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Transmuted Sushila Distribution and its Application to Lifetime Data

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Abstract

The Sushila distribution is generalized in this article using the quadratic rank transmutation map as developed by Shaw and Buckley (2007). The newly developed distribution is called the Transmuted Sushila distribution (TSD). Various mathematical properties of the distribution are obtained. Real lifetime data is used to compare the performance of the new distribution with other related distributions. The results shown by the new distribution perform creditably well.

Keywords: Sushila distribution, Lindley distribution, lifetime distribution, reliability function.
2010 MSC: 60E05, 62E10, 62G30, 62H10, 97K50.

1. Introduction

Last few years have witnessed generalizations of various lifetime distributions. This is achieved by compounding the distribution with any of new families of distributions. The process involves introduction of new shape parameter(s) to improve flexibility of the baseline distribution. Among well-known generalized families of distributions are: Marshall-Olkin family of distributions [1]; Beta G distributions [2]; Quadratic Transmuted family of distributions [3]; Kumaraswamy G distributions [4]; Gamma G distributions [5]; Exponentiated generalized G distributions [6]; Weibull G distributions [7]; and Alpha Power Transformation [8]. Researchers in sciences and engineering have applied these families of distribution to improve modelling of various lifetime data.

In this article, we generalize the Sushila distribution [9] using the Quadratic Transmuted family of distributions [3] and the new generalization is called the Transmuted Sushila Distribution (TSD). A random variable X is said to have the Sushila distribution if its distribution function (CDF) is given as:

$$G(x) = 1 - \frac{\lambda\,(1+\theta) + \theta\,x}{\lambda\,(1+\theta)} e^{-\frac{\theta x}{\lambda}}. \tag{1.1}$$

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It is observed that is a special case of the Lindley distribution [10] when $\lambda = 1$.

Given a baseline distribution with the *CDF* $G(x)$, the Quadratic Transmuted **(**QT**)** family of distributions has the *cdf*:

$$F(x) = (1+a)G(x) - a\Big(G(x)\Big)^2. \tag{1.2}$$

QT was applied to some probability distributions by [11, 12] with the resulting distributions offering more flexibility. [13] also discussed some mathematical properties for the QT family of distributions. With generally acceptability, researchers have applied (1.2) and introduced different new members of the QT family for diverse lifetime distributions. List of some QT distributions was provided by [14].

2. Transmuted Sushila Distribution

The Transmuted Sushila Distribution **(**TSD**)** is obtained by putting (1.1) into (1.2). Hence, a random variable **X** is said to have the TSD, i.e. $\mathbf{X} \sim TSD(\theta, \lambda, a)$ if its CDF is given as:

$$F(x) = (1+a)\left(1 - \frac{\lambda\,(1+\theta) + \theta\,x}{\lambda\,(1+\theta)} e^{-\frac{\theta\,x}{\lambda}}\right) - a\left(1 - \frac{\lambda\,(1+\theta) + \theta\,x}{\lambda\,(1+\theta)} e^{-\frac{\theta\,x}{\lambda}}\right)^2. \tag{2.1}$$

Figures 1 illustrates the cdf of the TSD for some selected values of scale and shapes parameters. The probability distribution function **(**PDF**)** of TSD is obtained by differentiating (2.1) once. Hence, the PDF of $\mathbf{X} \sim TSD(\theta, \lambda, a)$ is:

$$f(x) = \frac{\theta^2}{\lambda^3(1+\theta)^2}(\lambda + x)\Big(2ae^{-\frac{\theta x}{\lambda}}(\lambda\theta + \theta x + \lambda) - \lambda(1+\theta)(a-1)\Big)e^{-\frac{\theta x}{\lambda}}. \tag{2.2}$$

Note:

(i) The TSD becomes the Sushila Distribution due to [9] if $a = 0$.
(ii) The TSD becomes the Lindley Distribution due to [10] if $a = 0$ and $\lambda = 1$.
The pdf of the TSD for some selected values of scale and shapes parameters is shown in figure 2.

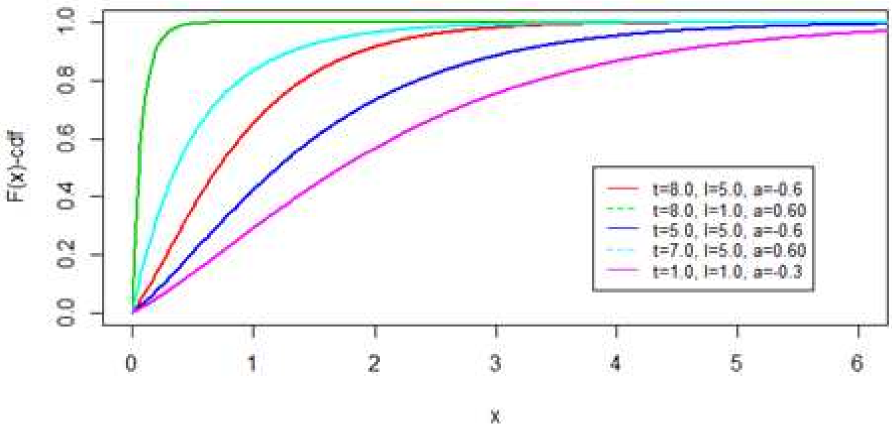

Figure 1: CDF of Transmuted Sushila Distribution

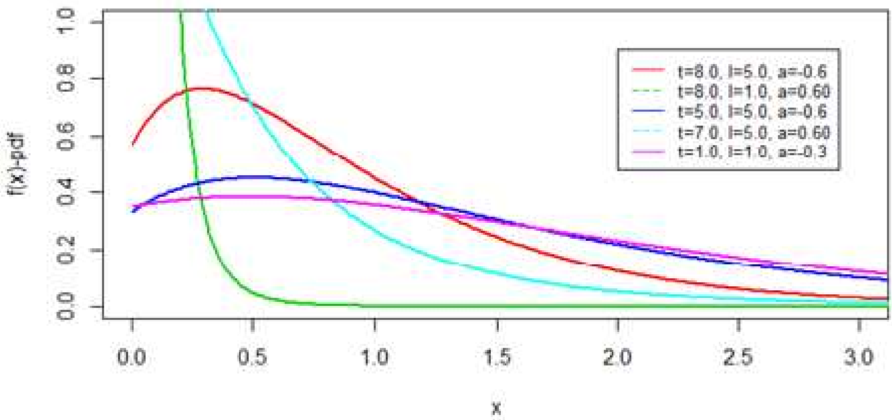

Figure 2: PDF of Transmuted Sushila Distribution

3. Reliability Analysis

Survival Function: The probability of an item not failing prior to a particular time is defined by its reliability or survival function $S(x)$. This is defined by $S(x) = 1 - F(x)$. Therefore, if a random variable $\mathbf{X} \sim TSD(\theta, \lambda, a)$, then its survival function is given by:

$$S(x) = 1 - (1+a)\left(1 - \frac{\lambda\,(1+\theta) + \theta\,x}{\lambda\,(1+\theta)} e^{-\frac{\theta x}{\lambda}}\right) + a\left(1 - \frac{\lambda\,(1+\theta) + \theta\,x}{\lambda\,(1+\theta)} e^{-\frac{\theta x}{\lambda}}\right)^2. \quad (3.1)$$

Hazard Rate Function (HRF): This is the risk a system has in experiencing and event in an instantaneous time provided it has not experienced it at present time. It is a measure of proneness to an event and it is obtained as: $h(x) = \frac{f(x)}{S(x)}$. For a random variable that has the TSD, the hrf is given as:

$$h(x) = \frac{\theta^2(x+\lambda)\left(2a\exp^{-\frac{\theta x}{\lambda}}(\lambda\theta+\theta x+\lambda)-\lambda(1+\theta)(a-1)\right)}{\lambda(\lambda\theta+\theta x+\lambda)\left((\lambda\theta+\theta x)a\exp^{-\frac{\theta x}{\lambda}}\right)-\lambda(1+\theta)(a-1)}. \tag{3.2}$$

Figure 4 shows the HRF of the TSD for some values of scale and shapes parameters.

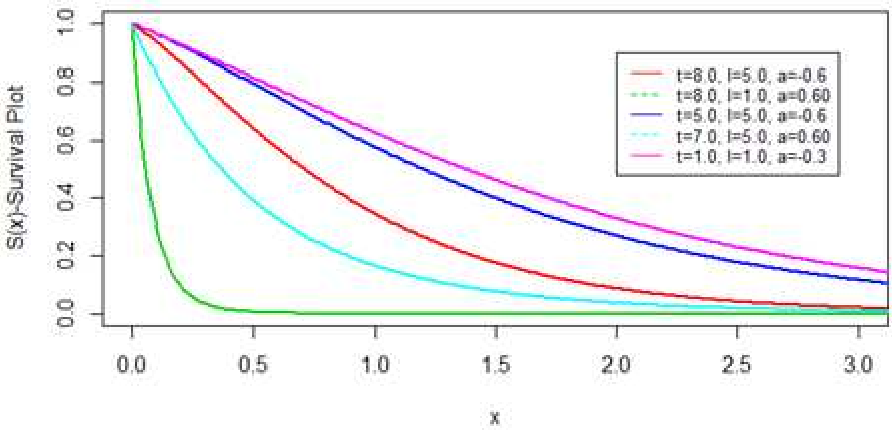

Figure 3: The Survival function of Transmuted Sushila Distribution

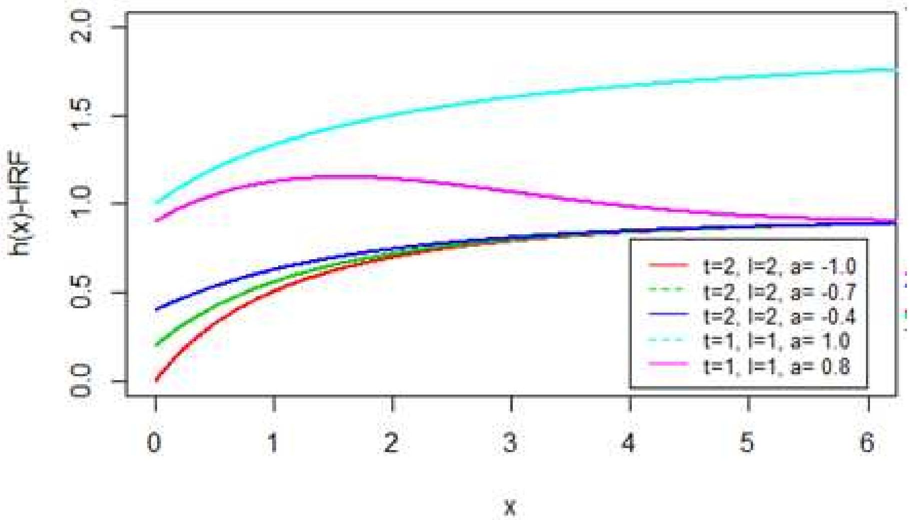

Figure 4: HRF of Transmuted Sushila Distribution

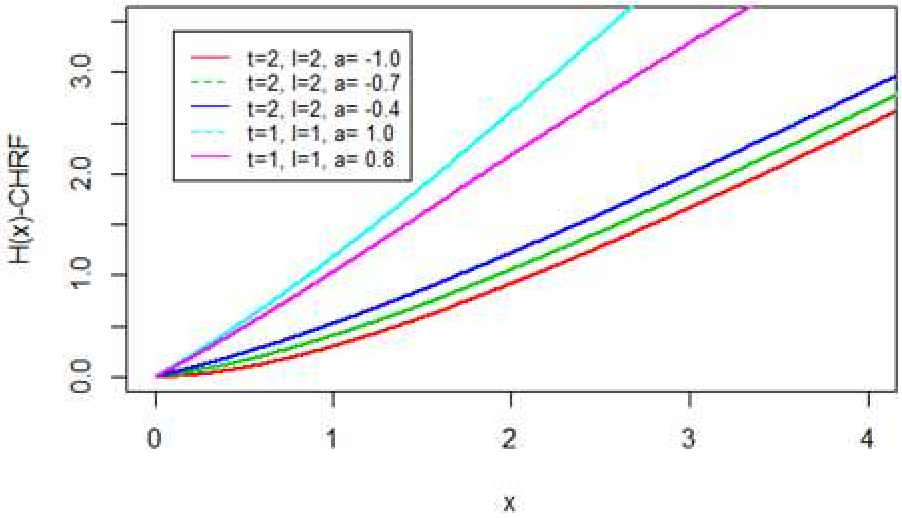

Figure 5: Cumulative HRF of Transmuted Sushila Distribution

Cumulative Hazard Rate Function, H(x): This is the overall risk rate from the onset up to a given time. The measure is obtained as $H(x) = -\ln(S(x))$.

$$H(x) = -\ln\left((\lambda\theta + \theta x + \lambda)\left(\frac{a\exp^{-\frac{\theta x}{\lambda}}(\lambda\theta + \theta x + \lambda) - a\theta\lambda + \theta\lambda + \lambda}{\lambda^2(1+\theta)^2}\right)\exp^{-\frac{\theta x}{\lambda}}\right). \quad (3.3)$$

The trend depicting the cumulative HRF of the TSD for some values of scale and shapes parameters is shown in figure 5.

4. Mathematical Properties

4.1. Order Statistics

Given that $X_{1,n} < X_{2,n} < ... < X_{n,n}$ is a set of ordered random variable of size **n**, if $\mathbf{X} \sim TSD(\theta, \lambda, a)$, then, the PDF of the r^{th} order statistics is given as:

$$f_{r,n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}.$$

Therefore, the rth order statistics of **X** is given as:

$$\begin{aligned} f_{r,n}(x) = {} & \frac{n!\theta^2(\lambda+x)}{(r-1)!(n-r)!}\left[\frac{\left(2ae^{-\frac{\theta}{\lambda}x}(\lambda\theta + \theta x + \lambda) - a\theta\lambda - a\lambda + \theta\lambda - \lambda\right)e^{-\frac{\theta}{\lambda}x}}{\lambda^3(1+\theta)^2}\right] \\ & \left[(1+a)\left(1 - \frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right) - a\left(1 - \frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right)^2\right]^{r-1} \\ & \left[1 - (1+a)\left(1 - \frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right) + a\left(1 - \frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right)^2\right]^{n-r}. \end{aligned} \quad (4.1)$$

If $\mathbf{r} = \mathbf{1}$ and $\mathbf{r} = \mathbf{n}$, the *1st* and *nth* order statistic for X are respectively given in (4.2) and (4.3).

$$f_{1,n}(x) = \frac{n!\theta^2(\lambda+x)}{(n-1)!}\left[\frac{\left(2ae^{-\frac{\theta}{\lambda}x}(\lambda\theta+\theta x+\lambda)-a\theta\lambda-a\lambda+\theta\lambda-\lambda\right)e^{-\frac{\theta}{\lambda}x}}{\lambda^3(1+\theta)^2}\right] \left[1-(1+a)\left(1-\frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right)+a\left(1-\frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right)^2\right]^{n-r}. \tag{4.2}$$

$$f_{n,n}(x) = \frac{n!\theta^2(\lambda+x)}{(n-1)!}\left[\frac{\left(2ae^{-\frac{\theta}{\lambda}x}(\lambda\theta+\theta x+\lambda)-a\theta\lambda-a\lambda+\theta\lambda-\lambda\right)e^{-\frac{\theta}{\lambda}x}}{\lambda^3(1+\theta)^2}\right] \left[(1+a)\left(1-\frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right)-a\left(1-\frac{\lambda(1+\theta)+\theta x}{\lambda(1+\theta)}e^{-\frac{\theta}{\lambda}x}\right)^2\right]^{n-1}. \tag{4.3}$$

4.2. Quantiles Function

The quantile function of a random variable $\mathbf{X} \sim \mathrm{TSD}(\theta,\lambda,a)$, is given as:

$$Q_{(u)} = -\left(\frac{\lambda}{\theta}\right)\left[\mathbf{W}\left(-\frac{e^{-(1+\theta)}(1+\theta)(a-1+\sqrt{1+2a+a^2-4ua})}{2a}\right)+1+\theta\right]. \tag{4.4}$$

Therefore, the first, the second, and the third quartiles for the random variable are obtained by respectively setting $\mathbf{u}$ to 0.25, 0.50, and 0.75. These are given by:

$$Q_{(\frac{1}{4})} = -\left(\frac{\lambda}{\theta}\right)\left[\mathbf{W}\left(-\frac{e^{-(1+\theta)}(1+\theta)(a-1+\sqrt{1+a^2+a})}{2a}\right)+1+\theta\right],$$

$$Q_{(\frac{1}{2})} = -\left(\frac{\lambda}{\theta}\right)\left[\mathbf{W}\left(-\frac{e^{-(1+\theta)}(1+\theta)(a-1+\sqrt{1+a^2})}{2a}\right)+1+\theta\right],$$

$$Q_{(\frac{3}{4})} = -\left(\frac{\lambda}{\theta}\right)\left[\mathbf{W}\left(-\frac{e^{-(1+\theta)}(1+\theta)(a-1+\sqrt{1+a^2-a})}{2a}\right)+1+\theta\right],$$

where the Lambert function $\mathbf{W}$ is a complex function with multiple values which is defined as the solution for the equation $W_{(u)}e^{W_{(u)}} = u$.

Equation (4.4) is obtained using Maple 2016 [15].

4.3. Skewness and Kurtosis

Variability in a data set can be investigated using skewness and kurtosis, and classical measures can be susceptible to outliers. Given the quantile function as in (4.4), the Moor's Kurtosis [16] based on octiles is given as:

$$K_M = \frac{Q_{\frac{7}{8}} - Q_{\frac{5}{8}} + Q_{\frac{3}{8}} - Q_{\frac{1}{8}}}{Q_{\frac{6}{8}} - Q_{\frac{2}{8}}}. \tag{4.5}$$

Also, the Bowley's measure of skewness [17] based on quartiles is given as:

$$SK_B = \frac{Q_{\frac{3}{4}} - 2Q_{\frac{2}{4}} + Q_{\frac{1}{4}}}{Q_{\frac{3}{4}} - Q_{\frac{1}{4}}}. \tag{4.6}$$

4.4. Moments

Proposition 4.1. If a random variable X follows TSD with *pdf* as given in (2.2), then the **kth** raw moment is given by:

$$\begin{aligned} E(x^k) = \frac{\lambda^k k!}{\theta^k(1+\theta)^2}\Bigg(&\frac{a\theta(1+\theta)}{2^k} + \frac{a(1+2\theta)(k+1)}{2^{k+1}} \\ &-(a\theta + a - \theta - 1)(\theta + k + 1) + \frac{a(k+1)(k+2)}{2^{k+2}}\Bigg). \end{aligned} \tag{4.7}$$

Proof.

$$\begin{aligned}
\mu_k^1 &= E(x^k) = \int_0^\infty x^k f(x)\,dx \\
&= \int_0^\infty x^k \frac{\theta^2}{\lambda^3(1+\theta)^2}(\lambda + x)\Big(2ae^{-\frac{\theta x}{\lambda}}(\lambda\theta + \theta x + \lambda) - \lambda(1+\theta)(a-1)\Big)e^{-\frac{\theta x}{\lambda}}\,dx \\
&= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\int_0^\infty \left(\lambda x^k e^{-\frac{\theta x}{\lambda}} + x^{k+1}e^{-\frac{\theta x}{\lambda}}\right) \\
&\quad \left(2ae^{-\frac{\theta x}{\lambda}}(\lambda\theta + \theta x + \lambda) - \lambda(1+\theta)(a-1)\right)dx \\
&= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\int_0^\infty \left(\lambda x^k e^{-\frac{\theta x}{\lambda}} + x^{k+1}e^{-\frac{\theta x}{\lambda}}\right)\Bigg(\lambda\theta 2ae^{-\frac{\theta x}{\lambda}} \\
&\quad + \theta x 2ae^{-\frac{\theta x}{\lambda}} + \lambda 2ae^{-\frac{\theta x}{\lambda}} - a\theta\lambda - a\lambda + \theta\lambda + \lambda\Bigg)dx \\
&= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\left(2a\theta\lambda^2 + 2a\lambda^2\right)P_1 + \left(4a\theta\lambda + 2a\lambda\right)P_2 \\
&\quad -\lambda^2\left(a\theta + a - \theta - 1\right)P_3 + 2a\theta P_4 - \lambda\left(a\theta + a - \theta - 1\right)P_5,
\end{aligned}$$

where

$$P_1 = \int_0^\infty x^k e^{-\frac{2\theta x}{\lambda}}\,dx = \frac{\lambda^{k+1}k!}{2^{k+1}\theta^{k+1}}; \quad P_2 = \int_0^\infty x^{k+1}e^{-\frac{2\theta x}{\lambda}}\,dx = \frac{\lambda^{k+2}(k+1)!}{2^{k+2}\theta^{k+2}};$$

$$P_3 = \int_0^{\infty} x^k e^{-\frac{\theta x}{\lambda}}\, dx = \frac{\lambda^{k+1} k!}{\theta^{k+1}}; \qquad P_4 = \int_0^{\infty} x^{k+2} e^{-\frac{2\theta x}{\lambda}}\, dx = \frac{\lambda^{k+3} k!}{2^{k+3}\theta^{k+2}};$$

$$P_5 = \int_0^{\infty} x^{k+1} e^{-\frac{\theta x}{\lambda}}\, dx = \frac{\lambda^{k+2}(k+1)!}{\theta^{k+2}}.$$

Therefore,

$$\begin{aligned} E(x^k) &= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\left(2a\theta\lambda^2 + 2a\lambda^2\right)\left(\frac{\lambda^{k+1}k!}{2^{k+1}\theta^{k+1}}\right) \\ &\quad + \left(4a\theta\lambda + 2a\lambda\right)\left(\frac{\lambda^{k+2}(k+1)!}{2^{k+2}\theta^{k+2}}\right) \\ &\quad - \lambda^2\left(a\theta + a - \theta - 1\right)\left(\frac{\lambda^{k+1}k!}{\theta^{k+1}}\right) + 2a\theta\left(\frac{\lambda^{k+3}k!}{2^{k+3}\theta^{k+2}}\right) \\ &\quad - \lambda\left(a\theta + a - \theta - 1\right)\left(\frac{\lambda^{k+2}(k+1)!}{\theta^{k+2}}\right). \end{aligned}$$

Hence,

$$E(x^k) = \frac{\lambda^k k!}{\theta^k(1+\theta)^2}\left(\frac{a\theta(1+\theta)}{2^k} + \frac{a(1+2\theta)(k+1)}{2^{k+1}} - (a\theta + a - \theta - 1)(\theta + k + 1) + \frac{a(k+1)(k+2)}{2^{k+2}}\right).$$

$\square$

Mean and Variance of TSD: The mean of a random variable **X** follows **TSD** is obtained by putting **k = 1** in (4.7) above. Hence the mean is given as:

$$E(x) = \frac{\lambda}{4\theta(1+\theta)^2}\left(2a\theta^2 + 6a\theta - 4\theta^2 + 3a - 12\theta - 8\right)$$

Also, the variance of **X** is obtained as $var(x) = E(x^2) - (E(x))^2$, where $E(x^2)$ is obtained by equating $\mathbf{k} = \mathbf{2}$ in (4.7).

$$E(x^2) = \frac{\lambda^2}{4\theta^2(1+\theta)^2}\left(6a\theta^2 + 24a\theta - 8\theta^2 + 15a - 32\theta - 24\right)$$

Hence, the variance of **X** is given by:

$$\begin{aligned} var(x) = \frac{\lambda^2}{4\theta^2(1+\theta)^2}\Bigg[&\frac{(2a\theta^2 + 6a\theta - 4\theta^2 + 3a - 12\theta - 8)^2}{4(1+\theta)^2} \\ &+ (6a\theta^2 + 24a\theta - 8\theta^2 + 15a - 32\theta - 24)\Bigg]. \end{aligned}$$

4.5. Moment Generating Function

Proposition 4.2. The moment generating function of a random variable **X** that follows the *TSD* is given as:

$$
\begin{aligned}
E(e^{tx}) = \left(\frac{\theta^2}{(1+\theta)^2}\right)\Bigg(\frac{2a(1+\theta)}{2\theta-\lambda t} + \frac{2a(1+2\theta)}{(2\theta-\lambda t)^2} + \frac{4a\theta}{(2\theta-\lambda t)^3} \\
- \frac{(a\theta+a-\theta-1)(\theta-\lambda t+1)}{(\theta-\lambda t)^2}\Bigg).
\end{aligned} \tag{4.8}
$$

Proof.

$$
\begin{aligned}
M_x(t) &= E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx \\
&= \int_0^\infty e^{tx}\frac{\theta^2}{\lambda^3(1+\theta)^2}(\lambda+x)\left(2ae^{-\frac{\theta x}{\lambda}}(\lambda\theta+\theta x+\lambda)-\lambda(1+\theta)(a-1)\right)e^{-\frac{\theta x}{\lambda}}\,dx \\
&= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\int_0^\infty(\lambda+x)e^{-\frac{\theta-\lambda t}{\lambda}x}\Bigg(2a\theta\lambda e^{-\frac{\theta x}{\lambda}}+2a\theta x e^{-\frac{\theta x}{\lambda}}+2a\lambda e^{-\frac{\theta x}{\lambda}} \\
&\quad -\lambda(a\theta+a-\theta-1)\Bigg)\,dx \\
&= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\left(2a\theta\lambda^2+2a\lambda^2\right)P_1+\Bigg(4a\theta\lambda \\
&\quad +2a\lambda\Bigg)P_2-\lambda^2\left(a\theta+a-\theta-1\right)P_3+2a\theta P_4-\lambda\left(a\theta+a-\theta-1\right)P_5,
\end{aligned}
$$

where

$$
P_1 = \int_0^\infty e^{-\frac{2\theta-\lambda t}{\lambda}x}\,dx = \frac{\lambda}{2\theta-\lambda t}; \qquad P_2 = \int_0^\infty x e^{-\frac{2\theta-\lambda t}{\lambda}x}\,dx = \frac{\lambda^2}{(2\theta-\lambda t)^2};
$$

$$
P_3 = \int_0^\infty e^{-\frac{\theta-\lambda t}{\lambda}x}\,dx = \frac{\lambda}{(\theta-\lambda t)}; \qquad P_4 = \int_0^\infty x^2 e^{-\frac{2\theta-\lambda t}{\lambda}x}\,dx = \frac{2\lambda^3}{(2\theta-\lambda t)^3};
$$

$$
P_5 = \int_0^\infty x e^{-\frac{\theta-\lambda t}{\lambda}x}\,dx = \frac{\lambda^2}{(\theta-\lambda t)^2}.
$$

Therefore,

$$\begin{aligned}
E(e^{tx}) &= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)(2a\theta\lambda^2+2a\lambda^2)\left(\frac{\lambda}{2\theta-\lambda t}\right)+(4a\theta\lambda+2a\lambda)\left(\frac{\lambda}{(2\theta-\lambda t)}\right)^2\\
&\quad -\lambda^2(a\theta+a-\theta-1)\left(\frac{\lambda}{\theta-\lambda t}\right)+4a\theta\left(\frac{2\lambda}{(2\theta-\lambda t)}\right)^3-\lambda(a\theta+a-\theta-1)\left(\frac{\lambda}{(\theta-\lambda t)}\right.\\
&= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\left[2a\lambda^2(1+\theta)\left(\frac{\lambda}{2\theta-\lambda t}\right)\right.\\
&\quad +2a\lambda(1+2\theta)\left(\frac{\lambda}{2\theta-\lambda t}\right)^2+4a\theta\left(\frac{\lambda}{2\theta-\lambda t}\right)^3\\
&\quad \left.-(a\theta+a-\theta-1)\lambda^3\left(1+\frac{1}{\theta-\lambda t}\right)\right]\\
&= \left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)\left[2a(1+\theta)\left(\frac{1}{2\theta-\lambda t}\right)\right.\\
&\quad +2a(1+2\theta)\left(\frac{1}{2\theta-\lambda t}\right)^2+4a\theta\left(\frac{1}{2\theta-\lambda t}\right)^2\\
&\quad \left.-(a\theta+a-\theta-1)\lambda^3\left(\frac{\theta-\lambda t+1}{\theta-\lambda t}\right)\right].
\end{aligned}$$

Hence

$$E(e^{tx})=\left(\frac{\theta^2}{(1+\theta)^2}\right)\left(\frac{2a(1+\theta)}{2\theta-\lambda t}+\frac{2a(1+2\theta)}{(2\theta-\lambda t)^2}+\frac{4a\theta}{(2\theta-\lambda t)^3}-\frac{(a\theta+a-\theta-1)(\theta-\lambda t+1)}{(\theta-\lambda t)^2}\right).$$

□

5. Parameter Estimation

Proposition 5.1. Given that $X_i, i=1,2,...,n$ are *iid* random variables from *TSD*, then the log-likelihood function of **X** is defined as:

$$\begin{aligned}
\log L &= 2n\Big(\log(\theta)-\log(1+\theta)\Big)-3n\log(\lambda)+\sum_{i=1}^{n}\log(\lambda+x_i)\\
&\quad+\sum_{i=1}^{n}\log\Big(2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)-a\theta\lambda+\theta\lambda+\lambda\Big)-\frac{\lambda}{\theta}\sum_{i=1}^{n}x_i.
\end{aligned} \tag{5.1}$$

Proof. The likelihood function of a random variable **X** that follows TSD is:

$$
\begin{aligned}
L &= \prod_{i=1}^{i=n}\left(\frac{\theta^2}{\lambda^3(1+\theta)^2}\right)(\lambda+x_i)\left(2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)-a\theta\lambda-a\theta\lambda-a\lambda+\theta\lambda+\lambda\right)e^{-\frac{\theta}{\lambda}x_i} \\
&= \left(\frac{\theta^{2n}}{\lambda^{3n}(1+\theta)^{2n}}\right)\prod_{i=1}^{i=n}(\lambda+x_i) \\
&\quad \prod_{i=1}^{i=n}\left(2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)-a\theta\lambda-a\theta\lambda-a\lambda+\theta\lambda+\lambda\right)e^{-\frac{\theta}{\lambda}\sum_{i=1}^{n}x_i} \\
&= \left(\frac{\theta}{1+\theta}\right)^{2n}\lambda^{-3n}\prod_{i=1}^{i=n}(\lambda+x_i) \\
&\quad \prod_{i=1}^{i=n}\left(2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)-a\theta\lambda-a\theta\lambda-a\lambda+\theta\lambda+\lambda\right)e^{-\frac{\theta}{\lambda}\sum_{i=1}^{n}x_i}.
\end{aligned}
$$

Hence, the log-likelihood function of a random variable **X** that follows TSD is:

$$
\begin{aligned}
\log L &= 2n\left(\log(\theta)-\log(1+\theta)\right)-3n\log(\lambda)+\sum_{i=1}^{n}\log(\lambda+x_i) \\
&\quad +\sum_{i=1}^{n}\log\left(2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)-a\theta\lambda+\theta\lambda+\lambda\right)-\frac{\lambda}{\theta}\sum_{i=1}^{n}x_i.
\end{aligned}
$$

□

The MLE of (θ,λ,a) can be obtained by maximizing (5.1). This gives the set of normal equations below:

$$
\begin{aligned}
\frac{\partial \log L}{\partial\theta} &= \frac{2n}{\theta}-\frac{2n}{1+\theta}-\frac{1}{\lambda}\sum_{i=1}^{n}x_i \\
&\quad +\sum_{i=1}^{n}\frac{-\frac{1}{\lambda}2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)+2ae^{-\frac{\theta}{\lambda}x_i}(\lambda+x_i+\lambda)-a\lambda+\lambda}{2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)-a\theta\lambda-a\lambda+\theta\lambda+\lambda} \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial\lambda} &= \frac{-3n}{\theta}+\sum_{i=1}^{n}\frac{1}{\lambda+x_i}+\frac{\theta}{\lambda^2}\sum_{i=1}^{n}x_i \\
&\quad +\sum_{i=1}^{n}\frac{-\frac{1}{\lambda^2}2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)+2ae^{-\frac{\theta}{\lambda}x_i}(1+\theta)-a\theta-a+\theta+1}{2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta+\theta x_i+\lambda)-a\theta\lambda-a\lambda+\theta\lambda+\lambda} \\
&= 0.
\end{aligned}
$$

$$\frac{\partial \log L}{\partial a} = \sum_{i=1}^{n} \frac{2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta + \theta x_i + \lambda) - \theta\lambda - \lambda}{2ae^{-\frac{\theta}{\lambda}x_i}(\lambda\theta + \theta x_i + \lambda) - a\theta\lambda - a\lambda + \theta\lambda + \lambda} = 0.$$

Obtaining solutions for the set of normal equations analytically is tedious. Using the **MaxLik** function in R language [18], the solutions are obtained numerically using algorithms like Newton-Raphson.

5.1. Asymptotic Confidence Bounds of TSD

Using the variance-covariance matrix I_o^{-1}, a $(100-\alpha)\%$ confidence intervals of the parameters θ, λ, a can be obtained. $I_n^{-1}[\Psi]$ is the inverse of the observed information matrix [19] given by:

$$I_n^{-1}[\Psi] = \begin{bmatrix} I_{\theta\theta} & I_{\theta\lambda} & I_{\theta a} \\ I_{\lambda\theta} & I_{\lambda\lambda} & I_{\lambda a} \\ I_{a\theta} & I_{a\lambda} & I_{aa} \end{bmatrix} = \begin{bmatrix} var(\hat{\theta}) & covar(\hat{\theta}, \hat{\lambda}) & covar(\hat{\theta}, \hat{a}) \\ covar(\hat{\theta}, \hat{\lambda}) & var(\hat{\lambda}) & covar(\hat{\lambda}, \hat{a}) \\ covar(\hat{\theta}, \hat{a}) & covar(\hat{\lambda}, \hat{a}) & var(\hat{a}) \end{bmatrix}$$

$$I_{\theta\theta} = \frac{\partial^2 \log L}{\partial\theta^2}; I_{\lambda\lambda} = \frac{\partial^2 \log L}{\partial\lambda^2}; I_{aa} = \frac{\partial^2 \log L}{\partial a^2}; I_{\theta\lambda} = \frac{\partial^2 \log L}{\partial\theta\partial\lambda}; I_{\theta a} = \frac{\partial^2 \log L}{\partial\theta\partial a}; I_{a\lambda} = \frac{\partial^2 \log L}{\partial a\partial\lambda}$$

Therefore, at specified level of significance, $(100-\alpha)\%$ confidence intervals of (θ, λ, a) are respectively given as: $\hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta})}$; $\hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\lambda})}$; $\hat{a} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{a})}$.

6. Application

The QT technique has been applied to obtain new set of distributions by compounding the Lindley distribution. The Transmuted Lindley distribution (TLD) was introduced by [20]. [21] introduced Transmuted Quasi-Lindley distribution (TQLD) while [22] introduced the Transmuted Two-Parameter Lindley distribution (TTPLD). The Transmuted Generalized Quasi Lindley distribution (TGQLD) was introduced by [23]. All these newly transmuted distributions have the Lindley distribution has special case. This attribute is also shared by the TSD newly introduced in this research. The probability distribution functions of all the models compared for data application are presented in *table 1* below.

The data set used to observe the performance of TSD is the remission times (months) of a sample of 128 bladder cancer patients. This data has been applied in various survival analysis [19, 20].

0.08, 2.09, 3.48,4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07,3.36, 6.93, 8.65, 12.63, 22.69

Table 1: Probability distribution functions of compared models

Distribution	PDF
TSD	$\frac{\theta^2}{\lambda^3(1+\theta)^2}(\lambda+x)\left(2ae^{-\frac{\theta x}{\lambda}}(\lambda\theta+\theta x+\lambda)-\lambda(1+\theta)(a-1)\right)e^{-\frac{\theta x}{\lambda}}$
SD	$\frac{\theta^2}{\lambda(1+\theta)}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\theta}{\lambda}x}$
LD	$\frac{\theta^2}{1+\theta}(1+x)e^{-\frac{\theta}{\lambda}x}$
TLD	$\frac{\theta^2}{1+\theta}(1+x)e^{-\frac{\theta}{\lambda}x}\left(1-\lambda+2\lambda\frac{1+\theta+\theta x}{1+\theta}\right)$
TQLD	$\frac{\theta}{1+\alpha}(\alpha+\theta x)e^{-\theta x}\left(1-\lambda+2\lambda e^{-\theta x}\left(\frac{\theta x}{1+\alpha}\right)\right)$
TTPLD	$\frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}\left(1+\lambda-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\right)\right)$
TGQLD	$\frac{a\theta}{1+\alpha}(\alpha+\theta x)e^{-\theta x}\left(1-e^{-\theta x}\left(1+\frac{\theta x}{1+\alpha}\right)\right)^{a-1}\left(1+\lambda-2\lambda\left(1-\frac{\theta x}{1+\alpha}e^{-\theta x}\right)^{\alpha}\right)$

Table 2: Descriptive Statistics for remission times in months

Min.	*Q1*	*Median*	*Q3*	*Max*	*Mean*	*Variance*	*Skewness*	*Kurtosis*
0.080	3.348	6.395	11.838	79.050	9.366	110.425	3.287	15.483

The descriptive statistics for the data set is presented in *table 2* while *table 3* shows the parameter estimates and model comparison criteria. To compare the models, -2LL (negative 2 Log-Likelihood), **AIC** (Akaike Information Criterion), and **CAIC** (Corrected Akaike Information Criterion), are used. Distribution with lowest criterion is the "*best*". Results from *table 3* shows that the newly proposed distribution (TSD) performs creditably well for the lifetime dataset presented.

7. Conclusion

We introduced a three-parameter generalization of the Sushila distribution using the Quadratic Transmuted technique championed by [3]. Shape of the distribution function and hazard rate function are investigated and various mathematical properties of the new distribution are presented. Since the Sushila distribution [9] is a special case of the Lindley distribution, performance of the new distribution is compared with the Sushila distribution and other transmuted Lindley distributions using data on cancer remission (in moths) of patients. Results show that the Transmuted Sushila Distribution (TSD) givers a better fit to the data set among the competing distributions.

Competing Interest

Author has declared that no competing interest exist.

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Table 3: Parameter estimates and model selection criteria

	θ	λ	a	α	**-2 LL**	**AIC**	**CAIC**	
TSD	0.742	6.323	0.712		825.884	831.884	832.037	
SD	52.650	484.078			828.694	832.694	832.766	
LD	0.196				839.040	841.060	841.064	
TLD	0.156	0.617			830.310	834.310	834.382	
TQLD	0.061	1138.000	0.862		826.994	832.995	833.147	
TTPLD	0.152	0.531			828.087	832.087	832.159	
TGQLD	0.121	1.218	0.500	1534.000	826.149	834.149	834.417	

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