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Bernstein polynomial induced two step hybrid numerical scheme for solution of second order initial value problems

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Abstract

This paper presents a two step hybrid numerical scheme with one off-grid point for numerical solution of general second order initial value problems without reducing to two systems of first order. The scheme is developed using collocation and interpolation technique invoked on Bernstein polynomial. The proposed scheme is consistent, zero stable and is of order four(4). The developed scheme can estimate the approximate solutions at both step and off step points simultaneously using variable step size. Numerical results obtained in this paper shows the efficiency of the proposed scheme over some existing methods of same and higher orders.

Keywords: Bernstein polynomial, Hybrid block method, Collocation, Interpolation, Zero Stability.
2010 MSC: 65L05, 65L06.

1. Introduction

Differential equations are important tools in solving real world problems and many physical phenomena are model into second order differential equations, such models may or may not have exact solutions, thus a need for a numerical solution.

In this paper, we consider a second order initial value problem of the form

$$y'' = f(x, y(x), y'(x)), \tag{1.1}$$

In order to solve equation (1.1), the conditions stated below need to be imposed

$$y(x_0) = y_0, \;\; y'(x_0) = y'_0 \tag{1.2}$$

where $a \leqslant x \leqslant b$, $a = x_0 < x_1 < x_2 < \cdots, < x_{N-1}$, $N = \frac{b-a}{h}$, $N = 0, 1, \cdots, N-1$ and $h = x_{n+1} - x_n$ is called the step length. where y_0 is the solution at x_0 and x_0 is the initial

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point , f is a continuous function within the interval of integration, the condition on the function f are such that existence and uniqueness of solution is guaranteed(Wend[1]), and prime indicates differentiation with respect to x, while y(x) is the unknown function to be determined.

The numerical solution of equation (1.1) coupled with equation (1.2) is still receiving a lot of attention due to the fact that many physical sciences and engineering problems formulated into mathematical equation result to equation of such type.

In most applications, equation (1.1&1.2) are solved by reducing it to a system of first order ordinary differential equations and appropriate numerical method(such as Runge Kutta mehod, Modified Euler method, e.t.c) could be used to solve the resultant system, this approach has setbacks which had been reported by scholars, among them are Awoyemi et al[2] and Bun and Vasil'yer[3].

Direct method of solving equation (1.1) has been shown to be more efficient and saves computational time rather than method of reduction to system of first order ordinary differential equation (Brown [4]) and this has led to many scholars to attempt to solve equation (1.1) directly without reduction to system of first order equation. Brown [4] proposed a multi-derivative method to solve equation (1.1&1.2) directly. Adeniran and Ogundare [5] propose a one step hybrid numerical scheme with two off grid points for solving directly second over order initial value problems, the scheme can estimate the approximate solution at both step and off step points simultaneously by using variable step size.

Adeniran, Odejide and Ogundare [6] developed a one step hybrid numerical scheme for the direct solution of general second order ordinary differential equations, the scheme was developed using the collocation and interpolation techniques on the power series approximate solution and augmented by the introduction of one offstep point, in order to circumvent Dahlquist zero stability barrier and upgrade the order of consistency of the method. Accuracy of the scheme was tested with numerical examples and the result shows a better performance over the existing schemes.

In recent years, the Bernstein polynomials have gained the attention of many researchers. It has been used to obtained approximate solutions of different differential equations, for example, a method for approximating solutions to differential equations, proposed by Bhatti used Bernstein [7] operational matrix of differentiation. Ojo and Okoro [8] use a Bernstein polynomial to develop one step hybrid scheme with one off-grid point via collocation and interpolation techniques for the direct solution of general second order ordinary differential equations. The paper extend the work of Ojo and Okoro [8] by developing a two step hybrid method for solution of equation 1.1&1.2.

2. Berstein Polynomial

Aysegul and Nese [9] define Berstein Polynomial of degree m on interval $[0, 1]$ as

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}$$

there are $(m+1)nth$ degree of Berstein Polynomial, for mathematically convenience, we usually set $B_{i,m} = 0$, if $i < 0$ or $i > 0$. In general, the approximation of any function $y(x)$ with the first $(m+1)$ Berstein Polynomial as

$$y(x) = \sum_{i=0}^{m} c_i B_{i,m}(x)$$

3. Development of the method

We seek numerical approximation of the analytic solution $y(x)$ by assuming an approximate solution of the form

$$y(x) = \sum_{k=0}^{c+i-1} a_k B_{k,n}(x) \tag{3.1}$$

where c and i are number of distinct collocation and interpolation points respectively and $B_{k,n}(x)$ is the Bernstein Polynomial derived from the recursive relation

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x) \tag{3.2}$$

Differentiating equation (3.1) twice and substituting into equation (1.1) gives:

$$f(x, y(x), y'(x)) = \sum_{k=0}^{c+i-1} a_k B''_{k,n}(x) \tag{3.3}$$

We consider a grid point of step length two(2) and off step point at $x = x_{n+\frac{3}{2}}$. Collocating (5) at $x = x_n$, x_{n+1}, $x_{n+\frac{3}{2}}$ and x_{n+2} and interpolating (3) at $x = x_n$ and $x_{n+\frac{3}{2}}$ give a system of six equations which are solved using Gaussian elimination method to obtained the parameters $a_j's$, $j = 0, 1, \cdots, 5$. The parameter $a_j's$ obtained are then substituted back into equation (3.3) to give a continuous two step hybrid method of the form

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+\frac{3}{2}} + h^2 \left[\beta_0 f_n + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \tag{3.4}$$

where α and β are continuous coefficients. The continuous method (3.4) is used to generate the main method. That is, we evaluate at $x = x_{n+1}$ and $x = x_{n+2}$

$$y_{n+1} = \frac{1}{3}y_n + \frac{2}{3}y_{n+\frac{2}{3}} - h^2\left[\frac{1}{36}f_n + \frac{13}{48}f_{n+1} - \frac{5}{72}f_{n+\frac{3}{2}} + \frac{1}{48}f_{n+2}\right] \tag{3.5}$$

$$y_{n+2} = -\frac{1}{3}y_n + \frac{4}{3}y_{n+\frac{2}{3}} + h^2\left[\frac{1}{36}f_n + \frac{7}{24}f_{n+1} + \frac{5}{36}f_{n+\frac{3}{2}} + \frac{1}{24}f_{n+2}\right] \tag{3.6}$$

In order to incorporate the initial condition at (1.2) in the derived schemes, we differentiate (6) with respect to x and evaluate at point $x = x_n$, $x = x_{n+1}$, $x = x_{n+\frac{3}{2}}$ and x_{n+2} to have:

$$hy'_n = -\frac{2}{3}y_n + \frac{2}{3}y_{n+\frac{2}{3}} - h^2\left[\frac{11}{40}f_n + \frac{60}{80}f_{n+1} - \frac{17}{40}f_{n+\frac{3}{2}} + \frac{9}{80}f_{n+2}\right] \tag{3.7}$$

$$hy'_{n+1} = -\frac{2}{3}y_n + \frac{2}{3}y_{n+\frac{2}{3}} + h^2\left[\frac{7}{120}f_n + \frac{90}{240}f_{n+1} - \frac{29}{120}f_{n+\frac{3}{2}} + \frac{13}{240}f_{n+2}\right] \tag{3.8}$$

$$hy'_{n+\frac{3}{2}} = -\frac{2}{3}y_n + \frac{2}{3}y_{n+\frac{2}{3}} + h^2\left[\frac{17}{320}f_n + \frac{99}{160}f_{n+1} + \frac{1}{20}f_{n+\frac{3}{2}} + \frac{9}{320}f_{n+2}\right] \tag{3.9}$$

$$hy'_{n+2} = -\frac{2}{3}y_n + \frac{2}{3}y_{n+\frac{2}{3}} + h^2\left[\frac{7}{120}f_n + \frac{131}{240}f_{n+1} + \frac{17}{40}f_{n+\frac{3}{2}} + \frac{53}{240}f_{n+2}\right] \tag{3.10}$$

Combining the schemes derived in equation $(3.5 - 3.10)$. The block method is employed to simultaneously obtain value for y_{n+1}, $y_{n+\frac{3}{2}}$, y_{n+2}, y'_{n+1}, $y'_{n+\frac{3}{2}}$ and y'_{n+2} needed to implement equation (1.1&1.2).

Definition 3.1. Let Y_m and F_m be defined by $Y_m = (y_n, y_{n+1}, \cdots, y_{n+r-1})^T$, $F_m = (f_n, f_{n+1}, \cdots, f_{n+r-1})^T$. Then a general k block, r-point block method is a matrix of finite difference equation of the form

$$Y_m = \sum_{i=1}^{k} A_i Y_{m-i} + h\sum_{i=0}^{k} B_i F_{m-i} \tag{3.11}$$

where all the A_i's and B_i's are properly chosen $r \times r$ matrix coefficient and $m = 0, 1, 2, \cdots$ represent the block number, $n = mr$ is the first step number of the mth block and r is the proposed block size. (Chu and Hamilton [10]).

In order to implement equation (3.5) to (3.10), we use a modified block method defined as follows:

$$h^\lambda \sum_{j=1}^{q} a_{ij} y_{n+j}^\lambda = h^\lambda \sum_{j=1}^{q} e_{ij} y_n^\lambda + h^{\mu-\lambda}\left[\sum_{j=1}^{q} d_{ij} f_n + \sum_{j=1}^{q} b_{ij} f_{n+j}\right], \tag{3.12}$$

where λ is the power of the derivative of the continuous method and μ is the order of the problem to be solved; q=r+s. In vector notation, (3.12) can be written as:

$$h^\lambda \overline{a} Y_m = h^\lambda \overline{e} y_m + h^{\mu-\lambda}\left[\overline{d} f(y_m) + \overline{b} F(Y_m)\right], \tag{3.13}$$

where the matrices $\overline{a} = (a_{ij})$, $\overline{e} = (e_{ij})$, $\overline{d} = (d_{ij})$ are constant coefficient matrices and $Y_m = (y_{n+v_i}, y_{n+1} y'_{n+v_i}, y'_{n+1})^T$, $y_m = (y_{n-(r-1)}, y_{n-(r-2)}, \cdots, y_n)$, $\overline{F}(Y_m = (f_{n+v_i}, f_j)^T$ and $F(y_m = f(f_{n-i}, \cdots, f_n)$, $i = 1, \cdots, q$. The normalized version of (3.13) is given by

$$\overline{A} Y_m = h^\lambda \overline{E} y_m + h^{\mu-\lambda}\left[\overline{D} f(y_m) + \overline{B} F(Y_m)\right]. \tag{3.14}$$

The modified block formulae (3.13) and (3.14) are employed to simultaneously obtain the values of y_{n+1}, $y_{n+\frac{3}{2}}$, y_{n+2}, y'_{n+1}, $y'_{n+\frac{3}{2}}$ and y'_{n+2} needed to implement (1.1&1.2). we obtain the block solution as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y'_{n+1} \\ y'_{n+\frac{3}{2}} \\ y'_{n+2} \end{pmatrix} = \begin{pmatrix} 1 & h \\ 1 & \frac{3}{2}h \\ 1 & 2h \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_n \\ y'_n \end{pmatrix} + \begin{pmatrix} \frac{89}{360}h^2 \\ \frac{33}{80}h^2 \\ \frac{26}{45}h^2 \\ \frac{1}{3}h \\ \frac{21}{64}h \\ \frac{1}{3}h \end{pmatrix} \begin{pmatrix} f_n \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{31}{60}h^2 & -\frac{16}{45}h^2 & \frac{11}{120}h^2 \\ \frac{189}{160}h^2 & -\frac{51}{80}h^2 & \frac{27}{160}h^2 \\ \frac{28}{15}h^2 & -\frac{32}{45}h^2 & \frac{4}{15}h^2 \\ \frac{7}{6}h & -\frac{2}{3}h & \frac{1}{6}h \\ \frac{45}{32}h & -\frac{3}{8}h & \frac{9}{64}h \\ \frac{4}{3}h & 0h & \frac{1}{3}h \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}.$$

4. Analysis of the method

we analyze the derived method which includes the order and error constant, Consistency zero stability, and convergence of the method.

4.1. *Order and error constant*

We adopted the method proposed by Fatunla [11] and Lambert [12] to obtain the order of our method as $(4,4,4,4,4,4)^T$ and error constant as $(-\frac{1}{160}, -\frac{1}{60}, -\frac{117}{10240}, -\frac{31}{2880}, -\frac{51}{5120}, -\frac{1}{90})^T$

4.2. *Consistency*

According to Gurjinder et al.[13] A linear multistep method is said to be consistent if it has an order of convergence, say $p \geqslant 1$. Thus, our derived methods are consistent, since all are of order four.

4.3. *Zero Stability*

To obtain the zero stability of the method, we consider the following conditions:

1. The block (3.12) is said to be stable if as $h \to 0$ the roots r_j, j=1(1)k of the first characteristics polynomial $\rho(R) = 0$, that is $\rho(R) = \det[\sum A^{(i)} R^{k-1}] = 0$, satisfy $|R| \leqslant 1$ and for those roots with $|R| \leqslant 1$, must have multiplicity equal to unity.(see Fatunla[11] for details).
2. If (3.12) be an $R \times R$ matrix then, it is zero stable if as $h^\mu \to 0$, $|RA^0 - A^i| = R^{r-\mu} = 0$. For those root with $|R_j| \leqslant 1$, the multiplicity must not exceed the order of the differential equation.

For our method

$$\lambda A^0 - A^i = \begin{pmatrix} \lambda & 0 & 0 & -1 & 0 & -1 \\ 0 & \lambda & 0 & -1 & 0 & -\frac{3}{2} \\ 0 & 0 & \lambda & 0 & 0 & -2 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{pmatrix} \quad (4.1)$$

As $h \to 0$, we have

$$\begin{pmatrix} \lambda & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda - 1 \end{pmatrix}$$

Taking the determinant of above, we have

$$\lambda^4(\lambda - 1) = 0 \tag{4.2}$$

solving equation (4.2) we obtain $\lambda = 0,, 1$.
Since all the two conditions above are satisfied, we conclude that the block method converges.

5. Numerical implementation of the scheme

The effectiveness and validity of our newly derived method was tested by applying it to some second order differential equations. All calculations and programs are carried out with the aid of Maple 2016 software.

Example 1

Considering a moderately stiff problem

$$y'' = y', y(0) = 0, y'(0) = -1$$

Whose exact solution is $y(x) = 1 - \exp(x)$.

Table 1: Showing the exact solutions, computed results and error from the proposed methods. $h = 0.1$.

x	Exact	Numerical	Error
0.1	−0.1051709180756476248	−0.10517092531230067983	7.237×10^{-9}
0.2	−0.22140277842597346028	−0.22140277842597346028	2.027×10^{-9}
0.3	−0.3498588075760031040	−0.34985885190094527583	4.432×10^{-8}
0.4	−0.4918246976412703178	−0.49182477470105403287	7.706×10^{-8}
0.5	−0.6487212707001281468	−0.64872139573589266565	1.250×10^{-7}
0.6	−0.8221188003905089749	−0.82211898595391881399	1.856×10^{-7}
0.7	−1.0137527074704765216	−1.0137529743378625233	2.669×10^{-7}
0.8	−1.2255409284924676046	−1.2255412943824773703	3.659×10^{-7}
0.9	−1.4596031111569496638	−1.4596036040850111275	4.929×10^{-7}
0.10	−1.7182818284590452354	−1.7182824729834857232	6.445×10^{-7}

Example 2

We consider a highly oscillatory test problem

$$y'' + \lambda^2 y = 0, y(0) = 1, y'(0) = 2,$$

Table 2: Comparison of error for proposed scheme with existing literature for Example 1. (Anake et al. [14],Yahaya and Badmus [15], Kayode and Adeyeye[16],Adeniran and Ogundare [5], New Proposed Method (NPM)

x	[14]	[15]	[16]	[5]	NPM
0.1	0.84×10^{-07}	0.87×10^{-04}	0.817×10^{-06}	2.22×10^{-08}	7.237×10^{-9}
0.2	0.53×10^{-05}	0.32×10^{-03}	0.31×10^{-5}	1.25×10^{-07}	2.027×10^{-9}
0.3	0.62×10^{-05}	0.22×10^{-02}	0.65×10^{-05}	3.250×10^{-07}	4.432×10^{-8}
0.4	0.16×10^{-05}	0.49×10^{-02}	0.66×10^{-05}	6.424×10^{-07}	7.706×10^{-8}
0.5	0.10×10^{-04}	0.91×10^{-02}	0.11×10^{-07}	1.099×10^{-06}	1.250×10^{-7}
0.6	0.29×10^{-04}	0.14×10^{-01}	1.80×10^{-04}	1.7213×10^{-06}	1.856×10^{-7}
0.7	0.59×10^{-04}	0.21×10^{-01}	0.26×10^{-04}	2.538×10^{-06}	2.669×10^{-7}
0.8	0.10×10^{-03}	0.29×10^{-01}	0.37×10^{-04}	3.583×10^{-06}	3.659×10^{-7}
0.9	0.15×10^{-03}	0.4×10^{-01}	0.51×10^{-04}	4.896×10^{-06}	4.929×10^{-7}
1.0	0.23×10^{-03}	0.52×10^{-01}	0.67×10^{-04}	6.522×10^{-06}	6.445×10^{-7}

Table 3: Numerical result for Example 2 with h=0.01

x	Exact	Numerical	Error
0.01	1.0197986733599108578	1.0197986733595032128	4.076×10^{-13}
0.02	1.0391894408476120998	1.0391894408465250404	1.087×10^{-12}
0.03	1.0581645464146487647	1.0581645464124141562	2.235×10^{-12}
0.04	1.0767164002717920723	1.0767164002681286455	3.663×10^{-12}
0.05	1.0948375819248539184	1.0948375819192781955	5.576×10^{-12}
0.06	1.1125208431427856122	1.1125208431350078814	7.778×10^{-12}
0.07	1.1297591108568736536	1.1297591108463964849	1.048×10^{-11}
0.08	1.1465454899898729124	1.1465454899763992427	1.347×10^{-11}
0.09	1.1628732662139455929	1.1628732661969654734	1.698×10^{-11}
0.10	1.1787359086363028466	1.1787359086155129853	2.079×10^{-11}

Table 4: Comparison of error for Example 2 with existing literature.

x	[17]	[5]	NPM
0.01	–	0.00	4.076×10^{-13}
0.02	0.26×10^{-05}	0.00	1.087×10^{-12}
0.03	0.40×10^{-05}	0.00	2.235×10^{-12}
0.04	0.53×10^{-05}	0.00	3.663×10^{-12}
0.05	0.66×10^{-05}	0.00	5.576×10^{-12}
0.06	0.79×10^{-05}	0.00	7.778×10^{-12}
0.07	0.93×10^{-05}	0.00	1.048×10^{-11}
0.08	0.11×10^{-04}	0.00	1.347×10^{-11}
0.09	0.12×10^{-04}	0.00	1.698×10^{-11}
0.10	0.13×10^{-04}	0.00	2.079×10^{-11}

with $\lambda = 2$ whose exact solution $y(x) = \cos 2x + \sin 2x$.

Example 3
We consider a highly stiff problem

$$y'' + 1001y' + 1000y = 0, y(0) = 1, y'(0) = -1,$$

whose exact solution is $y(x) = \exp(-x)$.

Table 5: Numerical result for Example 3 with $h = 0.05$

x	Exact	Numerical	Error
0.1	.90483741803595957316	.90483741805285210796	1.689×10^{-11}
0.2	.81873075307798185867	.81873075309216300049	1.418×10^{-11}
0.3	.74081822068171786607	.74081822069798894102	1.627×10^{-11}
0.4	.67032004603563930074	.67032004605227219651	1.663×10^{-11}
0.5	.60653065971263342360	.60653065972973741902	1.710×10^{-11}
0.6	.54881163609402643263	.54881163611127309314	1.725×10^{-11}
0.7	.49658530379140951470	.49658530380864066605	1.723×10^{-11}
0.8	.44932896411722159143	.44932896413427753661	1.706×10^{-11}
0.9	.40656965974059911188	.40656965975735890380	1.676×10^{-11}
1.0	.36787944117144232160	.36787944118780739436	1.637×10^{-11}

Example 4
We consider the non-linear initial value problem:

$$y'' - x(y')^2 = 0, \quad y(0) = 1, y'(0) = \frac{1}{2}$$

Table 6: Comparison of error for Example 3 with existing literature..

x	[17]	[5]	NPM
0.1	–	2.05×10^{11}	1.689×10^{-11}
0.2	0.26×10^{-05}	4.39×10^{11}	1.418×10^{-11}
0.3	0.40×10^{-05}	6.55×10^{11}	1.627×10^{-11}
0.4	0.53×10^{-05}	8.38×10^{11}	1.663×10^{-11}
0.5	0.66×10^{-05}	9.86×10^{11}	1.710×10^{-11}
0.6	0.79×10^{-05}	1.10×10^{10}	1.725×10^{-11}
0.7	0.93×10^{-05}	1.19×10^{10}	1.723×10^{-11}
0.8	0.11×10^{-04}	1.24×10^{10}	1.706×10^{-11}
0.9	0.12×10^{-04}	1.28×10^{10}	1.676×10^{-11}
1.0	0.13×10^{-04}	1.30×10^{10}	1.637×10^{-11}

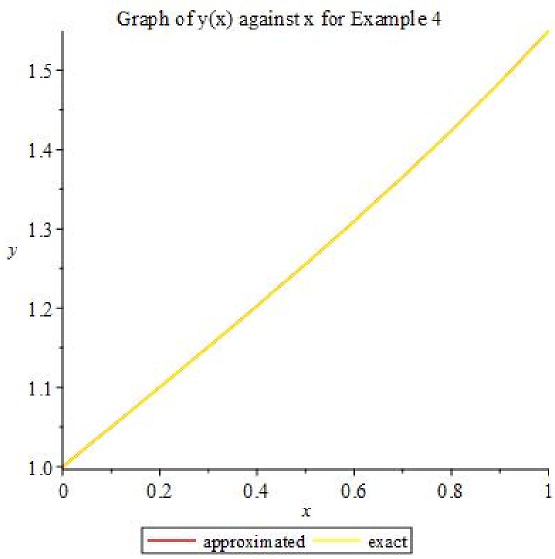
whose exact solution is given by $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$.

Table 7: Showing the exact solutions and the computed results from the proposed methods for Example 4, $h = 0.1$.

x	exact	Numerical	error
0.1	1.0500417292784912682	1.0500417198073274141	1.041×10^{-9}
0.2	1.1003353477310755806	1.1003353275055234711	1.541×10^{-9}
0.3	1.1511404359364668053	1.1511404029243392812	1.613×10^{-9}
0.4	1.2027325540540821910	1.2027325054111612616	1.578×10^{-9}
0.5	1.2554128118829953416	1.2554127435116525126	1.812×10^{-9}
0.6	1.3095196042031117155	1.3095195086259392942	1.352×10^{-8}
0.7	1.3654437542713961691	1.3654436330677690756	1.351×10^{-8}
0.8	1.4236489301936018068	1.4236487513499449584	1.121×10^{-8}
0.9	1.4847002785940517416	1.4847000151409032852	1.146×10^{-8}
1.0	1.5493061443340548457	1.5493057445240738187	1.138×10^{-8}

Table 8: Comparison of error for Example 4 with existing literature.

x	[18]	NPM
0.1	1.051251×10^{-8}	1.041×10^{-9}
0.2	2.176690×10^{-8}	1.541×10^{-9}
0.3	3.462528×10^{-8}	1.613×10^{-9}
0.4	5.022104×10^{-8}	1.578×10^{-9}
0.5	7.018369×10^{-8}	1.812×10^{-9}
0.6	9.700952×10^{-8}	1.352×10^{-8}
0.7	1.3471588×10^{-7}	1.351×10^{-8}
0.8	1.9005788×10^{-7}	1.121×10^{-8}
0.9	2.749090×10^{-7}	1.146×10^{-8}
1.0	4.1118559×10^{-7}	1.138×10^{-8}



6. Conclusion

We have proposed a two-step Bernstein polynomial fitted methods for the direct solution of general second order initial value problems. The method process a good accuracy with order 4, consistent and zero stable. The methods are implemented without the need for the development of predictors nor requiring any other method to generate starting values. Implementation of the method with numerical examples showed that the methods can compete favorably most of the existing multistep methods available for approximating similar class of problems.

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