\|_L = \int_a^b |g(s)| \, ds,$$

respectively.

Here we can suffice to refer to Banach's fixed point theorem [30] and Krasnoselskii's fixed point theorem [30].

3. Main results:

The next lemma deals with a linear variant of the BVP (1.1).

Lemma 3.1. *[26] Let* $1 < \sigma_1 < 2$, *and* $0 \leqslant \sigma_2 \leqslant 1$, *where* $\gamma = \sigma_1 + 2\sigma_2 - \sigma_1\sigma_2$, $w \in C(\mathfrak{J},\mathbb{R})$. *If*

$$\Upsilon = \frac{(b-a)^\gamma}{\gamma} - \sum_{j=1}^{m-2} \xi_j(\vartheta_j - a)^{\gamma-1} \neq 0, \tag{3.1}$$

then the function $\varkappa \in C(\mathfrak{J}, \mathbb{R})$ *is a solution of the FDE*

$$_H \mathbb{D}^{\sigma_1, \sigma_2} \varkappa(s) = w(s), \quad s \in \mathfrak{J}, \tag{3.2}$$

with nonlocal integro-multipoint boundary conditions

$$\varkappa(a) = 0, \quad \int_a^b \varkappa(\tau)d\tau + \lambda = \sum_{j=1}^{m-2} \xi_j \varkappa(\vartheta_j), \tag{3.3}$$

if and only if

$$\varkappa(s) = \mathbb{I}^{\sigma_1} w(s) + \frac{(s-a)^{\gamma-1}}{\gamma} \left[\sum_{j=1}^{m-2} \xi_j \mathbb{I}^{\sigma_1} w(\vartheta_j) - \int_a^b \mathbb{I}^{\sigma_1} w(\tau)d\tau - \lambda \right].$$

In light of Lemma 3.1, we consider the operator $\Pi : C(\mathfrak{J}, \mathbb{R}) \to C(\mathfrak{J}, \mathbb{R})$ defined by

$$\begin{aligned}
(\Pi\varkappa)(s) &= \mathbb{I}^{\sigma_1} g(\tau, \varkappa(\tau), _H \mathbb{D}^{\sigma_1, \sigma_2} \varkappa(\tau))(s) + \frac{(s-a)^{\gamma-1}}{\gamma} \\
&\times \left[\sum_{j=1}^{m-2} \xi_j \mathbb{I}^{\sigma_1} g(\tau, \varkappa(\tau), _H \mathbb{D}^{\sigma_1, \sigma_2} \varkappa(\tau))(\vartheta_j) \right. \\
&\left. - \mathbb{I}^{\sigma_1+1} g(\tau, \varkappa(\tau), _H \mathbb{D}^{\sigma_1, \sigma_2} \varkappa(\tau))(b) - \lambda \right].
\end{aligned} \tag{3.4}$$

It must be noted that problem (1.1) has solution if and only if Π has fixed points. Next , for the aim of suitability, we put a constant

$$\Psi := \left[\frac{(b-a)^{\gamma-1}}{|\gamma|} \left(\sum_{j=1}^{m-2} |\xi_j| \frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} + \frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \right) + \frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} \right]. \tag{3.5}$$

In the following subsections, we prove the existence and uniqueness results for the BVP (1.1) by employing the standard fixed point theorems due to Banach and Krasnoselskii.

3.1. Existence and uniqueness results for (1.1)

First, we consider the following assumptions:

(H$_1$) There exist constant $0 < \ell < 1$ such that

$$|g(s, \varkappa_1, \varkappa_1^*) - g(s, \varkappa_2, \varkappa_2^*)| \leqslant \ell \left(|\varkappa_1 - \varkappa_2| + |\varkappa_1^* - \varkappa_2^*| \right),$$

for any $\varkappa_1, \varkappa_1^*, \varkappa_2, \varkappa_2^* \in \mathbb{R}$ and $s \in \mathfrak{J}$.

(H$_2$) Let $g \in C(\mathfrak{J} \times \mathbb{R}^2, \mathbb{R})$ and $\mathfrak{f} \in C(\mathfrak{J}, \mathbb{R}^+)$ such that

$$|g(s, \varkappa, \varkappa^*)| \leqslant \mathfrak{f}(s), \quad \forall (s, \varkappa, \varkappa^*) \in \mathfrak{J} \times \mathbb{R}^2.$$

Theorem 3.2. *Assume that* (H_1) *holds. If*

$$\frac{\ell}{1-\ell}\Psi < 1, \tag{3.6}$$

then the BVP (1.1) has a unique solution on $\mathfrak{J}$.

Proof. We transform the BVP (1.1) into a fixed point problem, i.e., $\varkappa = \Pi\varkappa$, where Π is defined by (3.4). Note that the fixed points of Π are solutions of the problem (1.1). Using Banach theorem [30], we will show that Π has a unique fixed point. Indeed, we set $\sup_{s\in\mathfrak{J}}|g(s,0,0)| = N < \infty$ and select

$$\epsilon \geqslant \frac{N\Psi + \frac{(1-\ell)(b-a)^{\gamma-1}|\lambda|}{|\Upsilon|}}{1-\ell-\ell\Psi}.$$

First, we prove that $\Pi\mathcal{B}_\epsilon \in \mathcal{B}_\epsilon$, where $\mathcal{B}_\epsilon = \{\varkappa \in C(\mathfrak{J},\mathbb{R}) : \|\varkappa\| \leqslant \epsilon\}$. By applying (H_1), we get

$$
\begin{aligned}
|g(s,\varkappa(s),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(s))| &\leqslant |g(s,\varkappa(s),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(s)) - g(s,0,0)| + |g(s,0,0)| \\
&\leqslant \ell|\varkappa(s)| + \ell|_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(s)| + N \\
&= \ell|\varkappa(s)| + \ell|g(s,\varkappa(s),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(s))| + N,
\end{aligned}
$$

which implies

$$|g(s,\varkappa(s),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(s))| \leqslant \frac{\ell}{1-\ell}|\varkappa(s)| + \frac{N}{1-\ell}.$$

For any $\varkappa \in \mathcal{B}_\epsilon$, we have

$$
\begin{aligned}
|(\Pi\varkappa)(s)| &\leqslant \sup_{s\in\mathfrak{J}}\Bigg\{\mathbb{I}^{\sigma_1}|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(s)| + \frac{(s-a)^{\gamma-1}}{|\Upsilon|} \\
&\quad \times \Bigg[\sum_{j=1}^{m-2}|\xi_j|\,\mathbb{I}^{\sigma_1}\big|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau)(\vartheta_j))\big| \\
&\quad +\mathbb{I}^{\sigma_1+1}|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(b)| + |\lambda|.\Bigg]\Bigg\} \\
&\leqslant \frac{\ell}{1-\ell}\mathbb{I}^{\sigma_1}\|\varkappa\| + \frac{N}{1-\ell}\mathbb{I}^{\sigma_1} + \frac{(b-a)^{\gamma-1}}{|\Upsilon|} \\
&\quad \times \Bigg[\sum_{j=1}^{m-2}\Big[|\xi_j|\frac{\ell}{1-\ell}\mathbb{I}^{\sigma_1}\|\varkappa\| + \frac{N}{1-\ell}\mathbb{I}^{\sigma_1}\Big] + \frac{\ell}{1-\ell}\mathbb{I}^{\sigma_1+1}\|\varkappa\| + \frac{N}{1-\ell}\mathbb{I}^{\sigma_1+1} + |\lambda|\Bigg] \\
&\leqslant \frac{\ell}{1-\ell}\|\varkappa\|\frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} + \frac{N}{1-\ell}\frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} + \frac{(b-a)^{\gamma-1}}{|\Upsilon|} \\
&\quad \times \Bigg[\sum_{j=1}^{m-2}\Big[|\xi_j|\frac{\ell}{1-\ell}\|\varkappa\|\frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} + \frac{N}{1-\ell}\frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}\Big] \\
&\quad +\frac{\ell}{1-\ell}\|\varkappa\|\frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)} + \frac{N}{1-\ell}\frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)} + |\lambda|\Bigg]
\end{aligned}
$$

$$\leqslant \left[\frac{(b-a)^{\gamma-1}}{|\Upsilon|}\left(\sum_{j=1}^{m-2}|\xi_j|\frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}+\frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\right)+\frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]\frac{\ell}{1-\ell}\|\varkappa\|$$

$$+\left[\frac{(b-a)^{\gamma-1}}{|\Upsilon|}\left(\sum_{j=1}^{m-2}\frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}+\frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\right)+\frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]\frac{N}{1-\ell}$$

$$+\frac{(b-a)^{\gamma-1}}{|\Upsilon|}|\lambda|$$

$$\leqslant \frac{\ell}{1-\ell}\Psi\epsilon+\frac{N}{1-\ell}\Psi+\frac{(b-a)^{\gamma-1}}{|\Upsilon|}|\lambda|\leqslant\epsilon,$$

which means that $\Pi\mathcal{B}_\epsilon\in\mathcal{B}_\epsilon$.

Next, we take $\varkappa,\varkappa^*\in\mathbb{R}$. Then for $s\in\mathfrak{J}$, we obtain

$$|(\Pi\varkappa)(s)-(\Pi\varkappa^*)(s)|$$

$$\leqslant \mathbb{I}^{\sigma_1}\left|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(s)-g(\tau,\varkappa^*(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa^*(\tau))(s)\right|+\frac{(b-a)^{\gamma-1}}{|\Upsilon|}$$

$$\times\sum_{j=1}^{m-2}\left|\xi_j\right|\mathbb{I}^{\sigma_1}\left|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau)(\vartheta_j))-g(\tau,\varkappa^*(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa^*(\tau)(\vartheta_j))\right|$$

$$+\frac{(b-a)^{\gamma-1}}{|\Upsilon|}\mathbb{I}^{\sigma_1+1}\left|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(b)-g(\tau,\varkappa^*(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa^*(\tau))(b)\right|.$$

Here, by (H_1) we have

$$|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(s)-g(\tau,\varkappa^*(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa^*(\tau))(s)|$$
$$\leqslant \ell\,|\varkappa-\varkappa^*|+\ell\,|_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa-_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa^*|,$$

which gives

$$|g(\tau,\varkappa(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(s)-g(\tau,\varkappa^*(\tau),_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa^*(\tau))(s)|$$
$$\leqslant \frac{\ell}{1-\ell}|\varkappa-\varkappa^*|.$$

Consequently,

$$|(\Pi\varkappa)(s)-(\Pi\varkappa^*)(s)|$$

$$\leqslant \frac{\ell}{1-\ell}\mathbb{I}^{\sigma_1}\|\varkappa-\varkappa^*\|+\frac{(b-a)^{\gamma-1}}{|\Upsilon|}\sum_{j=1}^{m-2}|\xi_j|\frac{\ell}{1-\ell}\mathbb{I}^{\sigma_1}\|\varkappa-\varkappa^*\|$$

$$+\frac{(b-a)^{\gamma-1}}{|\Upsilon|}\frac{\ell}{1-\ell}\mathbb{I}^{\sigma_1+1}\|\varkappa-\varkappa^*\|$$

$$\leqslant \left[\frac{(b-a)^{\gamma-1}}{|\Upsilon|}\left(\sum_{j=1}^{m-2}|\xi_j|\frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}+\frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\right)+\frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]\frac{\ell}{1-\ell}\|\varkappa-\varkappa^*\|$$

$$\leqslant \frac{\ell}{1-\ell}\Psi\|\varkappa-\varkappa^*\|,$$

which means that $\|\Pi\varkappa - \Pi\varkappa^*\| \leqslant \frac{\ell}{1-\ell}\Psi\|\varkappa - \varkappa^*\|$. As (3.6) Π is a contraction. Therefore, Banach theorem [30] shows that Π has a fixed point which is the unique solution of the (BVP) (1.1). $\qquad\square$

Our second existence result for the problem (1.1) depends on the Krasnoselskii theorem [30].

Theorem 3.3. *Assume that* (H_2) *holds. If*

$$\Psi^* := \frac{(b-a)^{\gamma-1}}{|\Upsilon|}\left[\sum_{j=1}^{m-2}|\xi_j|\frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} + \frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)} + \frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} + |\lambda|\right] < 1, \quad (3.7)$$

then the BVP (1.1) *has at least one solution on* $\mathfrak{J}$.

Proof. Putting $\sup_{s\in\mathfrak{J}}|f(s)| = \|f\|$, and choosing

$$\sigma \geqslant \Psi\|f\| + \frac{(b-a)^{\gamma-1}}{|\Upsilon|}|\lambda|, \qquad (3.8)$$

where Ψ is defined by (3.5). Consider the ball $\mathcal{B}_\sigma = \{\varkappa \in C(\mathfrak{J},\mathbb{R}) : \|\varkappa\| \leqslant \sigma\}$. Then we define the operators Π_1, Π_2 on $\mathcal{B}_\sigma$ by

$$(\Pi_1\varkappa)(s) = \mathbb{I}^{\sigma_1}g(\tau,\varkappa(\tau),{}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(s), \quad s \in \mathfrak{J},$$

and

$$\begin{aligned}(\Pi_2\varkappa)(s) &= \frac{(s-a)^{\gamma-1}}{\Upsilon}\left[\sum_{j=1}^{m-2}\xi_j\mathbb{I}^{\sigma_1}g(\tau,\varkappa(\tau),{}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau)(\vartheta_j))\right.\\&\quad\left.-\mathbb{I}^{\sigma_1+1}g(\tau,\varkappa(\tau),{}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(b) - |\lambda|\right].\end{aligned}$$

For any $\varkappa,\varkappa^* \in \mathcal{B}_\sigma$, we obtain

$$\begin{aligned}&|(\Pi_1\varkappa)(s) + (\Pi_2\varkappa^*)(s)|\\&\leqslant \sup_{s\in\mathfrak{J}}\left\{\mathbb{I}^{\sigma_1}|g(\tau,\varkappa(\tau),{}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(s)| + \frac{(s-a)^{\gamma-1}}{|\Upsilon|}\right.\\&\quad\times\left[\sum_{j=1}^{m-2}|\xi_j|\,\mathbb{I}^{\sigma_1}\left|g(\tau,\varkappa(\tau),{}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau)(\vartheta_j))\right|\right.\\&\quad\left.\left.+\mathbb{I}^{\sigma_1+1}|g(\tau,\varkappa(\tau),{}_H\mathbb{D}^{\sigma_1,\sigma_2}\varkappa(\tau))(b)| + |\lambda|\right]\right\}\\&\leqslant \left[\frac{(b-a)^{\gamma-1}}{|\Upsilon|}\left(\sum_{j=1}^{m-2}|\xi_j|\frac{(\vartheta_j-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} + \frac{(b-a)^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\right) + \frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]\|f\|\\&\quad+\frac{(b-a)^{\gamma-1}}{|\Upsilon|}|\lambda|\\&= \Psi\|f\| + \frac{(b-a)^{\gamma-1}}{|\Upsilon|}|\lambda| \leqslant \sigma.\end{aligned}$$

This proves that $\Pi_1 \varkappa + \Pi_2 \varkappa^* \in \mathcal{B}_\sigma$. By using (3.7) one can observe that Π_2 is a contraction map.

The continuity of g gives that Π_1 is continuous. Also, Π_1 is uniformly bounded on $\mathcal{B}_\sigma$ as

$$\|\Pi_1 \varkappa\| \leqslant \frac{(b-a)^{\sigma_1}}{\Gamma(\sigma_1+1)} \|f\|.$$

Now, we show the compactness of the operator Π_1.

We define $\sup_{(s,\varkappa) \in (\mathfrak{J}, \mathcal{B}_\sigma)} |g(s, \varkappa, \varkappa^*)| = g^* < \infty$, and let $s_1, s_2 \in \mathfrak{J}$ such that $s_1 < s_2$. Consequently,

$$
\begin{aligned}
&|(\Pi_1 \varkappa)(s_2) - (\Pi_1 \varkappa^*)(s_1)| \\
&\leqslant \frac{1}{\Gamma(\sigma_1)} \left| \int_a^{s_1} [(s_2-\tau)^{\sigma_1-1} - (s_1-\tau)^{\sigma_1-1}] g(\tau, \varkappa(\tau), {}_H \mathbb{D}^{\sigma_1,\sigma_2} \varkappa(\tau)) d\tau \right. \\
&\quad \left. + \int_{s_1}^{s_2} (s_2-\tau)^{\sigma_1-1} g(\tau, \varkappa(\tau), {}_H \mathbb{D}^{\sigma_1,\sigma_2} \varkappa(\tau)) d\tau \right| \\
&\leqslant \frac{g^*}{\Gamma(\sigma_1+1)} [2(s_2-s_1)^{\sigma_1} + |(s_2-a)^{\sigma_1} - (s_1-a)^{\sigma_1}|].
\end{aligned}
$$

The last inequality with $s_2 - s_1 \to 0$, gives

$$|(\Pi_1 \varkappa)(s_2) - (\Pi_1 \varkappa^*)(s_1)| \to 0, \ \forall \ |s_2 - s_1| \to 0, \ \varkappa \in \mathcal{B}_\sigma.$$

So Π_1 is relatively compact on $\mathcal{B}_\sigma$. An application of the Arzel-Ascoli theorem, Π_1 is compact on $\mathcal{B}_\sigma$. Hence, all the assumptions of Krasnoselskii theorem [30] are satisfied. So, we deduce that the problem (1.1) has at least one solution on $\mathfrak{J}$. $\qquad\square$

4. Examples:

Example 4.1. Consider the BVP of Hilfer-type implicit FDE

$$
{}_H\mathbb{D}^{\frac{5}{3},\frac{2}{3}} \varkappa(s) = \begin{cases} \frac{8}{3(87+8s)} \left(\frac{\varkappa^2(s)+2|\varkappa(s)|}{1+|\varkappa(s)|} + \sin\left({}_H\mathbb{D}^{\frac{5}{3},\frac{2}{3}}\varkappa(s)\right) \right) + \frac{4}{3}, \ s \in [0,1], \\ \varkappa(0) = 0, \ \int_0^1 \varkappa(\tau) d\tau + \frac{4}{3} = \frac{3}{4}\varkappa(\frac{4}{9}) + \frac{4}{5}\varkappa(\frac{5}{9}), \end{cases} \tag{4.1}
$$

Here $\sigma_1 = \frac{5}{3}$, $\sigma_2 = \frac{2}{3}$, $\xi_1 = \frac{3}{4}$, $\xi_2 = \frac{4}{5}$, $\vartheta_1 = \frac{4}{9}$, $\vartheta_2 = \frac{5}{9}$, $\lambda = \frac{3}{2}$, $a = 0$ and $b = 1$. From these settings, we compute constants as $\gamma = 1.8889$, $\Upsilon = -0.30979$, $\Psi = 1.7255$. Let

$$g(s, \varkappa, {}_H\mathbb{D}^{\frac{5}{3},\frac{2}{3}}\varkappa) = \frac{8}{3(87+8s)} \left(\frac{\varkappa^2(s)+2|\varkappa(s)|}{1+|\varkappa(s)|} + \sin\left({}_H\mathbb{D}^{\frac{5}{3},\frac{2}{3}}\varkappa(s)\right) \right) + \frac{4}{3}.$$

Then, for each $s \in [0,1]$, $\varkappa, \varkappa^* \in \mathbb{R}$

$$\left| g(s, \varkappa, {}_H\mathbb{D}^{\frac{3}{2},\frac{1}{2}}\varkappa) - g(s, \varkappa^*, {}_H\mathbb{D}^{\frac{3}{2},\frac{1}{2}}\varkappa^*) \right| \leqslant \frac{1}{11} \left(|\varkappa - \varkappa^*| + \left| {}_H\mathbb{D}^{\frac{3}{2},\frac{1}{2}}\varkappa - {}_H\mathbb{D}^{\frac{3}{2},\frac{1}{2}}\varkappa^* \right| \right).$$

Hence, (H$_1$) holds with $\ell = \frac{1}{11}$. Also, the condition (3.6) is fulfilled, i.e., $\frac{\ell}{1-\ell}\Psi \approx 0.17255 < 1$. Therefore, by the applying of Theorem 3.2, the problem (4.1) has a unique solution $\varkappa(s)$ on $[0,1]$.

Example 4.2. Consider the BVP of Hilfer-type implicit FDE

$$
{}_H\mathbb{D}^{\frac{7}{2},\frac{1}{3}}\varkappa(s) = \begin{cases} \frac{3N}{2(3s+2)}\left(\frac{|\varkappa(s)|}{1+|\varkappa(s)|} + \tan^{-1}({}_H\mathbb{D}^{\frac{7}{2},\frac{1}{3}}\varkappa(s))\right) + \frac{1}{2}, \ s \in [0, \frac{1}{5}], \\ \varkappa(0) = 0, \ \int_0^{\frac{1}{5}} \varkappa(\tau)d\tau + \frac{1}{5} = \frac{1}{3}\varkappa(\frac{1}{2}) + \frac{2}{3}\varkappa(\frac{1}{3}) + \frac{3}{4}\varkappa(\frac{1}{4}). \end{cases} \tag{4.2}
$$

Here $\sigma_1 = \frac{7}{2}$, $\sigma_2 = \frac{1}{3}$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{2}{3}$, $\xi_3 = \frac{3}{4}$, $\vartheta_1 = \frac{1}{2}$, $\vartheta_2 = \frac{1}{3}$, $\vartheta_3 = \frac{1}{4}$, $\lambda = \frac{1}{5}$, $a = 0$, $b = \frac{1}{5}$, and N is a given constant. For all $(s, \varkappa, \varkappa^*) \in \mathfrak{J} \times \mathbb{R}^2$, we have

$$
\begin{aligned}
\left| g(s, \varkappa, {}_H\mathbb{D}^{\frac{4}{3},\frac{1}{4}}\varkappa) \right| &\leqslant \frac{3N}{2(3s+2)}\left(\frac{|\varkappa(s)|}{1+|\varkappa(s)|} + \tan^{-1}({}_H\mathbb{D}^{\frac{7}{2},\frac{1}{3}}\varkappa(s))\right) + \frac{1}{2} \\
&\leqslant \frac{3N(2+\pi)}{4(3s+2)} + \frac{1}{2}.
\end{aligned}
$$

Hence (H$_2$) holds with $\mathfrak{f}(s) \in C(\mathfrak{J}, \mathbb{R}^+)$. Next, we can find that $\gamma = 4.6667$, $\Upsilon = -0.0042651 \neq 0$. Since $\Psi^* := 0.13644 < 1$, the condition (3.7) is fulfilled. Therefore, by the applying of Theorem 3.3, the problem (4.2) has has at least one solution on $[0, \frac{1}{5}]$.

5. Conclusions

In this article, we have studied a kind of nonlinear implicit FDEs with nonlocal integral-multipoint boundary conditions in the frame of Hilfer FD. The existence and uniqueness results are proved by using some fixed point theorems of Banach and Krasnoselskii.

In future work, we are thinking about investigating the existence and stability of solutions for the proposed problem (1.1) involving a generalized fractional derivative with respect to another function.

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