STUDENTS' VAN HIELE LEVELS OF GEOMETRIC THOUGHT AND CONCEPTION IN PLANE GEOMETRY: A COLLECTIVE CASE STUDY OF NIGERIA AND SOUTH AFRICA

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ABSTRACT

This study is inspired by and utilises the van Hiele theory of geometric thought levels, currently acclaimed as one of the best frameworks for studying teaching and learning processes in geometry. The study aims both to explore and explicate the van Hiele levels of geometric thinking of a selected group of grade 10, 11 and 12 learners in Nigerian and South African schools. The study further aims to provide a rich and in-depth description of the geometry instructional practices that possibly contributed to the levels of geometric conceptualisation exhibited by this cohort of high school learners.

This collective case study, presented in two volumes, is oriented within an interpretive research paradigm and characterised by both qualitative and quantitative methods. The sample for the study comprised a total of 144 mathematics learners and 6 mathematics teachers from Nigeria and South Africa. They were selected using both purposive and stratified sampling techniques.

In using the van Hiele model to interrogate both learners' levels of geometric conceptualisation and teaching methods in geometry classrooms, the study employs a qualitative and qunatitative approach to the data-collection process, involving the use of questionnaires (in the form of various pen-and-paper tests, hands-on activity-based tests), interviews and classroom videos. Although the data analysis was done largely through descriptive statistics, the whole process inevitably incorporated elements of inferential statistics (e.g. ANOVA and *Tukey HSD post-hoc test*) in the quest for in-depth analysis and deeper interpretation of the data. Learners were assigned to various van Hiele levels, mainly according to Usiskin's (1982) *forced* van Hiele level determination scheme. The whole process of analysing the classroom videos involved a consultative panel of 4 observers and 3 critical readers, using the checklist of van Hiele phase descriptors to guide the analysis process.

Concerning learners' levels of geometric conceptualisation, the results from this study reveal that the most of the learners were not yet ready for the formal deductive study of school geometry, as only 2% and 3% of them were respectively at van Hiele levels

3 and 4, while 47%, 22% and 24% were at levels 0, 1 and 2, respectively. More learners from the Nigerian subsample (53%) were at van Hiele level 0 than learners from the South African subsample (41%) at this level. No learner from the Nigerian subsample was at van Hiele level 4, while 6% of the South African learners were at level 4. In general, learners from the Nigerian subsample had a poorer knowledge of school geometry than their peers from the South African subsample, as learners from the latter subsample obtained significantly higher mean scores in the van Hiele Geometry Test (VHGT) and each of the other tests used in this study.

Results relating to gender differences in performance generally favour the male learners in this study.

For each of the participating schools, learners' van Hiele levels (as determined by their scores on the VHGT) strongly correlate with their performance in geometry content tests and mathematics generally. For each of the Nigerian and South African subsamples, for $n \leq 2$, learners at van Hiele level n obtained higher means on nearly all the tests administered in this study than their peers at level $n-1$. This finding provides support for the hierarchical property of the van Hiele levels.

Given the van Hiele model of geometry instruction, observed teaching methods in geometry classrooms of the South African subsample offer greater opportunities for the learners to learn geometry than observed teaching methods in geometry classrooms of the Nigerian subsample.

On the strength of the findings from this study, some tentative recommendations are made and areas for future research are specified.

DEDICATION

This thesis is dedicated to my parents

DEBORAH ATEBE

and

AUGUSTINE ATEBE

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LIST OF ABBREVIATIONS AND ACRONYMS

ADC	Academic Development Centre
ANOVA	Analysis of Variance
CDASSG	Cognitive Development and Achievement in Secondary School Geometry
CLM	Constructivist Learning Model
CPGT	Conjecturing in Plane Geometry Test
DoE	Department of Education
FRN	Federal Republic of Nigeria
GIST	Geometric Items Sorting Test
HSD	Honestly Significantly Different
IEB	Independent Examination Board
JAMB	Joint Admissions and Matriculation Board
MoE	Ministry of Education
NCS	National Curriculum Statement
NCTM	National Council of Teachers of Mathematics
NEB	National Examination Board
NPE	National Policy on Education
NS	Nigerian School
SAS	South African School
SDASS	Statistical Data Analysis Software System
SEM	School Examination in Mathematics
SVHGT	Supplementary Van Hiele Geometry Test
TIMSS	Third International Mathematics and Science Study
TPGT	Terminology in Plane Geometry Test
VHGT	Van Hiele Geometry Test
WAEC	West African Examinations Council

CHAPTER ONE

INTRODUCTION

1.1 Introduction to the study

Achievement in Science, Technology and Mathematics (STM) are increasingly recognised as one of the most reliable indicators for measuring socio-economic and geo-political development among nations (Justina, 1991). Thus, today, in modern societies the world over, there is a strong emphasis on the provision of good quality STM education (Igbokwe, 2000).

Mathematics plays a pivotal role in STM education: as Azikiwe puts it, "mathematics is the bedrock of science while science is the necessity for technological and industrial development" (Betiku, 1999, p.49). Mathematics enhances creative and logical reasoning about problems in our inherently geometric world (Clements & Battista, 1992).

In Nigeria and South Africa, mathematics is regarded as a cardinal factor in the nations' scientific and technological advancement because of its useful links to many other fields of human endeavour (South Africa, Department of Education [DoE], 1995; 2003; Federal Republic of Nigeria [FRN], Ministry of Education [MoE], 1985).

Students' mathematical competencies have been closely linked to their levels of geometric understanding (van Hiele, 1986; French, 2004). My study focuses on the geometric thinking levels of selected Nigerian and South African mathematics learners in the context of the Nigerian and South African mathematics curricula, as mandated by the Nigerian National Policy on Education (NPE) (FRN, MoE, 1998) and the South African National Curriculum Statement (NCS) Grades 10–12 (2003).

The teaching of high school geometry in many countries (including Nigeria and South Africa) was for a long period of time based on the formal axiomatic geometry (see section 2.3) that Euclid created over 2000 years ago (Greenberg, 1974; Bell, 1978; Adele, 1989; van Hiele, 1999; French, 2004). In his era, Euclid's logical construction of geometry with its axioms, postulates, definitions, theorems, and proofs was, indeed, an admirable mathematical achievement (van Hiele, 1999). However, van Hiele (1999) expresses the view that school geometry that is presented in the traditional Euclidean fashion assumes that school children also think on a formal deductive level. Empirical evidence, however, indicates that this is not the case, as many students experience difficulty with geometry when it is presented in the Euclidean way (Fuys, Geddes & Tischler, 1988; de Villiers, 1997; van Hiele, 1999).

In response to many years of students' experiencing problems with Euclidean formal axiomatic geometry, many countries (e.g. the U.S., the Netherlands and Russia) began to advocate reform in approaches to school geometry in their mathematics curriculum (Allendoerfer, 1969; Hoffer, 1983). The changes that were implemented reflected, for the most part, changes in didactics in the light of the research conducted in the late 1950s by two Dutch mathematics educators, Pierre van Hiele and his wife, Dina van Hiele-Geldof (see section 2.8).

The van Hieles were experienced teachers in a Montessori secondary school in the Netherlands who noticed with disappointment the difficulties that their learners had with geometry, particularly in formal proofs. They therefore conducted research on thought and concept development among their school children. Their work was first reported in 1957 in companion doctoral dissertations at the University of Utrecht. The van Hiele model identifies five sequential levels of thinking that learners pass through in geometry. According to the model, the learner, assisted by appropriate instructional experiences, passes through these levels in a hierarchical order, beginning with recognition of shapes as a whole (level 1), progressing to discovery of properties of shapes and informal reasoning about these shapes and their properties (levels 2 and 3), and culminating in a formal deductive and rigorous study of axiomatic geometry (levels 4 and 5) (van Hiele, 1986; Fuys et al., 1988).

In the years since 1957, the van Hiele model has motivated considerable research which has resulted in changes in geometry curricula in many developed countries. In Russia, for example, results from the van Hieles' research have been applied to the school mathematics curriculum, producing appreciable improvement in students' understanding of school geometry (Hoffer, 1983; Fuys et al., 1988). In the U.S., three similar federally-funded investigations (the Oregon Project, the Brooklyn Project, and the Chicago Project) were conducted in 1979–1982 (Hoffer, 1983). The purpose of the Oregon Project was to investigate the extent to which the van Hiele levels can serve as a model to access learners' understanding of geometry. The Brooklyn Project aimed at determining whether the van Hiele model adequately describes how students learn geometry, and implemented four instructional modules that were detailed in accordance with the van Hiele levels and phases (see Fuys et al., 1988; also see section 2.8.2 for the van Hiele phases). The focus of the Chicago Project was to determine whether the van Hiele levels are useful to predict students' achievement in standard geometry concepts and proofs (Usiskin, 1982; Hoffer, 1983). In all these projects, the van Hiele model proved to be a useful framework for accessing and unravelling students' difficulties with school geometry (Hoffer, 1983).

Despite the widespread application of the van Hiele theory to improve mathematics curricula in many Western countries, only a few studies have utilised this model in an African context. My literature search indicates that there has been little investigation involving the van Hiele model in Nigeria and South Africa, specifically. Yet more specifically, published research includes very few van Hiele studies in South Africa as a whole and the Eastern Cape in particular (e.g. de Villiers, 1994; 1997; 1998; van der Sandt & Nieuwoudt, 2003; Feza & Webb, 2005; Siyepu, 2005). And as far as I have been able to ascertain, not one study has applied the van Hiele theory to determine the level of geometric conceptualisation of Nigerian high school learners. Yet evidence abounds that many students in both countries encounter severe difficulties with school geometry (see sections 2.7.2 and 2.7.3).

In acknowledging the difficulties experienced by South African school children with geometry, and affirming the relevance of the van Hiele model in ameliorating these difficulties, de Villiers (1997, p.43), for example, asserted that "unless [and until] we [South Africans] embark on a major revision of the primary school geometry curriculum along van Hiele lines, it seems clear that no amount of effort at the secondary school will be successful". Although there is some evidence that the current South African NCS at the intermediate phase (grades 4–6) now reflects, to some extent, the van Hiele levels (Feza & Webb, 2005), the same cannot be said of the NCS at the Further Education and Training (FET) phase (Siyepu, 2005). Further, in order to embark on any major revision of the secondary school curriculum in line with the van Hiele model in an African context, it would seem necessary first to determine the van Hiele geometric thinking levels of the learners concerned. Information on the instructional practices that produced these learners would also play a useful role in the revision process. Hence, this study was undertaken to explore both the van Hiele levels of geometric thought among grades 10–12 learners in Nigeria and South Africa, and geometry classroom instructional practices that may have contributed to these levels.

It might be asked why I chose to investigate the geometric thinking levels of students in Nigerian and South African secondary schools. First, I wished to make my research study relevant not only to my own country, Nigeria, but also to the country in which I was registered for this degree. Secondly, Nigeria and South Africa share many common features. For example, both are multi-ethnic societies in which ethnicity and social stratification have been prominent in their historical development. South Africa is still grappling with the devastating effects of decades of apartheid and Nigeria is still grappling with the effects of decades of military dictatorship.

Further, Nigeria and South Africa have begun to enter into some formal partnerships in the development of human and natural resources. In October 1999, for example, South Africa and Nigeria established the South African–Nigerian Bi-National Commission (BNC), thereby formalising a strategic accord between the two countries. A major objective of the commission is to advance and hasten both countries' transformation and national reconstruction (Zuma, 2000). To aid in the understanding of a common educational challenge would make a contribution to this partnership. Moreover, their individual histories afford evidence that there are sufficient parallels between the countries to constitute a basis for this research (see Chapter 2 for the similarities in the mathematics curricula contents in both countries).

1.2 Research goals

This study sought to achieve three major goals. These are the following:

1. To explore and determine the van Hiele levels of geometric thinking of selected grade 10, 11 and 12 learners in Nigeria and South Africa;

2. To explore and explicate the possible correlations that might exist between the van Hiele levels and general mathematics achievement of the participating learners;

3. To provide information on geometry teaching in selected Nigerian and South African high schools, and hence to elucidate what possible learning opportunities observed instructional methods offer learners in geometry classrooms.

1.3 Research questions

In pursuance of the research goals, this study sought answers to and is structured around three main research questions. These are as follows:

1. What van Hiele level of geometric thinking do selected grade 10, 11 and 12 learners attain by the end of the study year in their respective grades?

2. How does a learner's van Hiele level of geometric thinking correlate with his/her achievement in school mathematics generally and in school geometry specifically?

3. What learning opportunities are evident in selected observed geometry classroom instructions in the participating schools?

Although there tends to be a one-to-one correspondence between the research questions and the research goals, it must be pointed out that the questions were not necessarily intended to set limits on what this study aimed to achieve. Rather, they were intended mainly (but not only) to provide a sharper focus for achieving the broader goals of this study.

1.4 Methodology

Seeking as it does to understand the participants' subjective world of geometry classroom experiences through direct engagement, this study is oriented within an interpretive research paradigm (Lincoln & Guba, 1985; Connole, 1998; Terre Blanche & Kelly, 1999; Schwandt, 2000). It utilises both quantitative methods (in the form of basic descriptive statistics) and qualitative methods (in the form of interviews and classroom video study analyses) (Creswell, 2003) in its attempt intensively to study, describe and interpret (Schunk, 2004) participating learners' van Hiele geometric thinking levels, as well as instructional methods in geometry classrooms. The study makes use of data from diverse sources (Cohen, Manion & Morrison, 2000) to ensure credibility (Lincoln & Guba, 1985; Denzin, 1988).

This research can also be described as a collective case study (Stake, 2000) conducted in Nigeria and South Africa. It employs purposive, simple random and stratified sampling techniques (Cohen et al., 2000) to select a cohort of 144 high school mathematics learners (from across grades 10–12) and 6 mathematics teachers for involvement in the study (see sections 3.3.1 and 3.3.2). In order to answer the research questions and, hence, meet the goals of this study, various data gathering techniques were employed. These included questionnaires (in the form of both traditional pen-and-paper tests and hands-on activity tests), interviews and classroom videos (see Chapter 3).

1.5 Significance of the study

This study is unique and significant on many counts. Because the significance of the study is described in some detail in section 10.4, the present section only highlights some aspects of this.

This study is significant and novel as it represents, as far as I have been able to ascertain, the first scholarly attempt simultaneously to compare the mathematical performances of high school learners and teaching methods in geometry classrooms in Nigeria and South Africa using the van Hiele model. It is of great value, if for no other reason, because it furnishes a baseline of comparison for subsequent studies.

There is a paucity of published research in which both aspects of the van Hiele theory (i.e. the thought levels and the instructional cycles) have been investigated in a single study, particularly in an African context (see, for example, Hoffer, 1983). This study thus owes its unique and significant attributes to being the first, as far as I am aware, that attempts to link learners' exhibited van Hiele levels to their instructional experiences in geometry classrooms in Nigeria and South Africa. It must quickly be acknowledged that the result of being comprehensive has, however, added to the volume of this thesis.

Furthermore, there appears to be a dearth of empirical evidence in the literature linking students' van Hiele levels with their mathematical knowledge in general (see, for example, Senk, 1989). By correlating learners' van Hiele levels with their general mathematical performance, this study has made a significant contribution towards closing the perceived gap in the existing literature. In addition, the finding that learners' van Hiele thought levels correlate significantly with their performance in school mathematics as a whole (see Chapter 8) should be of interest both to mathematics educators and curriculum developers. In order to improve the mathematical performance of their learners, teachers could, for example, attempt first to raise their van Hiele geometric thinking levels through the instructional cycles of the van Hiele model.

1.6 Limitations of the study

This study only represents portraits of selected learners' mathematical performance and of teaching methods in geometry classrooms in Nigeria and South Africa based on the van Hiele model. It does not claim to have captured and related the entire story about learners' van Hiele geometric thinking levels, nor does it purport to discuss instructional practices that represent the whole educational landscapes of the countries concerned. Consequently, as is typical with case studies, caution should be exercised in extrapolating and generalising from the findings of the study. Nevertheless, given the in-depth descriptions of the cases treated in this study, it is hoped that many of the results obtained in the research will resonate in similar contexts. A more comprehensive discussion of the limitations of this study is presented in section 10.5 of the last chapter.

1.7 Thesis overview

This thesis is presented in two volumes. Volume 1 is the *main body* of the thesis, beginning with the title page and ending with the list of references. Volume 2 comprises the Appendices, and it is intended to contribute to our understanding of the numerous data collection tools and how these contributed to the various findings reported in Volume 1.

Chapter 2 contextualises the study within the relevant literature and provides its theoretical underpinning. The chapter begins with a review of the several conceptions of geometry and how each of these relates to the study. This is followed by a discussion of Euclidean geometry as an important, even dominant subject in high school mathematics curricula in many countries, including attention to aspects of its historical development. Next, the study of school geometry is problematised against the objectives of geometry teaching and learning in the Nigerian and South African contexts. An examination of the conceptual difficulties encountered by learners in school geometry and their manifestations (e.g. misconceptions, imprecise terminology and classification of shapes) is then presented. This is followed by discussion of the three major causal aspects of students' poor performance in geometry – curricular, textual, and instructional factors – as these are identified in the literature.

Chapter 2 also presents the van Hiele model of geometric thinking levels as the overarching theory informing the study. It then discusses both aspects of the van Hiele theory (i.e. the thought levels and the instructional phases) in an attempt to identify and specify the frameworks within which data collected in this study were analysed.

The chapter ends with a critique of the van Hiele theory and explains how the major criticisms of the theory were addressed in this study.

Chapter 3 describes the research methodology. It specifies the paradigm within which the study is located, its overall design, the research process, and the techniques employed. The chapter also explains the procedures for data collection and analysis, and further highlights the validity and reliability measures adopted. Issues relating to research ethics (e.g. participants' rights to confidentiality, anonymity and informed consent) and how these were handled in this study are also discussed in this chapter.

Chapters 4 – 7 present and discuss the results of the analyses of learners' performance in both the pen-and-paper tests and hands-on activity test (see Chapter 3 for these tests), and relate the findings to the literature reviewed in Chapter 2. The analyses of the results presented and discussed in these chapters focus on school, grade level as well as gender differences in performance of the participating learners. These chapters seek answers to the first research question, and hence attempt to achieve the first goal of this study.

Chapter 8 provides a comprehensive analysis of the results of the correlations between learners' van Hiele levels and their performance in school geometry as well as in mathematics generally. The analysis of the results presented in this chapter also includes a determination of whether learners at adjacent van Hiele levels performed significantly different from each other on geometry content tests and on school mathematics examinations. This chapter attempts to realise the second goal of this study by seeking answers to the second research question.

Chapter 9 provides the results of the analysis of instructional methods in Nigerian and South African geometry classrooms and further interrogates and discusses the possible learning opportunities that observed teaching methods offer the learners to learn geometry. The chapter also attempts to relate learners' exhibited van Hiele levels to their instructional experiences in geometry classrooms. Chapter 9, therefore, seeks to answer the third research question so as to achieve the third goal of this study.

Chapter 10 is the concluding chapter and it provides a synopsis and summary of the major findings of the study. The chapter highlights the findings concerning the van Hiele geometric thinking levels of the participating learners and the instructional practices evident in the observed geometry classrooms. It further outlines the significance of the study and articulates its limitations, offers some recommendations, and ends with a final word of personal reflection on the whole research process.

Volume 2 consists of the Appendices which are documentations of the entire tools used for data collection in this study. It contains the letters with which access to the research sites was negotiated, the various testing instruments and the data each yielded, as well as the transcripts of the videotaped geometry lessons.

CHAPTER TWO

THEORY UNDERPINNING THE STUDY

2.1 Introduction

In the course of searching for relevant literature to support my study, I found it convenient to distinguish among three broad categories of studies in geometry (or more generally, mathematics) education. These are:

1. Those concerned with the formulation of theories, for example, Piaget and Inhelder (1969), van Hiele (1986).
2. Those that focus on theory verification, for example, Hoffer (1981), Usiskin (1982), Burger and Shaughnessy (1986), Fuys et al. (1988).
3. Those that deal with the application of theories, for example, Mayberry (1983), Shaughnessy and Burger (1985), Senk (1989), Feza and Webb (2005)..

It should, however, be noted that these three categories implicitly entail each other and should not be construed as occupying discrete compartments.

The third category distinguished above is of particular importance on two accounts: First, there are benefits to be obtained from applying a theory in a particular context through an intervention program. The results of such an application could yield insights enabling improvement of the status quo. Secondly, during the application of a theory in a given context, further insight about the phenomenon being studied could be gained, which could then inform either a refinement of the existing theory or the formulation of a new theory. It is in this third category of educational studies that I situate my study. As pointed out in chapter 1, since very few studies have utilised van Hiele's model of geometric thinking in Nigeria and South Africa, this project is of particular importance to these countries.

In this study, I seek to explore students' van Hiele levels of geometric thinking in relation to mathematics achievement and geometry classroom instruction in an

African context. The review of the literature pertinent to my study, therefore, begins with an exploration of the concept of geometry. Second, I discuss the enviable place of geometry in secondary school mathematics, both internationally and in the Nigerian and South African contexts. Third, a review of some research evidence illustrating the difficulties encountered by students in school geometry is presented. Fourth, I discuss some earlier models suggested for the teaching and learning of school geometry. Finally, I present the van Hiele theory and identify it as the conceptual basis for the present study.

The main purpose of this literature review, therefore, is properly to situate my study within the context of current research grounded in an ever-growing number of theories of teaching and learning.

2.2 The Concept of Geometry

> Geometry is the Science which treats of the shape, size and position of figures: it is based on definitions, axioms and postulates: these granted, all the rest follows by pure reasoning.
>
> Nixon (1887, p.1).

Considering the difficulties associated with any attempt to define a concept (Orton, 2004), and the fact that most concepts are better understood through the listing of a few examples, one might wish to conclude that providing such a definition is not necessary (van Hiele, 1986). This is particularly true of the concept of geometry, given its sheer extent as a field of mathematical study. Nevertheless, in order to give my study a sharp focus and to provide common ground for the understanding among various readers of the concept of geometry as it relates to this study, I deem it expedient to examine a few definitions of geometry.

Borowski and Borwein (1989, p.246) conceptualize geometry as "the elementary study of the properties and relations of CONSTRUCTIBLE [emphasis in the original] plane figures". It is the specific mathematical axiomatization of the properties and relations of plane shapes as studied, for example, under Euclidean geometry. An aspect of my study utilizes Borowski and Borwein's notion of geometry by exploring, through geometrical construction, students' understanding of the properties and relations of simple geometrical shapes, like triangles, squares, rectangles, rhombuses, trapeziums and circles (see section 3.3.4.1.3 of Chapter 3).

Generally, geometry is the study of the properties of spatial objects and the relations between those properties. Van Hiele (1986, p.60) describes geometry, seen as a science, as a form of study in which "we [as teachers] have no concern for space, nor for geometric figures in space, but [rather] only for the relations between properties of those figures". In terms of this notion of geometry, van Hiele (1986, p.76) proposes an "intuitive introduction" to the study of geometry in which learners are given the opportunity for direct observation/manipulation of geometric figures, such as triangles and quadrilaterals, so as to enable them to abstract the relations between the properties of those shapes. A good part of my study explores students' ability to recognize, describe, classify, and abstract properties of triangles and quadrilaterals, based on direct observation/manipulation of those shapes in the form of cardboard cut-outs.

Clements and Battista (1992, p.420) offer a very formal and highly comprehensive definition of school geometry by describing geometry as the "study of spatial objects, relationships, and transformations that have been formalized (or mathematized) and the axiomatic mathematical systems that have been constructed to represent them". In linking geometry with spatial reasoning, Clements and Battista (ibid.) state that spatial reasoning consists of the set of intellectual processes through which mental representations of spatial objects and the relationships between the properties of those objects are constructed and manipulated. This for me seems to indicate that geometry and spatial reasoning are two interrelated mathematical concepts, with the latter, for the most part, a tool for exploring the former. In this study, learners' understanding of spatial/geometrical objects and the relationships between the properties of those objects are explored using various research instruments.

The *Chambers Dictionary* (1998) defines geometry as "that part of mathematics which deals with the properties of points, lines, surfaces and solids, either under the classical Euclidean assumptions, or (*in the case of elliptic, hyperbolic, etc geometry*) involving postulates not all of which are identical with Euclid's". In this study, the properties of lines such as parallelism, perpendicularity and angle relations with respect to two-dimensional plane figures are explored.

The conceptions of geometry offered above appear to show that there are different approaches to the study of geometry. Indeed there are many approaches ─ Euclidean/synthetic, analytic/coordinate, transformational, even vectorial─ but when the term "school geometry" is used, it is almost universally understood to mean Euclidean geometry (Clements & Battista, 1992, p.420). Hence the excerpt at the beginning of this subsection refers to the study of a body or system of mathematical enquiry commonly referred to as Euclidean geometry. The pedagogical preference for Euclidean geometry might be due to the highly logical system of deductive reasoning that it promises to develop in learners (Suydam, 1985; Filimonov & Kreith, 1992).

The various conceptions of geometry incorporate a number of technical terms that are used and understood in different ways. It is hoped that an examination of the structure of Euclidean geometry may provide some useful pointers to a common understanding of some of this terminology.

2.3 The Structure of Euclidean Geometry: An Historical Perspective

> Thirty years ago high school plane geometry differed little from the geometry which Euclid unified and structured about 300 B.C. For two thousand years after Euclid, geometry to mathematicians was Euclidean geometry, and for twenty-two hundred years the geometry studied by students was that of Euclid.
>
> Bell (1978, p.78)

A discussion of the structure of Euclidean geometry would seem imperative, if only because its study dominated the mathematical world for over 2000 years (Adele, 1989). Euclidean geometry is a mathematical system that was developed by a Greek mathematician, Euclid of Alexandria, around 300 B.C. (Greenberg, 1974; Adele, 1989). In his seminal work, *The Elements*, Euclid developed a formal and somewhat rigid approach to the study of geometry that relied almost exclusively on logico-deductive reasoning.

Euclid adopted a method that consisted of assuming a small set of intuitively appealing axioms from which many other theorems (or propositions) could be proved (Casey, 1889). First, Euclid gave a list of definitions, and followed this up with five postulates and five axioms (Euclid, 1952). From these, Euclid deduced a number of propositions, and these constituted the geometry that was studied for over two millennia (Eves, 1972; Bell, 1978). Euclidean geometry was thus indeed a synthetic geometry in that it made use of definitions, axioms and theorems to establish mathematical truth in geometry (French, 2004). Euclidean geometry relied largely on a set of well reasoned and highly logical axioms, postulates and deductions in proving propositions or theorems. This postulational approach (even though it has been modified) is still the approach in terms of which high school geometry is studied in many countries, including Nigeria and South Africa (Bell, 1978).

Euclid's remarkable achievement in geometry is attributed to his success in singling out a few postulates, from which he deduced no less than 465 propositions that contained "all the geometric knowledge of his time" (Greenberg, 1974, p.9). These propositions are contained in his book, *The Elements*.

Euclid's *Elements* is divided into thirteen Books. The first six of these Books are on elementary plane geometry (Boyer, 1968). Book 1 opens, without any preamble, with a list of 23 definitions, 5 postulates and 5 axioms (or common notions, in Euclid's terms), from which Euclid deduced 48 propositions (Euclid, 1952; Boyer, 1968). The first 26 propositions deal mainly with the properties of triangles (Eves, 1953).

As stated earlier, for over two millennia Euclidean geometry was the only type of geometry that was studied in high schools in many countries (Greenberg, 1974; Bell, 1978). In the U.K., for example, it dominated the mathematics curriculum until the end of the nineteenth century (French, 2004), and in South Africa, it was the only geometry that was studied in many schools (de Villiers, 1997). My experience as a mathematics teacher in Nigeria for over a decade reveals that Euclidean geometry similarly forms the core of the geometry that is studied in secondary education in that country. Algebra had little place in geometry until the seventeenth century when Descartes (1596–1650) created analytic geometry, which has afforded an alternative approach to the study of geometry in many countries even to this day (Struik, 1967; Boyer, 1968; Adele, 1989; French, 2004).

Although Euclidean geometry dominated the mathematical world for over 2000 years, the end of the nineteenth century and early part of the twentieth century witnessed the development of forms of non-Euclidean geometry (Eves, 1953). Euclidean geometry came under severe criticism due to the controversies that surrounded the parallel postulate (see section 2.4.5.). The existence of Euclidean geometry was thus threatened, as mathematicians showed that it was possible to create geometries other than Euclid's through which mathematical truths could be established (Gittleman, 1975).

Despite these threats, Euclidean geometry has survived (see section 2.7.1.1), and today remains a core subject in many mathematics curricula around the world, including those of Nigeria and South Africa (de Villiers, 1997). It should, however, be noted that although Euclid's postulational geometry is still being studied in many countries (Netherlands, Russia, U.K., U.S., Nigeria and South Africa), strict adherence to Euclid's axiomatic approach has either been modified or replaced with alternative approaches (Bell, 1978; de Villiers, 1997; French, 2004). Before taking the discussion to non-Euclidean geometry, however, it is necessary to clarify some of the terminology that is likely to be encountered in this study. In discussing this terminology (sections 2.4.1–2.4.5), reference is made mostly (but not exclusively) to ancient authors in order further to spotlight the historical significance (section 2.3) and centrality of geometry in the mathematics curriculum (section 2.5.2), even in times far removed from the present.

2.4 Terminology

2.4.1 Plane Geometry

This is the geometry in two dimensions commonly referred to as Euclidean geometry that is taught in Nigerian and South African schools. Plane geometry investigates the properties of and the relationships between plane figures (Borowski & Borwein, 1989). Plane figures are two-dimensional shapes that are described by straight lines or curves (Nixon, 1887). Examples of plane figures include circles, triangles, quadrilaterals and other polygons. Figures that are bounded by straight lines are called rectilineal figures, and with the exception of the circle, all figures in Euclidean geometry are rectilineal (Deighton, 1886).

The first six Books of Euclid's *Elements* (see section 2.3) are on plane geometry. Of the 48 propositions in Book 1, the first 26 treat the properties of triangles. Propositions 27 through to 32 concern the theory of parallels and the proof of the angle sum of a triangle. According to Eves (1953), the rest of Book 1 focuses on parallelograms, triangles and squares. Book 3 concerns circle theorems and chord and tangent properties of circles. The theory of proportion is the subject of Book 5, while Book 6 focuses on similar triangles (Eves, 1953).

In sum, plane/Euclidean geometry concerns the study of such geometrical structures as points, lines, angles and plane figures. In this dissertation, plane geometry refers to the study of the properties of and relations between four different sets of geometrical configurations within the ambits of Nigerian and South African curricular prescriptions for senior secondary education. These geometrical configurations are discussed in subsections 2.4.1.1 through 2.4.1.4.

2.4.1.1 Lines

A line, according to (Deighton, 1886), has neither breadth nor thickness, but length only. Euclid (1952, p.1) defines a line as "breadthless length". With the exception of the line forming a circle, lines in this study refer to both intersecting and non-intersecting straight lines, be they parallel or perpendicular.

2.4.1.2 Triangles

Euclid (1952) refers to triangles as trilateral figures. According to Nixon (1887, p.2), "if three straight lines are drawn in a plane so as to intersect two and two, the plane figure formed is called a triangle". In this study, a triangle refers to any plane figure bounded by three straight lines (whether drawn on paper or cut out of cardboard). Triangles are described in terms of both side and angle properties. For example, a triangle could be described as a right-angled isosceles triangle if it is a right-angled triangle in which two of the sides are equal, or if it is an isosceles triangle with one of its angles a right angle.

2.4.1.3 Quadrilaterals

A quadrilateral is a rectilineal figure contained by four straight lines (Euclid, 1952). For the purpose of this study, any plane figure (drawn or cut-out) bounded by four straight lines is called a quadrilateral. The quadrilaterals explored in this study are squares, rectangles, rhombuses, parallelograms and trapeziums.

2.4.1.4 Circles

A circle is a plane figure whose boundary points are equidistant from a fixed point within it (Nixon, 1887). In this study, a circle (drawn or cut-out) is similarly defined.

2.4.2 Definition

A definition states the meanings which are to be attached to certain words, concepts and geometrical configurations (Nixon, 1887). A definition, once given, provides a ground for common interpretation and understanding about the sense or usage of a word or concept among a group of users. In his *Elements*, Euclid (1952) provided definitions for a number of geometrical terms prior to stating his 465 propositions. In this study, to define a geometrical term means to state the discernible general characteristics of such a term, or to list the distinguishing properties of such a term or geometrical configuration. The definitions explored in this study are those of triangles and quadrilaterals of the various types.

2.4.3 Postulate and Axiom

There appears to be a general tendency for contemporary mathematicians not to make a distinction between an axiom and a postulate. The early Greeks, however, did differentiate between the two terms. According to Eves (1972, p.17), “an axiom is an initial assumption common to all studies, whereas a postulate is an initial assumption pertaining to the study at hand”. Boyer (1968) describes an axiom as something known or accepted as an obvious truth, while a postulate is less obvious and does not presuppose the assent of the learner.

Deighton's (1886) notion of axioms is that axioms are theorems that are accepted without demonstration. Greenberg (1974) considers axioms and postulates as synonyms and states that an axiom (or a postulate) is a statement requiring no justification before being accepted as true, true in the sense that it is logically correct. This means that an axiom (or a postulate) is a statement that the learner already knows and accepts as logically correct. From this knowledge and acceptance, the truth of other statements (propositions or theorems), hitherto unaccepted, can be established.

The literature seems to indicate that Euclid himself did not make any essential distinction between axioms (which he called common notions) and postulates (Eves, 1953; Boyer, 1968). Nevertheless, Euclid's five common notions (or axioms) appear to be more general and obvious truths than his five postulates, especially, the fifth postulate (see section 2.4.5). Euclid's (1952) postulates and axioms as contained in his *Elements* are as follows:

Postulates:

1. it is possible to draw a straight line from any point to any point.
2. it is possible to produce a finite straight line continuously in a straight line.
3. it is possible to describe a circle with any centre and distance.
4. all right angles are equal to one another.
5. if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which the angles are less than the two right angles.

Axioms (or Common Notions):

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

It was from the above axioms and postulates alone that Euclid synthetically deduced all the 465 propositions contained in his *Elements* (see section 2.3).

2.4.4 Theorem

A theorem is a hypothesis in which a specified conclusion has to be demonstrated (Nixon, 1887). Borowski and Borwein (1989, p.589) define a theorem as "a statement or formula that can be deduced from the AXIOMS of a formal system by recursive application of its RULES OF REFERENCE" (emphasis in the original). Mogari (2002) states that a theorem is a statement that is accepted if the grounds on which it is based are adequate for its assertion. This means that a theorem is an assertion that has to be proved or demonstrated through the use of axioms and postulates. The truth of a theorem can be demonstrated by a sequence of deductive reasoning or experimentation. Euclid made use of the former in a synthetic fashion.

In Nigeria and South Africa, as in many other countries, theorems in high school Euclidean geometry are verified largely by employing a logical sequence of deductive reasoning. In this study, problems involving the proofs of theorems required the learners to apply this deductive (or synthetic) approach.

2.4.5 The Parallel Postulate and the Development of Non-Euclidean Geometries

The fifth of Euclid's postulates (see section 2.4.3), commonly referred to as the Euclidean parallel postulate, deserves a special mention, because consideration of alternatives to it resulted in the development of non-Euclidean geometries (Greenberg, 1974). The fifth Euclidean postulate, unlike the first four, lacks the characteristic of being "self-evident", and for over two thousand years, geometers were preoccupied with attempts either to derive it from the other postulates and axioms or at least to replace it with a more acceptable equivalent (Eves, 1953, pp.122–123). Of the several substitutes considered, the one most commonly used is *Playfair's postulate*, named after John Playfair (1748–1819), who published his work on Euclidean geometry in 1795 (Eves, 1953; Greenberg, 1974). According to Eves (1953, p.123), the *Playfair's postulate* states that "*Through a given point* [not on a given line] *can be drawn only one line parallel to* [the] *given line*" (emphasis in the original). Many high school geometry texts state the Euclidean parallel postulate in Playfair's formulation.

The controversy that has surrounded the Euclidean parallel postulate is that the fifth postulate, unlike the first four, neither lends itself readily to empirical verification nor can it be readily abstracted "from our experiences" (Greenberg, 1974, p.17). Euclid had postulated that lines *m* and *l* (Figure 2.1), if produced sufficiently far (or indefinitely), will meet on that side of line *t* for which $\alpha + \beta < 180°$.

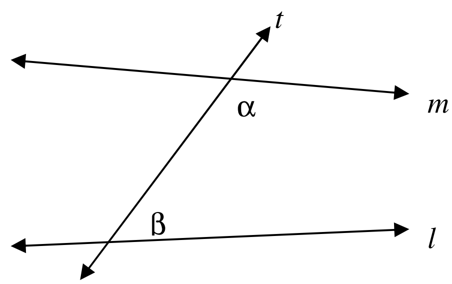


Figure 2. 1 Illustrating Euclid's parallel postulate

Many geometers have argued that the validity of this Euclidean assertion may depend upon the surface on which one is working, and therefore requires empirical investigation (Mogari, 2002).

Although several attempts were made to derive the parallel postulate from Euclid's other postulates and axioms, it was only in 1733 that Saccheri (1667–1733) published what could be termed the first scientific investigation of the postulate (Eves, 1953). Saccheri adopted a method of indirect proof commonly referred to as *reductio ad absurdum* (Eves, 1953, p123). By this method, Saccheri first assumed the postulate to be false and then worked toward a contradiction (Boyer, 1968; Burton, 1985; Mogari, 2002). Saccheri's proof, according to Eves (1953), is as follows: If in a quadrilateral ABCD (Figure 2.2), angles A and B are both right angles, and sides AD and BC have equal measure, then angles D and C are equal with three possibilities:

1. Angles D and C are both equal acute angles;
2. Angles D and C are both equal right angles; and
3. Angles D and C are both equal obtuse angles.

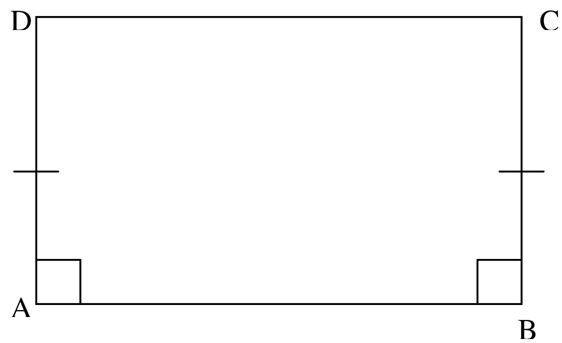


Figure 2. 2 Saccheri's proof of Euclid's parallel postulate

According to Eves (1953, pp.123–124), Saccheri referred to the above three possibilities respectively as "the *hypothesis of the acute angle*, the *hypothesis of the right angle*, and the *hypothesis of the obtuse angle*" (emphasis in the original). Saccheri's grand plan was to show that the hypotheses of the acute angle and that of the obtuse angle will both lead to a contradiction, and so, by *reductio ad absurdum*, the hypothesis of the right angle will carry with it a proof of the parallel postulate (Eves, 1953). Although Saccheri succeeded in eliminating the hypothesis of the obtuse angle, he would not admit his failure to find a contradiction in the case of the hypothesis of the acute angle. Instead, Saccheri, being too eager to arrive at a contradiction, twisted reasoning and forced a contradiction into the development of his proof even when this was not evident (Eves, 1953).

Saccheri, nevertheless, obtained results that proved consistent with Euclidean geometry, and had he not twisted reasoning by forcing into his proof an unconvincing contradiction, he would unquestionably have been credited with the discovery of non-Euclidean geometry (Eves, 1953).

Other notable eighteenth-century mathematicians who attempted the proof of the parallel postulate, and hence contributed to the development of non-Euclidean geometries, were Johann Heinrich Lambert (1728–1777), a German, and Adrien-Marie Legendre (1752–1833), a Frenchman (Boyer, 1968). According to Mogari (2002, p.50), Lambert's contribution was his success in the use of "an acute angle hypothesis to show that on spherical surfaces the sum of [the] angles of a triangle is inversely proportional to the area of the triangle". Although Legendre was unable to eliminate the acute angle hypothesis, he nevertheless obtained some useful results that he published in his book entitled "*Éléments de géométrie*" in 1794, and which were adopted as a substitute for Euclidean geometry in Central Europe and the USA (Eves, 1953, p.125).

It is now known that the geometry that is developed from a collection of axioms constituting a basic set together with the acute angle hypothesis is as consistent as the Euclidean geometry developed from the same basic set together with the hypothesis of the right angle. Therefore, the parallel postulate is independent of the other postulates, and cannot be derived from it (Eves, 1953). According to Eves (1953), Johannes Bolyai (1802–1860), a Hungarian, and Nicolai Ivanovitch Lobachevsky (1793–1856), a Russian, were the first to discover this fact. Working independently, both men carried out an extensive development of the acute angle hypothesis which culminated in the creation of a consistent non-Euclidean geometry.

The hypothesis of the obtuse angle was further investigated and from it was invented another consistent non-Euclidean geometry by Riemann in 1854 (Boyer, 1968).

Eves (1953) states that the above three geometries – the one developed by Euclid, the one developed by Bolyai and Lobachevsky, and the one developed by Riemann – are respectively referred to as parabolic geometry, hyperbolic geometry, and elliptic geometry. In Nigeria and South Africa, elementary and high school geometry is largely parabolic (i.e. Euclidean), and it is this kind of geometry that forms the focus of investigation in this study. Krause (1986) believes that a basic knowledge of Euclidean geometry is in any event necessary for the study of non-Euclidean geometries.

2.5 Geometry within the Wider Framework of School Mathematics

> The arguments for including geometry in the mathematics curriculum are closely linked to the reasons why mathematics as a whole is studied. It has been increasingly recognised that mathematics should have a central place in the education of all students and that geometry in some form has a vital role in the wider mathematics curriculum.
>
> French (2004, p.2).

The above excerpt from French pleads for a brief examination of the importance of mathematics as the grounding for a full appreciation of the usefulness of geometry in the overall mastery of mathematics. Geometry and mathematics appear to be inextricably united in their power to promote the development of logical reasoning in learners (French, 2004).

2.5.1 Importance of Mathematics

It would seem that mathematics is a universal subject, and in all cultures, one form of mathematics or the other has been studied over the ages (D'Ambrosio, 1997). All cultures appear to have accepted the general belief that knowledge of mathematics in some form is important for the training of the individual and for the development of the society because of its utilitarian values (South Africa, DoE, 2003). Kleiman (1995), for example, believes that to be human entails an ability to think creatively and communicate effectively, and that mathematics provides both a vehicle for creative thinking and a language for effective communication.

The view that mathematics is necessary for human and societal development dates back to antiquity. Plato (427–348 B.C.), for instance, stated that "the study of mathematics develops and sets into operation a mental organism more valuable than a thousand eyes, because through it alone can truth be apprehended" (Greenberg, 1974, p.7). According to Greenberg (*ibid.*), Plato believed that the universe of ideas is more important than the material world of the senses, and that the errors of the senses must be corrected by concentrated thought which, in Plato's view, is best acquired through the study of mathematics. Mathematics thus trains the mind to think, and to think creatively, an ability of great importance in all human endeavours.

Children are today expected to demonstrate a high level of competence in mathematics because it is generally regarded as one of the most important subjects in the school curriculum. In the U.K. (just as in Nigeria and South Africa), for example, the National Curriculum has always designated mathematics (not exclusively, though) as a 'core' subject (in South Africa, a 'fundamental' subject) (Orton & Frobisher, 1996).

There appears to be consensus among authors that mathematics is highly esteemed in all cultures because of the common belief that it fosters the development in learners of logical thinking and problem-solving abilities (Coxford, 1995; Hodgson, 1995; Kleiman, 1995; Orton & Frobisher, 1996; Jones, Langrall, Thornton & Nisbet, 2002). According to Orton and Frobisher (1996), and Kleiman (1995), the reasons why mathematics is considered a very important subject in the school curriculum may be summarised as follows:

- Mathematics has direct applications in a variety of real-life everyday human experiences, for example counting, locating, measuring, designing, building and so forth.
- Mathematics is important because of its value in enhancing the development in learners of critical and logical thinking, and problem-solving skills.
- Mathematics (self-evidently) fosters the development of basic mathematical skills. According to Sherard (1981, p.19), basic skills in mathematics enable learners "to function successfully as informed consumers, as concerned citizens, and as competent members of the working force".
- Mathematics aids communication. It is rich in vocabulary and therefore gives precision to our descriptions of our inherently geometric world.
- Mathematics underpins advances in science and technology, and this has led to its dominance in the school curricula of many countries (Atebe, 2005).
- Mathematics is also important because it forms part of man's cultural heritage. Mathematical symbols (and to a lesser extent, concepts) have evolved differently in different cultures (Oliver, 2003). In South Africa, for example, the decimal marker (.) is represented as a comma (,) different from how it is symbolized in Nigeria and many other countries. Thus, a South African grade 10 learner, very likely, would express, for example, the fraction, $\frac{1}{4}$ as 0, 25 as a decimal fraction, while a Nigerian grade 10 learner would express it as 0.25. It is probable that this symbolism has some cultural antecedence in South Africa.
- Lastly, mathematics is important for its aesthetic values. It can be enjoyed for its beauty and elegance (Orton & Frobisher, 1996).

It would appear from the foregoing discussion that nations differ only slightly in terms of the overall objectives of mathematics teaching in schools. Atebe (2005) expresses the view that since the learners for whom curriculum objectives are developed are expected to live in and contribute to the development of the society, curriculum objectives inevitably take into account the needs of the learners and the kind of society that is envisaged.

2.5.1.1 Objectives of Mathematics Teaching in Nigeria

According to Badmus (as cited in Atebe, 2005, pp.93–102), the objectives of mathematics teaching in Nigerian schools as outlined in the National Mathematics Curriculum are:

a) Developing originality, creativity and curiosity in the learners.
b) Acquisition of manipulative skills.
c) Discovering and appreciating the beauty and elegance of mathematics.
d) Demonstrating the applicability of mathematics to various fields.

Badmus (1997), elaborating on the above objectives, identifies the following as the general objectives of secondary school mathematics teaching in Nigeria:

i. To generate an enduring interest in mathematics and to lay a solid foundation for everyday living.
ii. To promote the acquisition of the mathematical skills and processes necessary for further education in mathematics and related fields.
iii. To apply the knowledge acquired in mathematics to solve the numerous and ever-increasing problems of human life.
iv. To foster the desire and ability to be accurate to a degree relevant to the problem at hand.
v. To develop the ability to engage in, and practise logical and abstract thinking.
vi. To stimulate and encourage creativity.
vii. To develop computational skills.

According to Obioma (as cited in Atebe, 2005), all the above objectives are aligned with the aim of fulfilling four major aspirations: personal, utilitarian, social, and cultural.

2.5.1.2 Objectives of Mathematics Teaching in South Africa.

It would seem that the objectives of mathematics teaching in South Africa are mostly consistent with those in Nigeria. In South Africa, the study of mathematics is largely predicated upon the belief that "mathematics enables creative and logical reasoning about problems in the physical and social world and in the context of mathematics" (South Africa, DoE, 2003, p.9).

The objectives of mathematics teaching in South Africa could be drawn from three inter-related headings outlined in the Revised National Curriculum Statement, Grades 10–12 (General). These are *purpose*, *scope*, and *educational and career links* (South Africa, DoE, 2003). A careful examination of the contents of these broad headings suggests that the objectives of mathematics teaching in South African are as follows:

- To develop learners' mathematical process skills that would empower them to make sense of the society, function successfully in the society, and contribute meaningfully to the development of the society.
- To develop learners' capacity for the creative and logical reasoning necessary for identifying, posing and solving real-world problems.
- To develop a repertoire of mathematical vocabulary that would enable the learners to communicate appropriately, whether verbally or pictorially.
- Developing in the learners what French (2004, p.3) calls mathematical "habits of mind", which entails the ever-growing mathematician's quest for conjecturing, investigating, proving, and generalizing.
- To enable the learners to acquire the knowledge and skills needed for further education.
- To develop learners' manipulative skills. This includes the manipulation of physical objects as well as the mental manipulation of concepts and images.

The above objectives comprise only an outline of the overall objectives of mathematics teaching in South Africa. For more details, reference should be made to the Revised National Curriculum Statement Grade 10–12 (General).

2.5.2 Importance of Euclidean Geometry

Let no one destitute of geometry enter my doors.
Plato (427–348 B.C).

Historically, the literature reveals that although geometry was developed with applications to measure the earth (Clements & Battista, 1992), Euclid in his *Elements* did not stress the practical utility of his geometry (Greenberg, 1974). Evidence could be drawn from the legend often told about Euclid concerning a beginning student of geometry who confronted him with this question: "what shall I get by learning these things?" Euclid, it is told, called his slave saying, "Give him a coin [Boyer, three pence], since he must make gain out of what he learns" (Eves, 1953, p.111; Boyer, 1968, p.111; Greenberg, 1974, p.7). Presumably, geometry was studied mainly for its aesthetic values – an attitude still held by many contemporary mathematicians (Greenberg, 1974). The inscription at the entrance to Plato's rooms – "Let no one destitute of geometry enter my doors" – nonetheless illustrates unequivocally the central role that geometry played in the mathematics enterprise of that era (Adele, 1989, p.461).

Euclidean geometry has, however, undergone remarkable refinements in many countries such as Russia, the Netherlands, the U.K., the U.S., Nigeria and South Africa, with the result that the rigid (and sometimes slavish) adherence to Euclid's axioms and postulates has been relaxed. Nevertheless, the fundamental structure of geometry "as a postulational mathematical system" is still retained, albeit with a shift of emphasis "to applications of the inductive and deductive techniques of geometry in [real-world] situations" (Bell, 1978, p.78; de Villiers, 1997; French, 2004). Perhaps these reforms account for the survival and continued dominance of Euclidean geometry in many school mathematics curricula across the world.

There seems to be a general consensus in the literature that the major objective of geometry teaching in schools is to "develop [students'] logical thinking abilities" (Hoffer, 1981, p.12; Suydam, 1985, p.481; French, 2004, p.2). It is this objective that underlies the reason for including geometry in the mathematics curriculum, and it parallels the major objective of mathematics teaching in schools (see sections 2.5.1.1 and 2.5.1.2). Geometry is thus a central component of the school mathematics curriculum in Nigeria, South Africa and elsewhere (FRN, MoE, 1985; South Africa, DoE, 2003), and cannot be separated from the mathematics curriculum as a whole (van Hiele, 1986; French, 2004).

Understanding geometry is an important mathematical skill since the world in which we live is "inherently geometric" (Clements & Battista, 1992, p.420). In the U.S., for example, the National Council of Teachers of Mathematics' working document, *Curriculum and Evaluation Standards for School Mathematics*, NCTM (1989, p.112), states that geometry "helps students [to] represent and make sense of their world". The Council further states that geometry is an important school subject because it provides perspectives for developing students' deductive reasoning abilities and the acquisition of spatial awareness (NCTM, 1989). Improving learners' geometric thinking levels is one of the major aims of mathematics education since geometric thinking is very important in many specific, technical, and occupational areas (Hoffer, 1981; Olkun, Sinoplu & Deryakulu, 2005).

In 1976, the National Council of Supervisors of Mathematics (NCSM) in the U.S. identified geometry as a basic skill in mathematics (Sherard, 1981). Basic skills in mathematics, according to Sherard (1981, p.19), should "be sufficient for our students so that they can function successfully as informed consumers, as concerned citizens, and as competent members of the working force". Sherard (*ibid.*) states that the NCSM recommended that:

> Students should learn the geometric concepts they will need to function effectively in the three-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular....

Explaining why geometry is a basic mathematical skill and why it should be taught in secondary school mathematics, Sherard (1981) advances the following seven reasons.

- Geometry is a basic skill because it is an important aid for communication. Our basic speaking and writing vocabularies are rich in many geometric terms, such as point, line, angle, parallel, perpendicular, plane, circle, square, triangle, and rectangle. This geometric terminology helps us to communicate our ideas to others in a precise form.

- Geometry has important applications to many real-life contexts. Measurements around our homes and many other aspects of our daily life activities require geometrical applications.
- Geometry has important applications to many topics in basic mathematics. Many arithmetical, algebraic, and statistical concepts are better understood when given geometric interpretations.
- Geometry provides a valuable mathematical background for further education. In the U.K., for example, Euclidean geometry was a prerequisite for university entrance (French, 2004).
- Geometry is a basic skill because it is part of the cultural heritage of humanity. It has an immediate intuitive appeal at a visual level. There are cultural and aesthetic values to be derived from its study. In South Africa, for example, designs in beadwork and many other aspects of ethno-mathematical study make use of a rich collection of geometric terms (Mogari, 2002).
- Geometry, like mathematics, provides a context for developing students' logical reasoning skills (Mogari, 2002; French, 2004).
- Geometry enhances the "development of students' spatial perception and understanding" (NCTM, 1989, p.49).

A conclusion that can reasonably be drawn from the above seven points is that geometry is an important mathematical skill because it serves, among other things, as a unifying theme to the entire mathematics curriculum and as a tool for developing students' skills in logical and deductive reasoning. Geometry provides opportunities for learners to develop spatial awareness, geometrical intuition, and the ability to visualise and use geometrical properties in a variety of real-world contexts (Jones, Fujita & Ding, 2006).

In Nigeria and South Africa, geometry is accorded a central position in the mathematics curriculum. In the South African mathematics curriculum for grades 10–12, for example, geometry is integral in the study of algebra, trigonometry, and even statistics (South Africa, DoE, 2003). The relatively greater number of questions assigned to geometry by examining bodies further exemplifies the centrality of geometry to the entire mathematics curriculum. This is illustrated below.

I analyzed selected past examination question papers in mathematics of four examining bodies, two each from Nigeria and South Africa, in order to determine the extent of their geometry contents. In Nigeria, the examining bodies under consideration were the West African Examinations Council (WAEC) and the Joint Admissions and Matriculation Board (JAMB), while in South Africa the examining bodies were the Independent Examinations Board (IEB) and the National Examinations Board (NEB). The years covered in this analysis were 1999 through to 2004, with the year 2000 omitted for lack of available data. For the purpose of this analysis, the category of questions considered to carry geometric content were those that included Euclidean and/or coordinate geometry. Questions that had the visual appeal of geometric shapes (for example, triangles) but tested students' knowledge of other mathematics concepts like trigonometry, for example, were not included in this category. The results of this analysis (see table 2.1) revealed that in both countries, approximately one-third of the mathematics questions in the Senior Certificate Examination and the University Matriculation Examination are based on concepts in geometry.

Table 2. 1 Question numbers having geometric content of four examining bodies

Year	**Question Number (Nigeria)**				**Question Number (South Africa)**			
	WAEC	% **n=50**	**JAMB**	% **n=50**	**IEB 1&2**	%	**NEB 1&2**	%
1999	17,18,22,32,33,34,35,36,37,38,39, 40,41,42,43,44,45,46, 47,48,49,50	**44**	18,22,23,24,25,26,27,28, 29,30,31,32,38,46,49	**30**	1,2,3,4, 10,11,12, 13,14	**31 n=29**	nd	
2001	36,37,38,39,40,41,42,43,44,45, 46,47,48,49,50	**30**	21,22,23,24,25,26,27,28, 29,30,31,32,37,39,40,41	**32**	1,2,3,7,10, 11,12,13	**35 n=23**	1,2,3,4p_1, 7,8,9,10	**40 n=20**
2002	1,6,11,20,23,29,30,32,34,38, 39,40,41,44,45,50	**32**	7,9,10,11,12,13,14,15,16, 17,18,19,20,21,47,50	**32**	1,2,3,4,8,9, 11,12,13	**43 n=23**	1,2,3,7, 8,9,10	**37 n=19**
2003	3,5,6,9,11,12,13,18,19,20,22, 23,26,27,28,29,31,32,34,39	**40**	18,20,22,23,24,25,27,29, 30,31,32,33,34,35,38,40	**32**	1,2,3,7,8,9, 9p_1,10	**38 n=21**	1,2,7,8, 9,10	**33 n=18**
2004	4,6,10,14,15,18,19,23,26,27, 32,33,34,38,45,46,49,50	**36**	1,2,3,4,5,6,7,8,9,10,11, 21,27,33	**28**	1,2,3,4,9, 10,11,12,13	**38 n=24**	1,2,7,8, 9	**29 n=17**

Note n = total number of questions set for the respective years.
4p_1 = paper 1, question number 4; and 9p_1 = paper 1, question number 9.

Similar results had been reported by Olkun, Tolu and Durmus (as cited in Olkun et al., 2005) who claim that in Turkey, for example, about one-third of the mathematics questions in the university entrance examination had geometric content. That these results are similar should come as no surprise, for according to French (2004, p.7), "there is much common ground between geometry syllabuses across the world in terms of topics". Hence, these uniform results further reinforce the notion of the dominance of geometry in mathematics curricula internationally.

It is evident from the foregoing that geometry is, indeed, a central component of the school mathematics curriculum in Nigeria and South Africa. It is appropriate next to highlight some of the objectives of high school geometry teaching in these countries.

2.5.2.1 Objectives of Geometry Teaching in Nigeria and South Africa

Generally, objectives in education are the expected behavioural and cognitive changes on the part of students as a result of their exposure to a program of learning experiences. According to Bloom (1956, p.12), educational objectives are "the *intended behavior* (emphasis in the original) of students…the ways in which individuals are to *act*, *think*, or *feel* (emphasis mine) as the result of participating in some unit of instruction". Decisions about, and modifications of, educational objectives mainly occur at three levels of the educational system, namely the national/regional level, the school level, and the classroom level (Cogan & Schmidt, 1999). The educational objectives referred to in this study are those formulated and pursued at the national level.

To attempt to synthesize the objectives of geometry teaching in two countries like Nigeria and South Africa (each with its separate, well-articulated set of objectives) poses a challenge. A logical way forward is to look first at the objectives of geometry teaching and learning in each country separately, and then attempt to synthesize these. Before taking this step, however, it may be helpful to outline the objects of geometry study common to the nations of the world.

2.5.2.2 The Objects of Geometry Study

French (2004, p.7) states that the "objects upon which the study of geometry is based" are to a large extent common to the curricula of many countries around the world, even though there may be variations in approach and in the degree of importance accorded to each of the objects. French (*ibid.*) identifies the following as the objects of geometry study, stating that variation may, however, occur in the last two among countries:

- Polygons and their properties, giving particular emphasis to triangles, quadrilaterals and regular polygons;
- Circles and their properties related to chords, tangents and angles;
- Three-dimensional figures such as polyhedra and the sphere, cylinder and cone;
- Other curves, such as the parabola and ellipse, and their properties.

My analysis of the Nigerian and South African geometry curricula indicates that the above list generally reflects the objects of geometry study in Nigeria and South Africa. The first two items on the list are particularly consistent with both the Nigerian and the South African curricular contents and will constitute the focus of this research study (see sections 2.4.1.2–2.4.1.4).

2.5.2.3 Objectives of Geometry Teaching in Nigeria

The Nigerian national mathematics curriculum for high school learners seems generally to have emphasized mastery of three basic mathematical skills in the learning of geometry. These are the skill of geometrical constructions using straightedges, compasses, protractors and setsquares; the skill of proving theorems in Euclidean Geometry; and the skill of solving riders based on the theorems. The curriculum indicates that learners are expected to establish, through geometrical constructions, the properties of and relationships between various geometrical shapes. For example, the national mathematics curriculum stipulates that learners should be able to “use the basic constructions of given angles, perpendicular and parallel lines to construct triangles, parallelograms, rhombus etc” (FRN, MoE, 1985, p.7).

Regarding the proof of theorems, the curriculum expects learners to follow step-wise logical deductions to arrive at valid conclusions. For example, the national mathematics curriculum states that teachers should “let learners place emphasis on dependence of the truth of any statement on theorems previously accepted [and] emphasize the step-by-step nature of deductive proof and the if – then relationships” (FRN, MoE, 1985, p.14).

According to the national mathematics curriculum, riders in Euclidean geometry should be aimed at promoting students' deductive reasoning. For example, the curriculum prescribes that teachers should give "exercises and riders to help students reproduce arguments based on reasons, theorems or axioms" (FRN, MoE, 1985, p.8).

Given the basic skills emphasized in the national mathematics curriculum, it would seem that the general objectives of high school (Euclidean) geometry teaching in Nigeria are as follows:

- Development of students' spatial awareness and visualization through geometrical constructions.
- Development of students' logical reasoning abilities through explicit teaching of deductive proofs in Euclidean geometry.
- Development of students' problem-solving ability in geometry that has wide applications in many other aspects of mathematics and related fields.

2.5.2.4 Objectives of Geometry Teaching in South Africa

> The learner is able to describe, represent, analyze and explain properties of shapes in 2-dimensional and 3-dimensional space with justification.
>
> (South Africa, DoE, 2003, p.13).

The South African National Curriculum Statement (NCS) for high school mathematics stresses four major learning components to be taught and learned in senior secondary mathematics. These learning components are referred to as 'Learning Outcomes'. A learning outcome, according to the NCS, "is a statement of an intended result of learning and teaching. It describes knowledge, skills and values that learners should acquire" (South Africa, DoE, 2003, p.7). This means that the learning outcomes coincide with the objectives of teaching and learning in the four learning areas. The learning outcomes as spelt out in the NCS are as follows.

- Learning Outcome 1: Number and Number Relationships.
- Learning Outcome 2: Functions and Algebra.
- Learning Outcome 3: Space, Shape and Measurement.
- Learning Outcome 4: Data Handling and Probability.

This study focuses on learning outcome 3, which is concerned almost exclusively with the study of Euclidean geometry. It stresses the development of students' skills in making and testing conjectures, investigating, justifying, proving, and generalizing in Euclidean geometry. Given the emphasis on these skills in the NCS, the objectives of high school (Euclidean) geometry teaching in South Africa may be summarized as follows:

- Development of students' spatial awareness and visualization through the use of various methods, including geometrical constructions, to investigate geometrical properties of 2-dimensional and 3-dimensional figures.
- Development of students' reasoning abilities through explicit teaching of processes such as experimentation, testing conjectures, justifying statements that would ultimately lead to the acquisition of skills in proof writing in Euclidean geometry.
- Development of students' problem-solving ability by using geometrical properties to solve a wide range of problems in many other aspects of mathematics, such as trigonometry and algebra and other related fields.

2.5.2.5 A Synthesis of the Objectives of Geometry Teaching in Nigeria and South Africa

There thus appear to be many similarities between Nigeria and South Africa in terms of the objectives of geometry teaching in high school mathematics.. The national mathematics curricula of both countries, for example, emphasize as a major objective of geometry teaching the development of students' reasoning abilities through the teaching of deductive proof writing in Euclidean geometry. In both countries there also appears to be a commitment to teaching students problem-solving skills in Euclidean geometry, with the aim of applying these skills in other learning areas.

Despite these similarities, however, there are distinct zones of mutual exclusivity between the Nigerian and South African mathematics curricula in terms of relative emphasis on *approaches* to Euclidean geometry. For example, in Nigeria, more than in South Africa, the geometry curriculum emphasizes the need for learners to use geometrical constructions to explore and establish the properties of geometric plane shapes. Thus, geometrical constructions had been, and still are, a regular feature in the Senior Certificate Examinations in Nigeria.

Another dissimilarity between the Nigerian and South African geometry curricula concerns the relative emphasis on the extent of connections between geometry and other aspects of mathematics. The South African geometry curriculum appears to have emphasized more explicitly than the Nigerian one the need to link geometry with other aspects of mathematics. For example, the NCS states that the learning of geometry should enable learners (among other things) to: "link algebra and geometric concepts through analytic geometry; link the use of trigonometric relationships and geometric properties to solve problems" (South Africa, DoE, 2003, pp.13–14).

It must, however, be pointed out that the South African national mathematics curriculum is also the teaching syllabus, and is as a result more comprehensive in its statement of objectives and learning area contents than the Nigerian one. In Nigeria, details pertaining to learning experiences are given in the teaching (or examination) syllabuses and not in the national mathematics curriculum. As an example, in South Africa (South Africa, DoE, 2003, p.32), the objectives of (Euclidean) geometry teaching in grade 10 read as follows:

We know this when the learner is able to:

(a) Through investigations, produce conjectures and generalizations related to triangles, quadrilaterals and other polygons, and attempt to validate, justify, explain or prove them, using any logical method (Euclidean, coordinate and/or transformation).
(b) Disprove false conjectures by producing counter-examples.
(c) Investigate alternative definitions of various polygons (including the isosceles, equilateral and right-angled triangle, the kite, parallelogram, rectangle, rhombus and square).

Whereas in Nigeria, the objectives of (Euclidean) geometry teaching in grade 10 as spelt out in the national mathematics curriculum (FRN, MoE, 1985, pp.7–9) read as follows:

Students will be able to:

i. perform further constructions using a pair of compasses and a ruler;
ii. write out formal proofs of some basic theorems in Euclidean geometry;

iii. apply the skills of deductive reasoning in proving rides [*sic*] in Euclidean geometry.

It should, however, be noted that in the case of the Nigerian mathematics curriculum, the contents to be covered for each of the above objectives are stated separately. For example, the content to be taught with regard to the second objective is “deductive proof of an angle sum of a triangle” (FRN, MoE, 1985, p.8). In South Africa, on the other hand, the objectives and contents appear together in a statement of “Assessment Standards” in the NCS (South Africa, DoE, 2003, p.32).

Given the above similarities and differences, the objectives of geometry teaching in Nigerian and South African high schools may be summarized as follows.

- Development of students’ logical reasoning with regard to specific concepts in both mathematical and non-mathematical learning fields.
- Development of students’ ability to visualize, represent pictorially, and apply geometric ideas to describe and answer questions about the real world.
- Development of students’ mathematical ‘habit-of-mind’, which relates to the mathematician’s consistent interest in conjecturing, investigating, proving, justifying, and generalising.

The objectives of geometry teaching in Nigeria and South Africa as outlined above appear to be consistent with what Suydam (1985) considers to be the general goals of geometry teaching in schools. Suydam (1985, p.481) opines that the goals of geometry teaching in schools are primarily to:

- develop logical thinking abilities;
- develop spatial intuitions about the real world;
- impart the knowledge needed to study more mathematics; and
- teach the reading and interpretation of mathematical arguments.

These appear to be consistent with the reasons for geometry teaching identified by French (2004, p.2), which are to:

- extend spatial awareness
- develop the skills of reasoning
- stimulate, challenge and inform.

There appears to be a great deal of inconsistency in the literature regarding the usage of terminology such as spatial awareness, spatial visualization, and other related concepts and terms used to describe students' understanding of geometry (Clements & Battista, 1992; Schäfer, 2003; Nickson, 2004). It is therefore necessary at this point to clarify some of this terminology and indicate the meaning of specific terms in the context of my study.

2.6 Terminology Used to Describe Students' Understanding of Geometry

2.6.1 Spatial Ability

Gardner (1993, p.173) states that spatial ability (which he calls "spatial intelligences") is an amalgam of abilities that includes the human "capacities to perceive the visual world accurately, [and] to perform transformations" on both physical and imagined objects in space. Schäfer (2003), in his study, defines spatial ability as an all-embracing concept that describes an individual's ability to be involved in a mental operation or problem-solving situation that is approximately spatial in nature. It is the ability of a person mentally to create geometric images and then manipulate these images in the mind. Gardner (1993, p.174) argues that although there are several components of spatial ability, the "ability to perceive a form or an object" is, however, the most fundamental, and can easily be tested by multiple-choice questions.

There is ample evidence in the literature supporting the view that there exists a relationship between cognitive variables of a spatial nature and learning that is related to geometrical concepts. There also appear to be clear indications that developing spatial abilities in children is important to this aspect of their learning (Nickson, 2004). The exact relationship between spatial ability and other aspects of mathematical learning seems less clear (Nickson, 2004), although Clements and Battista (1992, p.444) assert that "spatial ability is [equally] important in students' construction and use of [even] non-geometrical" concepts. Bishop (1980) identifies two major components of spatial ability that are relevant to students' learning of mathematics. These are spatial visualization and spatial orientation.

2.6.2 Spatial Visualization

This relates to a student's ability to understand and "perform the imagined movements of objects in two-dimensional and three-dimensional space" (Clements & Battista, 1992, p.444). Bishop (1983) believes that students' ability to interpret figural information and to understand visual representations and vocabulary is relevant to their mathematical learning.

2.6.3 Spatial Orientation

Thurstone (as cited in Schäfer, 2003) describes spatial orientation as the ability of students to recognize a given geometric shape viewed from different positions. Spatial orientation is the understanding of, and operating on, the relationships between the positions of spatial objects relative to the viewer's position (Clements & Battista, 1992). Bishop (1983) proposes that developing students' capacity for visual processing, such as the manipulation and transformation of visual representations and images, and the translation of abstract relationships into visual representation, is important for their acquisition of mathematical skills.

2.6.4 Spatial Perception

This relates to a student's ability to perceive (be it through the senses or in the imagination) and manipulate geometric objects (Schäfer, 2003; French, 2004).

2.6.5 Spatial Conceptualization

This is an umbrella term that describes the totality of a student's understanding of spatial objects. It includes an individual's spatial ability, visualization, orientation, and perception (Schäfer, 2004).

This research study investigates the van Hiele levels of geometric thinking among high school mathematics students in Nigeria and South Africa. As stated earlier (section 2.2), geometry *inter alia* concerns an understanding of the properties of and the relationships between spatial objects. Thus, for the purposes of this study, spatial ability, visualization, orientation, and perception are to be understood as those parts of spatial conceptualization that are measurable in a pen-and-paper test as well as a hands-on activity test. Therefore, learners' knowledge of school geometry is described not in terms of these concepts but rather in terms of their van Hiele levels, as indicated by their performance in the various tests used in this study. This approach is supported by Clements and Batista's (1992, p.444) view that many authors have argued that students' "performance on most spatial tests are best understood not in terms of imagery, but rather in terms of reasoning and problem-solving".

All through section 2.5.2 and its subsections, I discussed the importance of geometry and the reasons why it is learned/taught in secondary education, both globally and in the Nigerian and South African contexts. It is evident that geometry has always enjoyed a pride of place in school mathematics curricula, even in the ancient times of Egyptian and Greek mathematics. For example, 26 of the 110 problems in the Moscow and Rhind papyri (the two main sources of Egyptian mathematics acquired about 1850 B.C. and 1650 B.C. respectively) were geometric (Eves, 1976).

In response to the question why geometry should be included in the teaching of mathematics in secondary education, van Hiele argues that the teaching of geometry is central to the development of logical thinking, a key element of mathematical understanding (van Hiele, 1986). This stance underscores the importance of geometry in the overall mastery of mathematics, and further explains why geometry assumes a dominant place in the school curricula of many countries. But whether the emphasis on geometry teaching in secondary school mathematics has yielded results commensurate with the associated human and material investment (as measured by students' success rate in the subject) is an issue over which educators and stakeholders have expressed concern in many countries in recent years.

2.7 Students' Perception of and Achievement in Geometry

2.7.1 Students' Perception of Geometry

> Each year we ask many of our first-year students … to list the mathematics topics that they liked best and topics they liked least in their precollege classes. Although several subjects were "favorites", the subject that was almost universally disliked was geometry in high school.
>
> Hoffer (1981, p.11).

In the above excerpt, Hoffer is talking about his experience with students at the University of Oregon, U.S.A. But it is also my experience in Nigeria over many years of teaching, and in South Africa over a shorter period, that not only do many students dislike geometry, but that many teachers also do not feel comfortable teaching it. Setati (2002, p.4) reports a similar experience in which many professionals in South Africa declare almost with an air of pride "their inability [to do] mathematics".

Geometry, indeed, like mathematics more generally, has a widespread public image of being difficult, theoretical, abstract, but nevertheless important (Setati, 2002). The perception of geometry by students is very often couched in negative terms. Many students describe geometry as being "boring", "irrelevant" and "difficult" (Pegg, 1995, p.87). Some students even express the view that geometry involves too many theorems and proofs, all of which require deductive logical reasoning while they (the students) see themselves as "not too logical" (Hoffer, 1981; Shaughnessy & Burger, 1985, p.419). Even the few who were successful in their geometry course would still confess that they "got through the course by memorizing proofs", but "didn't understand" the course (Hoffer, 1981, p.11).

Many mathematics educators in recent times have expressed concern over students' apparent dislike for geometry and their inability to comprehend the deductive logical system of the subject (Hoffer, 1981; Shaughnessy & Burger, 1985; Fuys et al., 1988; Pegg, 1995; de Villiers, 1997; Shannon, 2002). Shaughnessy and Burger (1985), for example, observe that many students in high school geometry have a lot of difficulty with such essential elements as deduction and proof – the very tools for geometric exploration. Shaughnessy and Burger (1985, p.419) further lament:

> Despite our best efforts to teach [the students], even the most capable algebra students may struggle and get through geometry by sheer willpower and memorization but with little understanding of the logical system we have been developing all year.

The literature appears to offer some possible explanations for students' dislike or even outright hatred of geometry. Shannon (2002, p.26), for example, states that the geometry that was "based [strictly] on the Euclidean system whereby knowledge of shapes was derived almost exclusively from a set of axioms, using deductive reasoning" alone, was beneficial only to "the top 20% of pupils in secondary school". Because of the failure by many students to grasp the concept of geometry in the traditional Euclidean postulational fashion, many countries began a search for alternative approaches and Euclidean geometry came under severe criticism.

2.7.1.1 Criticisms of Euclidean Geometry

It was stated earlier on that Euclidean geometry dominated the mathematical world for over 2000 years (section 2.3), that consideration of alternatives to Euclid's parallel postulate led to the development of other geometries (section 2.4.5), and that there are today several approaches to the study of geometry (section 2.2, para.6). By criticisms of Euclidean geometry I do not mean the logical shortcomings of Euclidean geometry as an axiomatic mathematical structure (see Eves, 1953). Instead, I intend to foreground the reasons why alternative approaches to geometric exploration were sought after in many countries across the world.

There is a whole body of literature indicating that the geometry that is presented in a formal axiomatic fashion as Euclid did is accessible only to a small minority of learners in secondary education (Allendoerfer, 1969; Hoffer, 1981; Mayberry, 1983; Burger & Shaughnessy, 1986; van Hiele, 1986; Fuys et al., 1988; de Villiers, 1997; van Hiele, 1999; Shannon, 2002). Because of the negative results that were recorded over many years of its being taught in secondary education (Allendoerfer, 1969; van Hiele, 1999; Shannon, 2002), Euclidean geometry has been criticized as too formal, too complicated and even too difficult. Van Hiele (1999) expresses the view that school geometry that is presented in the traditional Euclidean fashion assumes that school children also think on a formal deductive level. But research evidence indicates that this is not the case, as many students experience basic difficulties with geometry presented in the Euclidean way (Fuys et al., 1988; Clements & Battista, 1992; de Villiers, 1997).

The failure by many students to understand geometry (in its strict Euclidean axiomatic form) generated debates in many countries, with some (e.g. the U.S., Netherlands and Russia) advocating reform in approaches to school geometry, and some others even calling for outright abandonment of Euclidean geometry in the mathematics curriculum (Allendoerfer, 1969; Shaughnessy & Burger, 1985; Fuys et al., 1988). The reforms that took place in many countries reflected for the most part changes in didactics in the light of the research conducted in the late 1950s by two Dutch mathematics educators, Pierre van Hiele and his wife, Dina van Hiele-Geldof. In Russia, for example, results from the van Hieles' research have been applied to the school mathematics curriculum with appreciable improvements in students' understanding of school geometry (Fuys et al., 1988).

As a consequence of improved student performance in geometry, coupled with advances in computer technology such as the geometer sketchpad, Euclidean geometry (even though with some modifications) is now experiencing an exciting revival in many countries (de Villiers, 1997). In the light of the van Hieles' research, de Villiers (1997) believes that in South Africa more informal geometry should be taught at the primary school level if students' achievement at the secondary school level is to be improved.

2.7.2 Students' Achievement in School Geometry

> Many students are quite unsuccessful in geometry. … For example, in the fall only 52% of the students could calculate the area of a square given its sides. … Many students are not learning even the simplest geometry notions … thus many students do not know these notions upon leaving high school.
>
> Usiskin (1982, p.86)

In the above excerpt, Usiskin (1982) is reporting schoolchildren's performance in geometry in the United States. The majority of the children concerned (96% of them) were aged between 14 and 17 years old, which is the same age group as the majority of high school students in Nigeria and South Africa.

Not surprisingly, students' negative perception of geometry (see section 2.7.1.1) seems to have been translated into low academic achievement.. A plethora of research exists that paints a somewhat depressing picture of students' knowledge of geometry (Usiskin, 1982; Mayberry, 1983; Burger & Shaughnessy, 1986; Fuys et al., 1988; Clements & Battista, 1992; van der Sandt & Nieuwoudt, 2003; WAEC, 2003; Feza & Webb, 2005; Siyepu, 2005). In the U.S., for example, students' knowledge of geometry at the elementary, middle, and high school levels appears to fall short of what is generally expected at these levels of education. As Clements and Battista (1992, p.421) put it, in the United States, elementary and middle school students are "failing to learn basic geometric concepts and geometric problem solving; they are woefully underprepared for the study of more sophisticated geometric concepts and proofs". At the high school level, U.S. students' knowledge of basic geometric shapes and class inclusions of shapes is equally unimpressive, as according to Clements and Battista (1992, p.421), "only 63% [of high school learners] were able to correctly identify triangles that were presented along with distractors" and "only 64% of the 17-year-olds knew that a rectangle is a parallelogram".

In Nigeria, little appears to have been reported in the available literature about students' geometry achievement in secondary education. However, a whole body of research exists in which the low mathematics performance of the Nigerian child is a common theme (Igwue, 1990; Adedayo, 2000; Agwagah, 2000; Bot, 2000; Igbokwe, 2000; Okonkwo, 2000). Going by the argument advanced by Usiskin (1982), van Hiele (1986) and French (2004), that performance in mathematics as a whole is a good indication of performance in geometry specifically, then Nigerian students' knowledge of geometry could be said to fall below expectations. Evidence could be drawn from WAEC Chief Examiner's Report, which states that "candidates were observed to be generally weak in the area of geometry" (WAEC, 2003, p.171). The WAEC Chief Examiner's Report (WAEC, 2003, p.175) indicates that a "question on the angle properties of a triangle was also unpopular" with the majority of the candidates. Many of the high school students in Nigeria could not comprehend "the concepts and principles of angle in the same segment as well as alternate and corresponding angles" (WAEC, 2003, p.175).

The weakness of students' knowledge of geometry is not much different in South Africa. For example, de Villiers (1997, p.42) states that in South Africa, "it is well known that on the average, pupils' performance in matric (Grade 12) geometry is far worse than in algebra". In KwaZulu and the Eastern Cape, for example, research indicates that the majority of high school learners have a weak understanding of many geometric concepts (de Villiers, 1997; Siyepu, 2005).

Generally, high school learners' mathematical performance in South Africa appears to be unimpressive, but it is even more so in geometry, since according to Roux (2003, p.362) "learners' performance [in South African high schools] is even poorer when it comes to items involving understanding of features and properties of shapes" – the very fundamentals of geometric understanding. Results from both national and international surveys of mathematical performance indicate that many "secondary learners [in South Africa] cannot identify and name shapes like kite, rhombus, trapezium, parallelogram and triangle" (Roux, 2003, p.362).

Further evidence concerning South African learners' low performance in mathematics generally, and in geometry specifically, can be drawn from the results of the Third International Mathematics and Science Study-Repeat (TIMSS-R) conducted by the Human Sciences and Research Council (HSRC) in 1999. The TIMSS-R results indicated that of the 38 countries that participated, South African learners obtained the poorest results in mathematics (Brombacher, 2001; Howie, 2001). The mathematics average score of 275 points out of 800 points was well below the international average of 487 points (Howie, 2001). According to Howie (2001, p.11), South African children in the TIMSS-R had "considerable difficulty dealing with … geometry questions … and in some cases were successfully distracted by questions testing misconceptions" in geometry.

The foregoing discussion paints a general picture of the problem of students' inadequate conceptual knowledge of geometry both internationally and in the Nigerian and South African contexts. But how are students' conceptual misunderstandings manifested in the study of school geometry? How do learners experience difficulties in geometry?

2.7.3 Conceptual Difficulty Experienced by Students in School Geometry

Students' conceptual misunderstandings in geometry come in various forms. Some of the ones commonly reported in the literature are: misconceptions, imprecise terminology, identification/classification of basic shapes, properties of shapes, class inclusions of shapes, parallel and perpendicular lines, the concept of angles, angle sum of a triangle, and proof writing (Usiskin, 1982; Mayberry, 1983; Senk, 1985; Shaughnessy & Burger, 1985; Burger & Shaughnessy, 1986; Fuys et al., 1988; Senk, 1989; Clements & Battista, 1992; Fuys & Liebov, 1997; Olson, Sakshaug & Olson, 1997; Andrews, 1999; Oberdorf & Taylor-Cox, 1999; Howie, 2001; Roux, 2003; French, 2004; Feza & Webb, 2005; Siyepu, 2005). These common misunderstandings, as they pertain to high school learners in Nigeria and South Africa, are the ones investigated in this study.

2.7.3.1 Misconceptions

Students' misconceptions about geometric concepts as reported in the literature are many and varied (Fuys & Liebov, 1997; Oberdorf & Taylor-Cox, 1999; French, 2004; Feza & Webb, 2005), yet they are interesting (Olson et al., 1997). Teachers' knowledge of students' misconceptions about geometry is important for remedial instructional design and delivery. The literature indicates that many schoolchildren in both primary and secondary education hold several misconceptions about geometric shapes and the relationships between their properties. Clements and Battista (1992, p.422), for example, listing some examples of students' misconceptions in geometry, state that many high school students reason that "a square is not a square if its base is not horizontal". That is, many students in secondary education are able to recognize shapes only in some standard orientation. This idea (orientation of shapes) was taken into account in the development of two of the instruments used in this study. A rather strange misconception concerns students' misunderstanding of diagonals. Figure 2.3 represents a student's response to a question that required stating the number of sides and diagonals in each of the two shapes (French, 2004). This unfortunate misconception noted by French was adapted and also tested in this study.

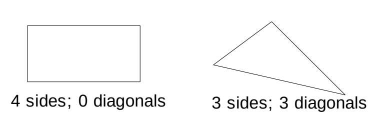


Figure 2. 3 Misconception about diagonal

Lack of exposure to proper terminology, too few authentic experiences in the primary school, together with misinformation by adults, have been identified as some of the possible reasons for students' misconceptions in geometry (Oberdorf &Taylor-Cox, 1999).

2.7.3.2 Imprecise Terminology

Language is undoubtedly a very important tool in communication, and perhaps geometry stresses the use of language more than any other mathematics course (Hoffer, 1981; Ashfield & Prestage, 2006). Geometric terminology is crucial to the communication of geometric ideas both inside and outside the classroom since, according to Feza and Webb (2005), lack of language competency impedes progress in geometric understanding. De Villiers (as cited in Feza & Webb, 2005, p.45) stresses the point that "success in geometry [indeed] involves acquisition of the technical terminology". Bloom (1956, p.63) asserts that "the most basic type of knowledge in any particular field is its terminology". But all too often, students lack the appropriate vocabulary to express the distinguishing properties of a figure or compare shapes in an orderly manner (Renne, 2004; Feza & Webb, 2005). Oberdorf and Taylor-Cox (1999, p.340) explain that "lack of exposure to proper vocabulary" is one of the reasons for students' misconceptions in geometry. Therefore, Hoffer (1981, p.12) suggests that precise terminology "may be thrust on students" early in their geometry course in order to remediate students' imprecise use of geometric terminology. One of the instruments used in this study focussed on students' knowledge of some basic geometric terminology in Nigerian and South African high schools.

2.7.3.3 Identification/Classification of Basic Shapes

A common activity in geometry is for students to identify, name, and classify various shapes (Mayberry, 1983; Burger & Shaughnessy, 1986; Fuys et al., 1988; Clements & Battista, 1992; Oberdorf & Taylor-Cox, 1999; van Hiele, 1999; Feza & Webb, 2005). The ability to recognize and name shapes has been recognized as important for geometric conceptualization (van Hiele, 1999). Research evidence, however, indicates that many high school learners lack the ability to correctly identify, name, and classify many simple geometric shapes. Mayberry (1983, p.64), for example, reports that some students in her study "had difficulty in recognizing a square with a non-standard orientation", while for some others, "giving the name of a concept seemed to be more difficult than choosing an example of the concept when the name was given and examples and non-examples were displayed". Usiskin, (as cited in Clements and Battista, 1992, p.421) further exemplifies students' difficulty with geometry by stating that in his study "only 63% [of high school learners] were able to correctly identify triangles that were presented along with distractors". In this study, students' ability to identify and classify triangles and quadrilaterals was investigated in Nigerian and South African high schools.

2.7.3.4 Properties of Shapes

High school students often do not perceive the properties of shapes (Mayberry, 1983). As a result, many of them cannot describe shapes explicitly in terms of their properties (Burger & Shaughnessy, 1986; Feza & Webb, 2005). Clements and Battista (1992, p.422), for example, state that in their study "less than 25% of 11th-grade [U.S.] students correctly identified which figures had lines of symmetry". In this study, Nigerian and South African high school learners' knowledge of the properties of circles, triangles and quadrilaterals was explored.

2.7.3.5 Class Inclusions of Shape

Empirical research reveals that many students (whether in the primary or secondary school) demonstrate a lack of ability to perceive class inclusions of shapes, contrary to general expectations (Mayberry, 1983; Burger & Shaughnessy, 1986; Fuys et al., 1988; Clements & Battista, 1992; Feza & Webb, 2005; Siyepu, 2005). Clements and Battista (1992, p.421), for example, state that "only 64% of the 17-year-olds [in the U.S.] knew that a rectangle is a parallelogram". On a sorting activity that involves different triangles and quadrilaterals, Burger and Shaughnessy (1986) disclose that the majority of the students sorted the shapes so as to prohibit class inclusions. Knowledge of class inclusions of shapes is important in geometry because it enables the learners to reason about the relationships between different geometric shapes and their properties.

2.7.3.6 Parallel and Perpendicular Lines

An understanding of the concept of parallel and perpendicular lines is an important aspect of mathematical knowledge to acquire because these concepts form a useful basis for the classification of many polygons, for the understanding of angle relationships, and in geometric proofs (Happs, 1992). The difficulty, however, arises when students are required to state the angle relationships of parallel lines and transversals. Determining and mastering the terminology associated with parallel lines and transversals – such as alternate angles, vertically opposite angles, corresponding angles, and co-interior angles – seems to pose a big challenge to many students. Usiskin (1982), for example, reports that only 30% of high school learners in his study could find the measure of angle *x* (Figure 2.4) given that lines m and n are parallel. Clements and Battista (1992, p.421) express the view that "students' performance with figures not frequently encountered in everyday life, such as perpendicular lines and the radius of a circle" is generally less impressive. Parallel and perpendicular lines and their angle properties constituted a good part of one of the instruments used to unpack students' conceptual understanding of geometry in this study.

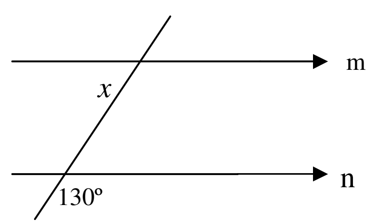


Figure 2. 4 Parallel lines and a transversal

2.7.3.7 The Concept of Angles

Although knowledge of angles is an essential requirement from an early stage in the study of geometry, evidence abounds concerning students' misconceptions about and difficulty with angle measurement, both in geometrical shapes and intersecting straight lines (Clements & Battista, 1992; French, 2004). French (2004), for example, states that many students in secondary education hold the misconception that the size of an angle is dependent upon the length of the two rays that form the angle – the longer the rays, the larger the angle. For students to struggle with determining the size of an angle using a protractor is not uncommon in the experience of many teachers. Even in this study, I had to demonstrate to the majority of the learners how to use a protractor to determine the measure of an angle in order to enable them to respond to questions that tested their knowledge of the angle properties of simple geometric shapes.

2.7.3.8 Angle Sum of a Triangle

According to French (2004, p.57), "triangles are the key building blocks for all geometrical configurations", and knowledge of the angle properties of triangles is necessary for proving and establishing relationships among many geometrical figures and their properties. For example, it would seem reasonable to expect that a student who already knows that the sum of the angles of a triangle is 180º should be able to deduce informally that the sum of the interior angles of quadrilateral ABCD (Figure 2.5) is 360º. This is because the diagonal AC divides the quadrilateral into two triangles. Research evidence, however, cast suspicion on students' knowledge of the angle sum of a triangle. Clements and Battista (1992, p.421), for example, report that "fewer than 10% of 13-year-olds could find the measure of the third angle of a triangle, given the measures of the other two angles".

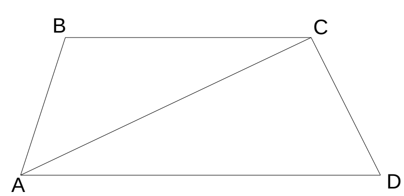


Figure 2. 5 Illustrating the angle sum of a quadrilateral to be 360° by drawing one of its diagonals to form two triangles

2.7.3.9 Proof Writing in Geometry

Teaching students how to write proofs forms an important objective of the geometry curricula of many countries (Senk, 1985; Siyepu, 2005). The Nigerian and South African high school geometry curricula, for example, specifically require that the learners should be able to "write out formal proofs of some basic theorems in Euclidean geometry" and apply the knowledge to solve riders (FRN, MoE, 1985, p.8;

South Africa, DoE, 2003, p.33). Research, however, shows that learning to write proofs in geometry "is one of the most difficult topics" for many high school learners (Hoffer, 1981; Senk, 1985, p.448; Senk, 1989, p.309; Siyepu, 2005). According to Usiskin (1982, p.88),

> of all high school students in the United States, approximately: …40% study proof. 11% study proof but cannot do anything with it; 9% can only do trivial proofs; 7% have moderate success with proof; 13% are successful with proof.

In Nigeria, testing students' ability to write proofs in geometry has been excluded from the examination syllabuses since about the late 1990s. For example, regarding plane geometry, the WAEC syllabus for 2005–2008, page 346, states that "the results of these standard [geometric] theorems…must be known BUT THEIR FORMAL PROOFS ARE NOT REQUIRED" (emphasis mine). This appears to contradict the position of the national mathematics curriculum on proofs because little is known about the proof-writing ability of Nigerian high school learners. Although the WAEC Chief Examiner's Report of 2003 indicates that many candidates in the School Certificate Examination in Nigeria could prove that "triangle ABC is isosceles by proving that two of its interior angles are equal", the question, however, only required the candidates to verify by substituting given and/or deduced numerical values (WAEC, 2003, p.175).

In South Africa, Siyepu's (2005) study indicates that the majority of 11th-graders encounter difficulties with the proofs of circle theorems. In this study, a grade-appropriate proof writing task was used to explicate high school learners' conceptual understanding of geometry in Nigeria and South Africa.

2.7.4 Causes of Learning Difficulty in School Geometry

Sources of learning difficulty in school geometry across the globe have been attributed to a number of factors. The literature, however, tends to have highlighted three major factors: curricular, textual, and instructional/pedagogical (Clements & Battista, 1992; Schäfer, 1996; Fujita & Jones, 2002; van der Sandt & Nieuwoudt, 2003; Olkun, et al., 2005; Siyepu, 2005). While of course these factors are not the only critical influences on students' learning of geometry, they do have a major impact and their evaluation does give some insight into the interplay among the expected, implemented, and attained educational objectives (Cogan & Schmidt, 1999; Fujita & Jones, 2002). The first two factors are discussed here only because of their interconnectedness with the third, which constitutes a major focus in this study.

2.7.4.1 Curricular Factor

The curriculum, with regard to both what topics are treated and how they are treated, has far-reaching implications for students' performance in geometry (Clements & Battista, 1992). It was pointed out, above, that countries differ only slightly in terms of their geometry curricular contents (see section 2.5.2.2). However, there may well be wide variations among countries with regard to the emphasis and timing of different elements, and in the "relative importance accorded to practical approaches, proofs, and applications" (French, 2004, p.7). How the major components of geometry are organised and presented in the geometry curriculum undeniably impinges directly on how school children experience basic geometric ideas, and hence has an influence on how they perform in the subject.

A recurring theme in the literature is the question of the amount of geometry in the primary school curriculum. There is evidence that one of the primary causes of students' poor performance in geometry at the secondary school level is the lack of a rich, coherent, and well-sequenced geometry curriculum at the primary school level (Clements & Battista, 1992; Pegg, 1995; de Villiers; 1997; Siyepu, 2005). Students enter secondary school with little or no encounter with geometry. In fact, Usiskin (as cited in Clements & Battista, 1992, p.422) states that "there is no curriculum at the elementary school level. As a result, students enter high school not knowing enough geometry to succeed".

De Villiers (1997, p.42) asserts that in South Africa, the geometry curriculum is "still heavily loaded in the senior secondary school with formal geometry, and with relatively little content done informally in the primary school". A cursory look at the Nigerian senior school geometry curriculum tends to indicate that de Villiers' assertion is equally true of the Nigerian situation. Indeed, my perusal of the Nigerian and South African geometry curricula for high school learners shows that there are many similarities between the two curricula, such as their spiral nature, a core of Euclidean geometry, and a commitment to enhancing students' logical thinking through explicit teaching of proofs. There are, however, areas of dissimilarity. For instance, analytical and transformation geometries are either excluded from or given only cursory treatment in the Nigerian General Mathematics[1] curriculum, while in South Africa, these topics form an integral part of the mathematics curriculum for secondary education.

2.7.4.2 Textual Factor

Generally, curricular content is experienced by learners through the textbooks that are used in the classroom and for homework. This is because textbooks are usually expected to reflect curricular prescriptions concerning the body of knowledge and skills that students are expected to master. Thus because of their content and the way in which it is organized, different geometry textbooks will tend to orient learners along different lines of competency in geometry problem solving (Fujita & Jones, 2002).

In a comparative study of the U.K. and Japan, for example, Fujita and Jones (2002, p.82) state that in the U.K. the textbooks analyzed were "designed around a set of exercises with mathematical theorems merely stated rather than developed or proved". Fujita and Jones (2002, p.82) claim that as a consequence, in the U.K., "even 14–15-year-olds show a consistent pattern of poor performance in constructing proofs", even when they excel in tasks that involve numerical calculations in geometry. By contrast, in Japan, "textbooks attempt to develop students' deductive reasoning through 'proof' using various approaches" (*ibid.*). Consequently, "most 14–15-year-old students (Japanese secondary 3rd grade) can write down a [geometry] proof" even though "around 70% [of the students] cannot understand why proofs are needed" (Fujita & Jones, 2002, p.81; Jones, Fujita & Ding, 2006).

[1] In Nigeria two mathematics curricula are usually implemented concurrently at the secondary level of education. These are 'General Mathematics' curriculum for all secondary school students, and 'Further Mathematics' curriculum for those secondary school students, who in addition to 'general Mathematics' desire a further knowledge of mathematics in preparation for mathematics and mathematics-related courses in university education. The curriculum referred to in this study is the 'General Mathematics' curriculum.

It has been observed that teachers generally "depend very heavily on the textbook; follow the text very closely for content and sequencing; and hold as a major objective the completion of the exercises at the end of each section" (Suydam, 1985, p.482). What this seems to imply is that teachers should select very carefully those geometry textbooks whose contents reflect the curricular objectives if learners are to have learning experiences consonant with the expectations of the geometry curriculum.

Van Hiele (1986, p.45) seems to suggest that an ideal geometry textbook is one in which "the subject matter is repeated many times, and each time it is dealt with from the very beginning". A textbook organised in this manner, in van Hiele's view, satisfies what he calls "telescoped reteaching". This seems to call for the implementation of a spiral curriculum through the organisation of the subject matter in textbooks. It is true that the Nigerian and South African geometry curricula are of a spiral nature (see section 2.7.4.1). However, the extent of the implementation of van Hiele's essential requirement that "each time it [the subject matter] is dealt with from the very beginning" in the geometry curricula and textbooks in these countries remains largely a matter of conjecture. It is my experiential conviction, however, that teachers can redress this situation through their instructional practices.

2.7.4.3 Instructional/Pedagogical Factor

The classroom remains one of the most important educational focal points where curricular intentions are transformed into potential learning experiences. Indeed, most school learning experiences are framed by a teacher who selects, prepares, and presents a wide range of instructional activities for the students (Cogan & Schmidt, 1999; Kilpatrick, Swafford & Findell, 2001). It is for this reason that Evans (1959, p.27) affirms that "the most important people in any educational system are the teachers in the classroom". Stoker (2003, p.11), too, believes that "learning is strongly and necessarily linked to teaching". This means that the amount of learning that takes place in the classroom depends for the most part on teachers' own knowledge of the subject matter to be learned. In fact, research seems to indicate that teachers' classroom behaviour is often "influenced by their knowledge" of the subject and subject matter-specific pedagogy (Cogan & Schmidt, 1999; van der Sandt & Nieuwoudt, 2003, p.199). Clearly, a teacher with good content knowledge of geometry coupled with a good teaching strategy in the subject would make learning much easier for the students (Hewson, 1999).

Shulman (1987) distinguishes between different categories of knowledge that a teacher should possess in order to each effectively. These include content knowledge – knowledge of the subject matter to be taught; pedagogical content knowledge – knowledge of specific strategies for the imparting of particular subject matter; and curriculum knowledge – knowledge of materials and programs by means of which instruction and assessment are to be carried out. Shulman (*ibid.*) is however of the opinion that pedagogical content knowledge is perhaps the most important "because it identifies the distinctive bodies of knowledge for teaching".

There is a whole body of research indicating that much of the difficulty that students experience with mathematics generally and with geometry specifically is due to teachers' lack of appropriate pedagogical content knowledge in these subjects (van Hiele, 1986; Shulman, 1987; Crawford & Adler, 1996; Mansfield & Happs, 1996; Stigler & Hiebert, 1999; Stoker, 2003; van der Sandt & Nieuwoudt, 2003; Feza & Webb, 2005; Mji & Makgato, 2006). It has been observed that "children's knowledge and teachers' understanding of that knowledge are central to instructional decision making" (Vacc & Bright, 1999, p.90). Unfortunately, many traditional teaching strategies do little to foster teachers' understanding of their learners' mathematical thought (Mansfield & Happs, 1996). Van Hiele (1986) asserts that the inability of many teachers to match instruction with their pupils' level of understanding in geometry more than anything else accounts for their failure to promote students' conceptual understanding in the subject.

In a study carried out in the North-West province in South Africa, for example, van der Sandt and Nieuwoudt (2003) report that grade 7 teachers and prospective teachers lacked the geometry content knowledge requisite for them to be successful teachers. This revelation holds important consequences for students' learning of geometry in a country like South Africa where "a function of the new curriculum [curriculum 2005] is for school learners to emerge with enhanced mathematical knowledge, skills and dispositions" (Stoker, 2003, p.11). But an improvement in students' learning assumes an improvement in the pedagogy enabling that learning. The fact is that teachers with inadequate geometry content knowledge tend to exacerbate the problems that students experience in learning the subject (Nieuwoudt & van der Sandt, 2003). Indeed, in South Africa, students' low mathematics achievement has largely been blamed on teachers' inadequacies in the classroom due to the system of apartheid education in the past era that ill-equipped the majority of the teachers for effective teaching (Stoker, 2003; Mji & Makgato, 2006). Crawford and Adler (1996, p.1196), for example, state that apartheid education left a legacy of "serious teacher shortages in mathematics…and for the majority of teachers a system of teaching and learning lamentably deficient".

In Nigeria, although a range of factors, such as teachers' qualifications, students' family background, school environment, and the curriculum have been identified as contributing to students' poor performance in high school mathematics, the literature appears to underscore instructional-related issues as the most critical influence on students' achievement (Igwue, 1990; Ivowi, 1990; Adedayo, 2000; Onabanjo & Akinsola, 2000). Research seems to indicate that all is not well with the teaching and learning of mathematics generally and geometry in particular in Nigerian secondary schools. Adedayo (2000), for instance, reports that none of the 20 secondary schools (making up 100% of her study sites) sampled in Lagos State had geoboards, and that only 8 (40%) of the schools had graph boards. In their study conducted in secondary schools in Oyo State, Nigeria, Onabanjo and Akinsola (2000) reveal that many schools in the state lacked common geometrical instruments necessary for the successful and effective teaching and learning of school geometry. As a result, geometry classroom instruction is often times dominated by verbal presentation and memorization of geometric concepts.

Further, there is the all-important issue of teachers' reluctance to opt for new and more progressive methods of teaching. Despite recognizing the pitfalls of the traditional behaviourist classroom (see section 2.8.4, para. 2) that is often characterized by regimented teacher-centred instruction, many teachers appear to be reluctant (for whatever reason) to embrace the generally acclaimed learner-centred social constructivist approach (see section 2.8.4, para. 3) (Yager, 1991; Schäfer, 1996). Olkun et al. (2005) suggest that teachers' instruction should aim at raising the level of students' mathematical thinking, and the learner-centred approach is the most appropriate for this purpose.

Teachers' geometry classroom instructional practices in Nigeria and South Africa constituted a major part of this study. The aim was to observe and report on geometry classroom instruction so as to provide some possible explanations for students' under-achievement in geometry in these countries. That is, the focus was on teaching methods rather than the mathematical proficiency of teachers. The impetus to embark on this rather ambitious goal was derived from Stigler and Hiebert's (1999, p.10) assertion that "teaching, not teachers, is the critical factor" in students' geometry learning. Stigler and Hiebert (1999, pp.10–13) claim that teaching is a "cultural activity", and that cross-cultural differences in the mathematics attainment of students are due more to variations in the pedagogical content knowledge of teachers across cultures than teachers' content knowledge of the subject.

Stigler and Hiebert's insight into the distinctive instructional practices characteristically prevalent in different cultures came from their analysis of the classroom videos used in the Third International Mathematics and Science Study (TIMSS) conducted in 1995. Classroom videos made in three countries, Germany, Japan and the U.S., were analyzed. From these analyses, Stigler and Hiebert (1999, p.10) conclude that "although variability in [teachers'] competence is certainly visible in the videos we collected, such differences were dwarfed by differences in *teaching methods* (emphasis in the original) that we see across cultures".

2.7.5 Some Earlier Models of Geometry Teaching in Schools

The idea that method of teaching is closely linked to achievement in school geometry seems to be a long-held and widespread one. Even the Mathematical Association of the U.K. in the 1920s, for example, asserted that students' achievement in geometry could be enhanced by structuring the course into stages according to method of teaching rather than subject matter. The Association states that although the "Euclidean way of teaching geometry recognised stages of subject matter, it however used the same method of teaching throughout" the course (Mathematical Association, 1923, p.14). As a result, learning geometry became an uphill task for many school children as “the method was not adapted to the psychology of young boys” (Mathematical Association, 1923, p.14). Accordingly, the Association attempted to structure instruction in geometry in ways that reflected students’ thinking processes in the early 1920s.

Consequently, the Mathematical Association prepared a report on the teaching of geometry in schools that focused on the difficulties encountered by students in Euclidean geometry (Pegg, 1995; French, 2004). The Association phased the teaching of geometry into five stages and provided recommendations on how learners might be taught at the various stages (Mathematical Association, 1923; French, 2004; Orton, 2004). The stages are:

Stage A: **The Experimental Stage**. This stage focuses on real-life problems. Common geometrical notions and shapes are to be studied with emphasis on the use of geometrical instruments. Oral presentation of facts relating to angles, lines and triangles was recommended. Deductions are to be made simple. The ideal age for the treatment of concepts at this stage is 12½ years (Mathematical Association, 1923).

Stage B: **The Deductive Stage**. In this stage students learn theorems, solve riders and write out proofs in geometry. The subject matter of this stage was “the whole of elementary plane geometry with occasional inroads upon the easier parts of solid geometry” (Mathematical Association, 1923, p.15). A practical approach that encourages the use of the deductive method as well as intuitive knowledge was recommended. Learners at this stage are yet to fully develop Euclid’s systematising process. This stage was recommended for students aged between 12½–15 years old.

Stage C: **The Systematizing Stage**. In this stage, theorems studied in Stage B are sequenced in logical order using a minimal number of axioms. Reasoning is meant to be rigorous. Not all students in secondary education are expected to attain this stage before leaving school. Students for whom this stage was recommended are those aged 16 or 17.

Stage D: **Modern Geometry**. In this stage, geometrical conics and formal solid geometry are studied alongside projective geometry and systems of circles. According to the Association, this stage is meant for students who are “reading for scholarship” (Mathematical Association, 1923, p.16). Very few students ever reach this stage (Orton, 2004).

Stage E: The Philosophy of Geometry. Geometry at this stage deals with higher levels of abstract formulation and analyses of theorems in different axiomatic systems (Orton, 2004). Non-Euclidean geometries can be studied. This stage is considered to be appropriate “for a few gifted specialists”, and for university education “rather than the school” (Mathematical Association, 1923, p.16).

Though the model advocated by the Mathematical Association found much support, it was criticized on two fronts: First, the categories were too broad to be very useful, and secondly, only a few students managed to reach Stage B because of the very broad divisions of subject matter and approach (Pegg, 1995; Orton, 2004). As a consequence, it was felt that an approach was needed which better reflects, in more detail, the growth in geometric understanding exhibited by students. One approach that addresses this is the van Hiele theory of geometric reasoning. Although the van Hiele theory was developed as far back as the late 1950s, the theory enjoys acclaim today as one of the best-known frameworks for studying teaching and learning processes in geometry (Battista, 2002). According to Battista (2002), the van Hiele theory offers the best description of students’ thinking about two-dimensional shapes.

2.8 The Van Hiele Theory

Pierre van Hiele and Dina van Hiele-Geldof were a husband-and-wife team of Dutch mathematics educators who did research in the late 1950s on thought and concept development in geometry among school children. As a result of many years of teaching experience, they noticed with disappointment the difficulties that their students had in learning geometry (Mason, 1998; Clements, 2004). From classroom observations, the van Hieles asserted that students pass through several levels of reasoning about geometric concepts (Shaughnessy & Burger, 1985). Consequently, the van Hieles developed a theory of levels of thought in geometry, now called the van Hiele theory, that suggests that students pass through numerous levels of geometric thinking as they progress from merely recognising geometric shapes to being able to construct a formal geometric proof (van Hiele, 1986; Teppo, 1991; van Hiele, 1999; Clements, 2004). The van Hiele theory enables insight into why many students encounter difficulties in their geometry courses, particularly with formal proofs. The theory also offers a model of teaching that teachers could apply in order to promote their learners' levels of understanding in geometry (van Hiele, 1986; Fuys et al., 1988; Pegg, 1995). In my study, both aspects of the van Hiele theory, that is, the van Hiele theory on the levels of geometric thinking, and the van Hiele theory on geometry instruction, were utilised to explore students' geometric understanding and the dominant patterns of geometry classroom instruction in Nigeria and South Africa.

The van Hiele theory originally posits the existence of five sequential and hierarchical discrete levels of geometric thought (Hoffer, 1981; Usiskin, 1982; Senk, 1989). Two different numbering schemes are commonly used in the literature to describe the van Hiele levels: level 0 through to 4, and level 1 through to 5 (Senk, 1989). The van Hieles originally made use of the level 0 through to 4 numbering scheme. However, Hoffer (1981) and van Hiele's (1986; 1999) more recent writings make use of the level 1 through to 5 numbering system. This, according to Senk (1989, p.310), permits the 0 to be used for students who do not operate even at the van Hiele's "basic" level. In this study, all references made to research studies that used the 0 to 4 scheme have been adapted to the 1 through to 5 numbering scheme. The van Hiele levels can be described as follows:

Level 1: Recognition (or visual level). At this level, students recognize geometric shapes as a whole (Shaughnessy & Burger, 1985). The students can identify, name and compare geometric shapes such as triangles, squares and rectangles only in their visible form (Fuys et al., 1988). No attention is given to the properties of these shapes (Mayberry, 1983). A figure is perceived as a whole recognizable by its visible form and only in some standard orientation. The student at this level makes "use of imprecise qualities to compare drawings and to identify, characterize, and sort shapes" (Burger & Shaughnessy, 1986, p.43). Descriptions are based purely on visual appeal. For example, a student at this level "will recognize a picture of a rectangle but likely will not be aware of many properties of rectangles" (Hoffer, 1981, p.13). If a student is asked why he or she called a figure a rectangle, the reply might be "because it looks like a rectangle; it is like a window or a door" (Shaughnessy & Burger, 1985, p.420).

Level 2: Analysis (or descriptive level). The student at this level is able to reason about a geometric shape in terms of its properties. The student now sees geometric shapes as collections of properties. Students can recognize and name properties of geometric figures, but they do not yet understand the relationships between these properties and between different figures (Hoffer, 1981; van Hiele, 1986; Mason, 1998). When asked why a figure is a rectangle, the student's response would be a litany of properties: "opposite sides are parallel, opposite sides are congruent, opposite angles are equal, you have four right angles" (Shaughnessy & Burger, 1985, p.420). The students have not yet mastered which properties are necessary and which are sufficient to describe a geometric shape (Mason, 1998). Class inclusion is not yet understood.

Level 3: Order (or theoretical level). At this level, the student can logically order the "litany" of properties of figures previously identified, and begins to perceive the relationships between these properties and between different figures (Pegg, 1995). Students use the properties that they already know to formulate definitions of simple geometric shapes, and class inclusions are understood (Mayberry, 1983; van Hiele, 1999). Simple inferences can be made. For example, in an isosceles triangle a student might be able to draw the inference that since the opposite sides are equal, then the opposite angles are also equal; and to give such definitions as "a square is a rectangle with all sides equal" (Pegg, 1995, p.91). The role and importance of formal deduction, however, is not yet understood (Mason, 1998).

Level 4: Deduction. At this level, deduction becomes meaningful. The student understands the significance of deduction and the role of postulates, axioms, theorems and proof (Hoffer, 1981). Students at this level should be able to supply the reasons for steps in a proof and also construct their own proofs, while the need for rote learning is minimized (Pegg, 1995). For example, when asked to describe a geometric shape, such as a rectangle, using the least amount of information, a student at this level may respond: "A rectangle is a parallelogram with an angle a right angle" (Pegg, 1995, p.91). While Pegg (1995, p.91) states that this level is likely to represent an upper bound on what might reasonably be expected in the secondary school, adding that "only about 25% of 18-year-olds will feel comfortable with problems of this level", Shaughnessy and Burger (1985, p.420) observe that "many high school courses approach the study of geometry at this level".

Level 5: Rigour. At this level, students can reason formally about mathematical systems. The necessity for rigour is understood and abstract deductions can be made (Usiskin, 1982). The students are able to analyze various deductive systems like establishing theorems in different axiomatic systems. Non-Euclidean geometries can be studied and different systems can be compared (Mayberry, 1983; Feza & Webb, 2005). For example, students at this level are able to establish that the locus of all points equidistant from a fixed point is a circle in Euclidean geometry, whereas, the same locus is a square in Taxicab geometry (Krause, 1986).

Clements and Battista (1992, p.429) argue that many school children exhibit thinking about geometric concepts more "primitive than, and probably prerequisite to, van Hiele's level 1". They therefore propose the existence of level 0, which they call pre-recognition. Students at this level can distinguish between curvilinear and rectilinear shapes, but not among shapes in the same class. A student at this level, for example, may differentiate between a circle and a rectangle, but not between a rhombus and a square. In this study, the existence of level 0 was taken into consideration in the assignment of van Hiele levels.

2.8.1 Properties of the van Hiele Levels

The van Hieles made certain observations about the nature of the levels of thinking in geometry and their relationships to geometry classroom instruction. In order to understand how an individual student progresses from one van Hiele level to the next, early van Hiele researchers like Usiskin (1982), Fuys et al. (1988) and, of course, the van Hieles themselves, identified the following properties as pertaining to the levels:

Property 1: Fixed Sequence/Hierarchy. A student cannot operate, with understanding, at van Hiele level $n+1$ without having gone through level n (van Hiele, 1986; Senk, 1989; Mason, 1998).

Property 2: Linguistic Character. An important consequence of the van Hiele theory is its pedagogical relevance to instruction in geometry. Each of the levels is characterized by its own linguistic symbols and network of relations. People reasoning at different levels speak different languages and the same term is interpreted differently. In a classroom situation, for example, one might find the teacher, the texts, and the students functioning at different levels and hence using different linguistic symbols or networks of relations. Consequently, neither the students nor the teacher would understand each other (van Hiele, 1986; Pegg, 1995; Mason, 1998). The mismatch between instruction and students' cognitive levels in geometry is caused largely by teachers' failure to deliver instruction to the pupils in a language that is appropriate to the students' thinking levels (van Hiele, 1986). Van Hiele (1986) sees this property as the most critical in the learning process:

> In education, teachers often give their students unsolvable problems. They use the language of the third level and the pupils often are not even able to use the language of the second level. Sometimes the pupils have not even formed a language of the first level that accompanies the visual structure. (p.90)

The result of such instruction is that the learners are obliged to imitate, but without understanding, the action structure of the teacher. Van Hiele (1986, p.45) therefore proposes that "a teacher beginning the teaching of geometry should address himself to the pupils in a language they understand". By this van Hiele means that teachers should use level-appropriate terminology, symbols, or general language in their geometry instructional practices.

Property 3: Adjacency. Concepts that were implicitly understood at level n become explicitly understood at level $n+1$ (Fuys et al., 1988).

Property 4: Discontinuity. According to van Hiele (1986, p.49), the most distinctive property of the levels of thinking "is their discontinuity, the lack of coherence between the network of relations". That is to say, the learning of geometry is a discontinuous process characterized by qualitatively distinct levels of thinking (Clements, 2004). Two persons who reason at different levels would not understand each other. A student having attained a given level remains at that level for a while, as if ‘maturing’ (Pegg, 1995). Attempts to force the student to perform at a higher level will not succeed until this maturation process has occurred.

Property 5: Retention. Research evidence indicates that students can be on different van Hiele levels for different concepts (Mayberry, 1983; Burger & Shaughnessy, 1986; Pegg, 1995; Mason, 1998). Some students also oscillate between levels (Burger & Shaughnessy, 1986; Orton, 2004). However, once a student’s thought has been raised to a certain level in one concept, it becomes easier for the student to think at that level in other concepts (Pegg, 1995; Mason, 1998).

Property 6: Ascendancy. By ascendancy, movement from a lower level to the next higher one is implied. Progress from one level to the next is more dependent on instructional experience than on age or biological maturation (Clements, 2004). Van Hiele himself stresses this property when he writes: “the transition from one level to the following is not a natural process; it takes place under influence of a teaching – learning program” (van Hiele, 1986, p.50). Van Hiele further posits that students progress through each level as a result of instruction that is sequenced into five phases of learning, beginning with information and moving through guided orientation, explicitation, free orientation, to integration (van Hiele, 1986; Mason, 1998).

2.8.2 The Van Hiele Phases

According to the van Hieles, there are five phases in the learning process that promote students’ progress from one van Hiele level to the next in geometry classroom instruction (van Hiele, 1986, p.53). These phases are as follows:

Phase 1: Information. Through discussion, the teacher leads the learners to become acquainted with the domain of investigation (Mason, 1998). For example, the teacher may show a picture (or diagram) of a rhombus to the class, and the students are asked to identify rhombuses from a collection of shapes and in composite figures (Pegg, 1995). According to Pegg (1995), the students should be allowed to use their own language (or imprecise terminology) in the discussion with minimal interference from the teacher. This enables the teacher to identify what the learners already known about the topic or concept being investigated (Mason, 1998).

Phase 2: Guided Orientation. The teacher guides the students to explore the object of instruction by assigning carefully structured but simple tasks that invite or permit only one solution (Mason, 1998; van Hiele, 1986). For example, simple activities involving folding, reflecting and measuring using a shape such as a rhombus may be assigned to the students. The students are expected to observe things about the angles, sides, and diagonals. The teacher still allows students to use their own language, but with occasional injections of the right terminology (Pegg, 1995).

Phase 3: Explicitation. The students describe what they have learned about the topic using their own language. For example, the students may discuss what they have discovered about the properties of a rhombus. The teacher introduces technical terms associated with the subject matter in order to promote accurate communication among the students.

Phase 4: Free Orientation. Activities here are designed to provide the students with problems that have multi-path solutions. The students are encouraged to find their own solution in the network of relations (van Hiele, 1986). For example, the students may be asked to construct a rhombus given some vertices and sides. Tasks in this phase are open-ended ones.

Phase 5: Integration. The students are conversant with the field of study and have arrived at an overview of the topic. They now have a clear sense of purpose with regard to the object of study, having developed a new network of relations pertaining to it, and are able to summarise and integrate what they have learned. The students have attained the next level. Van Hiele (1986, p.202) suggests that "problems set to check integration must be simple". For example, the students may be asked to "summarise and memorise the properties of the rhombus" (Pegg, 1995, p.96).

A major relevance of the van Hiele learning phases is their link with the level descriptions. The phases are invariant with respect to any two adjacent van Hiele levels. This offers teachers a chance to identify clear starting and ending points in their effort to raise students' thought at any given level to the next higher level during instruction in geometry. In sum, the van Hiele phases offer teachers the opportunity to compare their current instructional practice with a model of teaching that places "introductory discussion, straightforward exercises, language development, multi-path solution exercises and topic overview in a sequence" (Pegg, 1995, p.98).

One of the goals of my study was to explicate the pedagogical patterns of geometry instruction in Nigeria and South Africa using the van Hiele phases as a lens. Given the sequential framework of the van Hiele phases, this would seem an easy goal to achieve. However, a major challenge was the question of the number of lessons to be observed, since according to Dina van Hiele-Geldof, as many as 50 lessons are needed to move learners from level 2, for example, to level 3 (Pegg, 1995). The invariant quality of the phases across the levels provided an answer to this problem. The point stressed by the van Hiele theory on instruction for level attainment is that each unit of a teacher's geometry classroom instructional delivery should be organised in accord with the sequence of the phases. Therefore, even in a single lesson, one can easily observe the presence (or lack thereof) of this sequence.

The description of the van Hiele phases given above appears to be consistent with constructivism as a theory of instruction in education. It would seem necessary, therefore, to discuss the constructivist learning model within the context of my study. But first it is necessary to look more closely at some of the characteristics of the van Hiele theory.

2.8.3 Characteristics of the Van Hiele Theory

What is special about the van Hiele theory anyway? In one sentence, the van Hiele theory owes its unique character to being both a theory of learning and a theory of instruction. According to Bell (1978), Bruner distinguishes between these two categories of educational theories. Learning theories are descriptive in nature. A theory of learning (or intellectual development) describes what has happened and what can reasonably be expected to happen. It describes those mental activities which children can do at certain ages or stages of their intellectual development (Bell, 1978). For example, Piaget's theory of intellectual development describes the stages through which mental growth progresses among children (Piaget & Inhelder, 1969). Piaget's theory identifies cognitive activities which children are capable of doing at certain ages, but it does not prescribe procedures for teaching (Bell, 1978). Piaget's theory is therefore a learning theory.

A theory of instruction, on the other hand is prescriptive in nature (Bell, 1978). A theory of instruction is prescriptive if it "contains principles for the most effective procedure for teaching and learning facts, skills, concepts, and principles" (Bell, 1978, p.140). That is, within the theory there are prescribed processes and strategies for the attainment of the objectives of instruction. The van Hiele theory on levels of geometric thinking (see section 2.8) is descriptive, while the van Hiele theory on instruction (see section 2.8.2) is prescriptive, and hence the van Hiele theory in general is both a theory of learning and a theory of instruction. Bell (1978) expresses the view that a good theory of education is one that amalgamates learning and teaching.

Usiskin (1982) states that the van Hiele theory's ability to describe and predict behaviour, and to prescribe procedures for the attainment of levels (of thinking) are important attributes for any theory that purports to be scientific, as opposed to theories that are merely speculative. According to Usiskin (1982), the van Hiele theory possesses three appealing characteristics, namely elegance, comprehensiveness, and wide applicability. These are briefly discussed below.

1. Elegance: This refers to the rather simple structure of the theory. Movement from one level to the next follows the same basic principles, displaying elegance of form. According to Usiskin (1982, p.16), "the simplicity of structure is evident when one notes that the figures of level 1 are the building blocks for properties at level 2, which in turn are ordered in level 3", and so forth.

2. Comprehensiveness: By comprehensiveness, Usiskin (1982) refers to the descriptive and prescriptive attributes of the van Hiele theory. In his words, "any theory which covers the whole of learning of geometry, and which seeks to explain not only why students have trouble in learning but also what could be done to remove these stumbling blocks, must be called comprehensive" (Usiskin, 1982, p.17).

3. Wide Applicability: Having influenced major reforms in geometry curricula developments “in countries as diverse as the Netherlands, the Soviet Union, and the United States”, Usiskin (1982, p.17) believes that the van Hiele theory should “obviously [be] seen as both widely and easily applicable”. Despite this wide applicability, only a few studies have utilized the van Hiele model to explain students’ geometric thinking levels in an African context. There appears to be a particular dearth of research in the available literature concerning the use of the van Hiele theory on instruction to explicate geometry classroom instructional practices. This lack alone renders the present study a worthwhile endeavour, most especially in the Nigerian and South African contexts.

2.8.4 The Constructivist Approach to Instruction

The purpose of this section is to locate the van Hiele theory in the context of a much broader theory of education that places emphasis on learners as active participants in knowledge generation in the teaching–learning situation. According to Stoker (2003), the current South African national curriculum (Curriculum 2005) is based on a constructivist framework, with a shift in focus from teacher-centred instruction to an approach that places learners at the heart of instruction. Significantly, there is research evidence indicating that the South African national curriculum prescriptions for geometry in the intermediate phase are consistent with the van Hiele theory (Feza & Webb, 2005). Hence, by transitivity, the van Hiele theory can be regarded as a constructivist theory of teaching and learning.

Traditionally, mathematics instruction has been based on a transmission–absorption behaviourist model in terms of which pupils were expected to absorb unquestioningly mathematics structures invented by others (Orton, 2004). Teachers were perceived as holders of the mathematical knowledge to be learned, while pupils were treated, often through drills and practices, as passive recipients of knowledge (McInerney & McInerney, 2002; Orton, 2004). But this method of learning by rote was largely unsuccessful as the “transmitted knowledge was not comprehensively grasped” (Orton, 2004, p.195). Despite this, the behaviourist approach to instruction still permeates the classroom instructional practices of many teachers (see section 2.7.4.3) (Yager, 1991).

Constructivism developed from one of the fundamental assumptions of cognitive learning psychology, which is that new knowledge is for the most part constructed by the learners themselves (Orton, 2004). The early constructivists believed that knowledge is an individual construction and that each learner must construct knowledge for and by himself. According to Yager (1991), effective learning takes place only when learners have ownership of the body of knowledge being learned. Yager (1991) expresses the view that in teaching, knowledge cannot simply be transferred by means of words without there first being some form of agreement about meaning pertaining to some shared experiential base. That is, teachers should determine learners' prior knowledge and lead them, through negotiation, to construct their own meaning and understanding of the subject matter being taught. The importance of teachers' knowledge of learners' prior knowledge about the concept being taught is exemplified by Ausubel, one of the earliest proponents of cognitive psychology, when he said:

> If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly.
> (Ausubel, 1968, frontispiece).

From the constructivist perspective, knowledge is not acquired passively. The mind of the learner is not a blank slate ready only to receive impressions, information or knowledge 'copied' on it by the teacher. Rather, learners actively and creatively construct their own knowledge of the experiential world through organisation and reorganisation of their internal cognitive structures (Piaget & Inhelder, 1969).

There are many elements in the van Hiele theory that are consistent with constructivist ideas about teaching and learning. For example, van Hiele (1986) believes that learning would be inhibited if teaching were to proceed by indoctrination, with the teacher all-knowing and the learner simply there to be instructed. Van Hiele (1986, p.56) stresses the point that teachers "should treat pupils as dignified opponents, opponents capable of introducing new arguments". The role of the teacher, as shared by the van Hieles and the constructivists, should not be the "impartation of knowledge", but rather that of a facilitator who regulates and coordinates negotiations through group discussions in the process of constructing knowledge in the classroom (van Hiele, 1986, p.56).

Language is another important component of both van Hiele's theory and constructivism. Yager (1991, p.53) believes that "all learning is dependent upon language and communication", but that, in the constructivist perspective, language should not be used for transferring information alone. On the contrary, "language must have meaning and not [be] a source for it" (Yager, 1991, p.55). In his theory of cognitive thinking levels, van Hiele (1986) emphasises the view that language plays a central role in learning. He (ibid.) states that when a student has learned to understand a structure by direct contact with reality, he or she must learn the language associated with it, which will empower him or her to exchange views about it with other people. This is a phase in the learning process that van Hiele calls explicitation (see section 2.8.2).

One of the major goals of this study, as stated earlier, is to describe and assess teachers' geometry instructional practices in Nigeria and South Africa. The purpose of this review of literature has therefore been, in part, to establish a theoretical framework in terms of which to evaluate teachers' classroom practices in these countries. The van Hiele theory of instruction, that is, the component of the van Hiele theory concerned with teaching strategies (i.e. the van Hiele phases), has provided this framework (see section 2.8.2). Nevertheless, there remains the challenge of how to articulate what I call the van Hiele phase descriptors, as little seems to have been reported in the literature on what constitutes evidence of each of the van Hiele learning phases in a teaching–learning setting. Fuys et al. (1988), however, offer three different sets of instructional modules that seek to embody the van Hiele phases. In their view (and I concur), the development of these teaching modules reflects key elements of the van Hiele phases in moving learners from one thought level to the next. In his assessment of the modules, van Hiele himself expressed the view that "the modules embodied the levels" and utilized procedures that reflect descriptors of the phases (Fuys et al., 1988, p.15).

In Module 1, Fuys et al. (1988) treat basic geometric concepts (parallelism, angle, congruence etc.) and the properties of quadrilaterals. Angle measurement, i.e. angle properties of triangles and quadrilaterals, are the focus of Module 2. Module 3 treats area measurement of simple two-dimensional shapes. Several instructional activities were designed for each of the teaching modules. For example, Module 1 activities included classification, identification, sorting, listing properties, and explaining subclass relations of two-dimensional shapes. I found the contents as well as the activities of Modules 1 and 2 to be particularly relevant to my study, because they make full use of appropriate instruments for the assessment of students' geometric understanding. Although Fuys et al. (1988) did not draw up a list of what could be termed the van Hiele phase descriptors, the simple design of their teaching modules made it possible for me to isolate and articulate what in my opinion constitutes evidence of the van Hiele phases that can be used to analyze activities in an instructional setting according to the van Hiele theory. These descriptors will be discussed briefly.

The view has always been held that no single method of teaching is the best for all students and for all learning, because instructional procedures that are very effective in one context can be limited in others (McInerney & McInerney, 2002). This notion calls for broadness of mind in the quest for frameworks in terms of which teachers' instructional practices can be analyzed. Yager (1991, p.53) develops a model of learning according to constructivist principles which he calls "Constructivist Learning Model (CLM)", and offers a number of descriptors of what constitutes a constructivist teaching approach. The central idea in Yager's (1991, p.55) Constructivist Learning Model is that "knowledge is not acquired passively", and that in teaching the teacher should promote discussion of subject matter between himself and the learners, and among the learners themselves. The assumption here is that when teachers believe in this approach, they will no longer expect a mathematical problem to have only one solution strategy, and they will expect solution explanation from the learners. In other words, teaching should be directed towards students' development of what Kilpatrick et al. (2001, p.124) call strategic competence – a component of mathematical proficiency[2] which requires that students "should know a variety of solution strategies as well as which strategies might be useful for solving a specific problem".

[2] Mathematical proficiency was coined by Kilpatrick et al. (2001, pp.115–116) to describe what they believe is necessary for anyone to learn mathematics successfully. According to them, mathematical proficiency has five components namely, conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

Although Yager (1991) does not mention the van Hiele theory specifically, many elements of his CLM nevertheless appear to be consistent with the descriptors of the van Hiele phases. Some of the key elements of Yager's (1991) CLM which I found to be consistent with the van Hiele phase descriptors and which can be used for analyzing teaching activities and are therefore relevant to my study include the following:

- Teachers' recognition of, and building on learners' prior knowledge;
- Teachers' creation of an interactive learning environment;
- Teachers' encouragement of learners to challenge, contest and negotiate meaning;
- Teachers' promotion of collaborative work among learners;
- Teachers' use of open-ended questions to encourage learners to elaborate on their responses;
- Teachers' encouragement of learners to seek their own solution strategies;
- Teachers' encouragement of learners to reflect on and refine their ideas.

In using the van Hiele phases (see section 2.8.2) as the general framework for analyzing geometry classroom instructional practices, I have devised, for the purpose of this study, a checklist of what constitutes evidence of the van Hiele phases in an instructional setting. In drawing up this checklist, I have combined ideas from the instructional modules of Fuys et al. (1988) with key elements of Yager's (1991) CLM. The checklist of the van Hiele phase descriptors appears below.

2.8.4.1 Checklist of Van Hiele Phase Descriptors

1. Teacher introduces the topic by recognising and building on learners' prior knowledge.
2. Teacher delays introduction of formal vocabulary, and condones learners' use of common informal terms in the ensuing discussion.
3. Teacher asks questions that seek to clarify students' imprecise terminology and gradually introduces formal mathematical language.
4. Teacher creates an interactive learning environment and encourages learners to challenge, contest and negotiate meanings and solutions to mathematical problems.

5. Teacher asks questions that steer students' thought toward the central idea being developed.
6. Teacher uses open-ended questions and encourages learners to seek their own solution strategies.
7. Teacher encourages learners to elaborate on their responses.
8. Teacher uses questions that encourage learners to reflect on, refine and summarize their ideas about the concept learned.

In drawing up the above checklist, I acknowledge that teaching is a complex and dynamic activity, which means that the items in the list are not necessarily precisely ordered and mutually exclusive. That is, teaching does not necessarily progress in the sequence presented above. However, the first and the last items in the list seem to typify, respectively, the starting and the ending points for an instructional activity in the van Hiele phases.

2.8.5 A Critique of the Van Hiele Theory

As a result of its wide applicability, the van Hiele theory has over the years attracted comment from researchers across the globe. While some of these responses might appear superficial, others are objective and profound. These are some of the general observations that have been made about the van Hiele theory:

1. Although research evidence tends to support the hierarchical nature of the van Hiele levels (e.g. Usiskin, 1982; Mayberry, 1983; Burger & Shaughnessy, 1986; Fuys et al., 1988), there is doubt about the discreteness (i.e. discontinuity) of the levels "as a child can seem to act at different levels in different contexts and even change level within the same task" (Orton, 2004, p.183). Burger and Shaughnessy (1986, p.45), for example, observe that "although the van Hieles have theorised that the levels are discrete structures, [their] study did not detect that feature…Some [students] even oscillate from one level to another on the same task".

2. The van Hiele levels tend to be criterion dependent. By using two different level assignment criteria, Usiskin (1982), for example, demonstrates that a student's

assigned van Hiele level may depend upon the criterion for the level. That is, depending on the criterion used, a student's van Hiele level may change even when the questions or tasks are not changed. Usiskin (1982, p.32) sees this as a weakness of the van Hiele theory, because "if the theory is assumed, a student should have only one level". Nevertheless, Usiskin (1982) offers some suggestions that could help to diminish this weakness of the theory. First, the number of question items at each level could be increased in order to lessen the impact of guessing and of a response on just one item. Second, at higher van Hiele levels, the criteria for attainment could be made easier. For example, "an 80% criterion could be used for responses at levels 1 and 2, and a 60% criterion at levels 3, 4, and 5" (Usiskin, 1982, p.33).

3. There appears to be controversy in the literature regarding the existence or non-existence of van Hiele level 5. Zachos (as cited in Orton, 2004), for example, argues that since there is very little chance of finding any students who have attained level 5, there is little evidence to support the existence of this level. Usiskin (1982, pp.13–14) expresses a similar opinion when he said that "level 5 is of questionable testability" (p.13) and that "van Hiele [himself has] disavowed belief in the fifth level". Indeed, in his more recent writings, van Hiele (1986, p.47) appears to question the existence, or at least utility, of level 5:

> Some people are now testing students to see if they have attained the fifth or higher levels. I think this is only of theoretical value…So I am unhappy if, on the ground of my levels of thinking, investigations are made to establish the existence of the fifth and higher levels.

The above criticisms of the van Hiele theory were duly considered in my study. Concerning the discreteness or otherwise of the levels (criticism number 1), Hoffer (as cited in Fuys et al., 1988, p.181) argues that "these observations may not reflect continuity in learning but rather continuity in teaching". In other words, Hoffer maintains that the proposition that the van Hiele levels are discontinuous is valid. Accordingly, I have assumed discreteness to be an attribute of the levels in this study. In addressing criticism number 2, I accept Usiskin's (1982) suggestion that the number of items at each van Hiele level should be increased in order to lessen the impact of guessing and of the unreliability of a response on just one item. Finally, in line with van Hiele's (1986) changing view about the existence of level 5 (criticism number 3), I have investigated only the first four levels in this study.

In conclusion, whatever criticisms there might be, the literature remains optimistic about the likelihood of finding ways of improving the geometric understanding of school children by building on knowledge obtained from research into the van Hiele levels (Usiskin, 1982; Orton, 2004).

2.9 Chapter conclusion

This chapter has presented the theory in terms of which the data gathered for this study were analyzed and the results interpreted. In doing this, the chapter began with a consideration of the multiple conceptions of geometry. Euclidean geometry as an important and dominant subject in the school mathematics curricula of many countries was discussed and placed in historical perspective. The study of school geometry was problematized against the objectives of geometry teaching and learning in the Nigerian and South African contexts. The conceptual difficulties experienced by students in school geometry and their manifestations, for example, misconceptions, imprecise terminology, problems with the identification and classification of two-dimensional shapes, among others, were examined. Three major causes of students' underachievement in geometry highlighted in the literature – relating to curricular, textual, and instructional/pedagogical factors – were discussed.

The van Hiele theory of geometric thinking levels was presented as the overarching theory that underpins this study. Both aspects of the van Hiele theory – the aspect proposing levels of geometric understanding, and the aspect concerned with instruction in geometry – were discussed, with a view to identifying and specifying frameworks for analyzing the data generated in this study. I ended this chapter with a critique of the van Hiele theory and explained how the major criticisms of the theory were addressed in my study.

The following chapter contextualizes this study within its research paradigm, explaining the orientation, design and process that have shaped the study.

CHAPTER THREE

THE METHODOLOGY

3.1 Introduction

This chapter describes and explains the research process that informs this study. The chapter first gives a brief overview of the study, and then clarifies the research methodology. The methodology is articulated in terms of the orientation, overall design and conduct of the research. It is important to point out at once that these aspects of the research do not comprise a straightforward linear sequence, as one might be led to expect from the methodological structure they suggest (Southwood, as cited in Schäfer, 2003). On the contrary, the entire research process was a complex and dynamic exercise embracing several relationships involving interaction, and spanning three years of intensive study (July 2005–July 2008).

This research study was undertaken primarily to identify and explicate the van Hiele levels of geometric thinking of grade 10, 11 and 12 learners in Nigeria and South Africa, and to give a rich and in-depth description of the instructional processes most proximally related to the levels of geometric conceptualization exhibited by these learners. Thus, there are two aspects to this study. The first aspect concerns students' level of geometric understanding, while the second aspect deals with geometry classroom instructional practices. The study does not, however, necessarily assume a strong direct causal relationship between the two aspects, but rather tries to describe what learning opportunities/experiences students are exposed to within a given instructional strategy.

One of the critical issues that confronted me in this study was how to devise a convincing methodology for describing students' levels of thinking in geometry, and the instructional practices that possibly produced these levels. At first glance, one might think that this should not be a problem since *any pen-and-paper test that focuses on geometric concepts would suffice to determine levels of thinking*. Although most of the van Hiele writings (van Hiele, 1986; 1999), and many of the van Hiele earlier researchers (Hoffer, 1981; Usiskin, 1982; Mayberry, 1983; Burger & Shaughnessy, 1986; Fuys et al., 1988) have documented interesting and legitimate methodologies for accessing students' geometric thinking levels, van Hiele himself acknowledges the fact that the "tracing of levels of thinking that play a part in geometry is not a simple affair, for the levels are situated not in the subject matter but in the thinking of man" (van Hiele, 1986, p.41).

Van Hiele seems to be suggesting here that determining levels of thinking to elucidate what the learner knows in geometry is a complex process that requires more than the simple administration of a pen-and-paper test. This is shared by Schäfer (2003, p.45) who writes that many "traditional pen-and-paper tests do not [adequately] reflect an individual's spatial conceptualization". Driver (1978, p.58) similarly expresses the view that "applying a single test of drawing to describe the overall level of [mathematical] performance of a subject is unjustifiable". Responding to these informed views, I combined pen-and-paper tests with hands-on activity tests in my attempt to explore, describe and interpret levels of geometric understanding among high school learners. These tests, together with other measurement instruments used in this study, are described in detail towards the end of this chapter.

This research is a collective case study (Stake, 2000) conducted in two countries, Nigeria and South Africa. The study involved a cohort of 144 high school learners in the age range of 13 to 22 years, with an average age of 16.6 years. Most of the learners (85% of them) were aged between 15 and 19 years.

The case study as a research approach has generated a lot of debate among natural and social scientists, with some questioning its usefulness in obtaining generalizable results, and others favouring it as a suitable basis for generalization (Lincoln & Guba, 2000). Although I do not intend to immerse myself in this debate, I believe that a few comments from both camps could help to clarify my position on this issue as it relates to my study. Many (natural) scientists believe that generalizations are the be-all and end-all of inquiry, and assert that if one rejects the goals of achieving generalizations, all that can be left is knowledge of the particular, and they ask: "what value could there be in knowing only the unique?" (Lincoln & Guba, 2000, p.27). These scientists claim that case studies are of limited use because “they are not a suitable basis for generalization” (Stake, 2000, p.19).

On the other hand, there are many who hold the view that the case study research method is to be preferred, because the events reported in many such ‘cases’ are likely to be in harmony with the reader’s personal experiences, and thus provide for that person a natural basis for generalization (Stake, 2000). Therefore, Stake (2000, p.19) draws the conclusion that the “most effective means of adding to understanding for all readers [is] by approximating through the words and illustrations of our reports the natural experience attained in ordinary personal involvement”.

In the light of the above debate concerning case studies and generalizability, this study does not lay claim to generalization beyond the cases treated in the research. Nevertheless, especially in view of the in-depth descriptions of the cases provided, it is expected that some of the interpretations, results and conclusions generated in this research will prove useful in other comparable contexts.

3.2 Orientation

This study is oriented in the interpretive research paradigm. The interpretive paradigm holds the view that people have reasons why they behave/act the way they do, and that to understand the reasons behind human behaviour/action requires not detachment from, but rather direct interaction with the people concerned (Connole, 1998; Schwandt, 2000). This paradigm emphasizes the understanding and interpretation of the subjective (classroom) experiences of the participants involved in a study (Cantrell, 1993; Giddens, as cited in Connole, 1998).

Like other research paradigms (such as positivism and constructionism), the interpretive paradigm is characterized by its own ontology, epistemology and methodology (Terre Blanche & Kelly, 1999). Ontology specifies the nature of reality that is to be studied and what can be known about it; epistemology defines the nature of the relationship between the researcher (knower) and what can be known; and “methodology specifies how the researcher may go about practically studying whatever he or she believes can be known" (Terre Blanche & Durrheim, 1999, p.6). The interpretive tradition assumes that people's subjective experiences are real and should not be overlooked (ontology), that these experiences can be understood by interacting with the people concerned and listening to what they have to say (epistemology), and that qualitative research techniques are best suited to gaining an understanding of the subjective experiences of others (methodology) (Terre Blanche & Kelly, 1999).

Simply and somewhat crudely put, the ontological underpinning of this study consists of the geometric thinking levels of high school learners in grades 10–12 in Nigeria and South Africa, together with my desire to investigate these, to find out how the learners actually perceive, interpret and demonstrate their understanding of key concepts in school geometry. In order to better understand the geometric thinking processes attributable to a particular van Hiele level of these learners, I engaged in direct interaction with them and probed in-depth their choice of solution (path) in given geometry tasks (epistemology) (Greenwood, 1996; Dreyfus, 1999). In my efforts to explore, describe and interpret the levels of geometric thinking of the learner-participants, I devised and administered various pen-and-paper tests as well as hands-on activity tests (methodology). In sum, my interest in the sorts of interaction described above and my desire for in-depth probing of learner responses necessitated my choice of the interpretive paradigm in this study.

Given the goals of the study (see Chapter 1), it would be expected that both qualitative and quantitative data would be produced. Accordingly, various research instruments and techniques were employed that generated both qualitative and quantitative data. Creswell (2003) suggests that qualitative and quantitative data may be combined to expand an understanding from one data set to another or to confirm findings from different data sources. In this study, qualitative and quantitative data were used to in such a way as to corroborate each other.

A qualitative research approach is "characterized by intensive study, description of events, and interpretation of meanings" (Schunk, 2004, p.5). It utilizes first-hand accounts of participants' experiences and tries to describe events in rich detail (Terre

Blanche & Kelly, 1999). This study investigates students' geometric understanding in terms of the van Hiele levels and the findings are described in an elaborate manner.

Quantitative research techniques typically attempt to describe relationships among variables statistically and to present a numerical analysis of the social relations being studied (Creswell, 1994; Jackson, 1995). This study, in part, explores the relationship between a student's identified van Hiele level of geometric thought and his or her achievement in 'general' mathematics (algebra and geometry). This gives my study a correlational research element, which usually generates quantitative data to explore the relations that may exist between variables (Schunk, 2004). Given that students' achievement is usually measured and described in numerical terms (10%, 20%, and so forth), it became imperative for me to incorporate quantitative research techniques in this study in order to achieve its declared goals.

Combining qualitative and quantitative methods in a single study is not unproblematic (Brannen, 2004). The challenges that this approach poses for the researcher include the need to undertake extensive data collection, the time-intensive nature of analyzing both textual and numerical data, and more importantly, the need to be familiar with both forms of research (Creswell, 2003). Perhaps this latter point deserves special comment. Prior to undertaking this PhD study, my research experience centred largely on quantitative techniques. As a result, having to undertake a study that has a qualitative research element was, in the main, a huge challenge – a challenge that I managed to meet through extensive reading, attending seminars, and above all, as a result of good guidance from my supervisor.

With the above challenges in mind, I structured this study into three phases, with each phase coinciding approximately with how I sought answers to address each of the three major research questions 'driving' the study. Phase 1 concerns determining the van Hiele levels of geometric thinking of the participating learners. Phase 2 describes the correlations between the van Hiele levels and achievement in 'general' mathematics, and phase 3 examines instructional strategies in the geometry classroom. A more comprehensive description of the phases is presented later in this chapter. But first, the design of the study will be described.

3.3 Design

The research design is that of a collective case study (Stake, 2000), focusing on a total of 144 mathematics learners drawn from two high schools in Nigeria and South Africa – one school from each country. The study was carried out in the natural school setting of the participants. Yin (2003, p.13) states that a case study "is an empirical inquiry that investigates a contemporary phenomenon within its real-life context". In carrying out this study, I consulted the current mathematics curricula for the learner-participants in both countries in order to design questionnaires, worksheets, and tests relevant to the learning experiences expected of the learners. Stenhouse (as cited in Hammersley & Gomm, 2000, p.2), argues the need for case studies in education "as concerned with the development and testing of curricular and pedagogical strategies".

As a research approach, a case study typically investigates the "particularity and uniqueness of a single case, coming to understand its activity within important circumstances" (Stake, 1995, p.xi). However, Stenhouse (as cited in Schäfer, 2003, p.49) points out that in the case of "educational case studies", collective case study (what he calls multi-sited case study) approaches – such as that employed in this study – are increasingly being used. Using several data collection techniques, I interacted with the research participants in their respective classrooms and explored their geometric understanding in relation to the van Hiele levels.

As Jackson (1995, p.17) observes, the interpretive qualitative case study research utilizes "a small number of participants" to enable an in-depth understanding of how the participants experience and interpret their world. Thus a relatively small sample group of participants (24 learners from each of grades 10, 11 and 12 in each country) was involved in this investigation of van Hiele levels of geometric thinking.

3.3.1 Sample

The sample comprised a total of 144 Nigerian and South African high school learners with a mean age of 16.6 years. The ages of these learners ranged from 13 to 22 years, but with the majority, 85% of them, between the ages of 15 and 19. The wide age range of the learners did not pose a problem within the study as the tasks given to them were, for the most part, grade-specific. Where learners from across all three grades were required to perform a common task, I ensured that such a task was largely age-independent (but, nevertheless suitable for all senior[3] grade mathematics learners), as suggested by the van Hiele model.

Of the 144 learners, 72 were drawn from a State-owned high school in Ojo Local Education District (Ojo LED) in Lagos State, Nigeria, and the other 72 from a 'township' high school in Makana Educational District in Eastern Cape Province, South Africa. In each of these schools, 24 learners each from grades 10, 11 and 12 were selected for the study. The 24 learners from each grade comprised 8 high achievers, 8 average achievers and 8 low achievers in mathematics. This was done to ensure that the sample was not skewed towards one extreme in terms of the cognitive ability of the participants.

When this study commenced July, 2005, my initial plan was to involve learners from at least three high schools each in Nigeria and in South Africa. My assumption was that such a sample would be more constitutive and more representative of the cultural spectra of the Nigerian and South African educational landscapes. Doing this, however, would have meant focusing on learners drawn from a single grade (as is often the case in educational research) across the schools. However, since this study focuses on the entire senior stage of secondary education, and since it utilizes a case study approach, as the study evolved, my desire for comprehensiveness (i.e. involving learners from grades 10 through 12 in one study), and for depth (i.e. in-depth analysis of learners' geometric understanding) overrode the need for breadth. These criteria of comprehensiveness and in-depth analysis made a case for the involvement of only one high school in Nigeria and one high school in South Africa in this study.

Given this limitation, the choice of which schools to study was a critical factor. Although geographical accessibility and proximity, functionality, and sex composition (i.e. whether it is a single-sex or a co-educational school) were some of the important factors that influenced my choice of school, the primary criterion was their easy

[3]I use the phrase 'senior grade learners' to refer to Nigerian Senior Secondary School (SS 1–3) learners and South African learners at the Further Education and Training (FET) phase.

accessibility to the majority of Nigerian and South African learners because of the relatively low cost of attending them.

My choice of the senior grades came as a consequence of two considerations. In the first instance, these grade levels represent a complete phase of secondary education in both countries (FRN, NPE, 1998; South Africa, DoE, 2003). This research study, therefore, presents a comprehensive and in-depth view of the entire senior stage of secondary education of my sample in terms of geometric understanding. In the second instance, grade 12 represents a major transition point (from secondary to tertiary education) in each educational system, and a point at which the 'fruits' of secondary education can be assessed (Neville, 1969). The grade 12 learners are in their final year of secondary education and are thus ready for university entrance examinations. This study, therefore, is in a position to reveal the nature and the quality of students (cognitively speaking) who are accessing (or hoping to access) tertiary education in Nigeria and South Africa. For the purpose of this study and for ease of reference, the school involved in Nigeria is designated NS (i.e. Nigerian School) and the one involved in South Africa is designated SAS (i.e. South African School).

3.3.1.1 Description of Nigerian School (NS)

NS is a co-educational state secondary school in Lagos State. It is a 'day' school. The school has 4 buildings that barely accommodate its teeming 2,100 learners in grades 10–12 (Nigerian SS 1–3, i.e. senior school 1–3). Despite its large student population, the school has only 4 qualified mathematics teachers. The mathematics classes are very large, with numbers in excess of 75 learners. Typically of many state schools in Nigeria, the school buildings are not particularly colourful as nearly all are in a state of disrepair – no window panes, broken ceilings, and leaking roofs. As with all state schools in Lagos State, NS is a non-fee-paying school relying on State subsidy for its maintenance. Compared to many other state schools in Lagos State, NS is relatively well resourced as it has a computer laboratory with over 20 functional computers, and two separate science laboratories for chemistry and physics practicals.

In common with virtually all state senior secondary schools in Lagos State, NS houses a junior secondary school (referred to as NS Junior in this study) on the same plot of land. NS Junior has an enrolment of over 1,200 grade 7 – grade 9 (Nigerian JSS 1–3, i.e. Junior Secondary School 1–3) learners, accommodated in two large buildings. One of these buildings is a two-storey building and the other a bungalow. Both schools (NS and NS Junior) occupy a small plot of land that is less than 300 square metres, without any fence separating them. Given the population of over 3,300 learners relative to the size of the school compound, NS (and NS Junior) is often noisy with student chatter. It should, however, be pointed out that although NS and NS Junior share a common plot of land, they, like other high schools in Nigeria, are functionally and administratively distinct, each with its own separate teaching and administrative staff complements.

3.3.1.2 Description of South African School (SAS)

SAS is also a co-educational state 'township' secondary school located in East Grahamstown in the Makana Educational District in the Eastern Cape Province. The school has a total population of 1,200 learners in grades 8 through 12. It has 5 mathematics teachers, only two of whom are academically and professionally qualified. Like its Nigerian NS counterpart, SAS has no boarding facilities, being a 'day' school. The school has a total of 4 blocks, two of which are one-storey classroom blocks. The other two blocks are bungalows, one being the administrative building and the other a large hall that also houses the computer laboratory. The computer laboratory has about 20 computers, but only one of them is functional.

Although the average number of students (44) per mathematics class is above the recommended South African national average of 35 students per class, the situation in SAS is more tolerable than that in NS where, on average, 75 students are found in mathematics classes as against the recommended Nigerian national average of 40 students per class. SAS is comparatively well resourced – good-looking buildings; good chalkboards; good science laboratories for biology, chemistry and physics practicals; water and electricity; and a spacious school compound. As with other state 'township' schools in the Eastern Cape, SAS is non-fee paying, even though an annual stipend of R100 (about 1,650 Nigerian Naira) per student is levied to supplement the state's annual subvention to the school.

Common to many other state ‘township’ secondary schools in the Eastern Cape, SAS is an amalgam of two phases of secondary education in the South African education system. These are: (1) senior phase, i.e. grade 8 and grade 9 (Nigerian JSS 2 and 3), and (2) Further education and Training (FET) phase, i.e. grades 10–12 (Nigerian SS 1–3) – the phase involved in this study. Unlike in Nigeria where NS and NS Junior are separate schools, in South Africa, the senior and the FET phases are within one and the same school, being run by the same administrative and teaching staff. In SAS, the students do not have permanent classes. Instead, during each change-of-lesson time, the students move from one class to the other where the next lesson is to take place. This movement tends to generate noise-making from the students. By contrast, in NS students have their own permanent classes and the teachers move to teach the students during each change-of-lesson time.

3.3.2 Sampling Procedure

Sampling was carried out in two stages using a stratified sampling technique. In the first stage, all learners in each grade in each school were divided into three strata of equal numbers of students based on their cognitive abilities (high, average and low achievers in mathematics). The mathematics teachers in each grade were requested to assist in this selection process. As mentioned earlier, the schools involved were selected on the basis of their educational accessibility, functionality, geographical proximity and accessibility, and sex composition (only co-educational schools). The first stage of the sampling procedure thus employed a purposive sampling technique in which the researchers “handpick the cases to be included in the sample on the basis of their judgement of their typicality” and experience of the central phenomenon being studied (Cohen et al., 2000, p.103; Creswell, 2003).

The second stage employed the fishbowl simple random sampling technique (Omidiran & Sanni, 2001) to select 8 students from each stratum in each grade. The choice of the stratified sampling technique stemmed from the fact that my study is both qualitative and quantitative. Cohen et al. (2000, p.101) point out that “a stratified random sample is, therefore, a useful blend of randomisation and categorisation, thereby enabling both a quantitative and qualitative piece of research to be undertaken”.

Six mathematics teachers (one from each grade in each school) were also purposively selected for the purposes of examining teaching strategies in geometry classrooms in Nigeria and South Africa. The teachers were those involved with the learners participating in the study. Once constituted, the same sample of learners and teachers was involved throughout this study.

Not all the learners, however, took part in all the (testing) activities in this study, as a few of them were unavoidably absent from school on some of the test-taking days even though these days had been collectively agreed upon beforehand. Although my initial intention was to involve equal numbers of male and female students, it turned out that this was not feasible as participation was voluntary, and the sample was made up of only those students who indicated a willingness to participate in the study. In all, 39 male learners and 33 female learners participated in Nigeria, while 31 male and 41 female learners participated in South Africa. Tables 3.1 through 3.6 show the age and sex compositions of the learner-participants involved in this study.

Table 3. 1 Frequency age distribution of sample per school

Age	Number from NS	Number from SAS
13	5	0
14	8	1
15	12	12
16	21	14
17	16	12
18	6	15
19	4	10
20	0	4
21	0	3
22	0	1
Total	**72**	**72**

Table 3. 2 Grouped frequency age distribution of sample per school

Age	Number from NS	Number from SAS
13–14	13	1
15–19	59	63
20–22	0	8
Total	**72**	**72**

Table 3. 3 Frequency age distribution of sample per school per grade

NS Participants				SAS Participants			
Grade	10	11	12	Grade	10	11	12
Age	Number			Age	Number		
13	5	0	0	13	0	0	0
14	6	2	0	14	1	0	0
15	7	4	1	15	10	2	0
16	5	9	7	16	9	5	0
17	1	5	10	17	2	6	4
18	0	1	5	18	0	6	9
19	0	3	1	19	0	4	6
20	0	0	0	20	0	1	3
21	0	0	0	21	2	0	1
22	0	0	0	22	0	0	1
Total	24	24	24	Total	24	24	24

Table 3. 4 Frequency distribution of NS participants by sex per grade

Grade	**10**		**11**		**12**	
	Male	Female	Male	Female	Male	Female
Number	14	10	10	14	15	9

Table 3. 5 Frequency distribution of SAS participants by sex per grade

Grade	**10**		**11**		**12**	
	Male	Female	Male	Female	Male	Female
Number	14	10	5	19	12	12

Table 3. 6 Mean age distribution of sample by sex per school

NS (Nigeria)			**SAS (South Africa)**		
Sex	N	Mean age	Sex	N	Mean age
Male	39	15.6	Male	31	17.1
Female	33	16.3	Female	41	17.6

3.3.3 Personal Acquaintance and Research Ethics

As this study adopts an interpretive qualitative approach (see section 3.2, para. 2), establishing a direct personal relationship with the participating learners and their mathematics teachers was an issue of paramount importance to the entire data collection process. The names of the participating learners and those of their mathematics teachers as well as the names of the participating schools that appear in this research report are all pseudonyms.

The process of negotiating access to the study sites started on June 15, 2006 when I visited SAS with two letters: one from my supervisor, which introduced me to the school principal and sought permission to allow me do research in his school, and the other (written by myself) explaining my study and how I intended to carry it out with minimal interference with the school time-table (see Appendices 1.A and 1.B). On July 18, 2006, the principal granted my request to conduct research in his school, stating that he was interested in knowing the results of my study, since, according to him, "your study promises to address a major problem in my school". The principal then introduced me to Mr Andile, the grade 12 mathematics teacher in SAS who, incidentally, was a colleague with whom I had worked in 2005 in one of the mathematics development programs in the community where my study was conducted. This was an advantage as Mr Andile immediately pledged his support and commitment after I explained the purpose of my study to him.

Mr Andile then organised a meeting in which he and two other mathematics teachers – Mr John, the H.O.D. and grade 10 teacher, and Mr Shlaja, the grade 11 teacher – deliberated on the role that each of them would play in the project. After their meeting, they came up with a time-table detailing how my study should be conducted in the school in order for it to be as unobtrusive as possible. It was then agreed that I could start my fieldwork with SAS on August 14, 2006. Having agreed on this date, I then sought a way to meet with the learners. It was later suggested that teaching the learners some concepts in mathematics would be a useful way of establishing a cordial interpersonal relationship with them. Accordingly, I delivered two separate lessons (one on trigonometry and the other on geometry) to the grade 11 and 12 learners. For the grade 10 learners, I held an informal mathematics quiz with them in which we interacted freely. The learners enjoyed these sessions so much that when the sample for this study was to be constituted, almost every one of them wanted to be included. From the many learners who indicated their voluntary agreement to participate in the study, a sample of 24 learners from each of grade 10, grade 11 and grade 12 was selected, in the manner described in section 3.3.2.

The potential value of the study was explained to the participants during these initial contacts. For example, I informed the learners about how the questions in the research instruments relate to the learning experiences expected of them as prescribed by the geometry curriculum. The participants were assured of their right to confidentiality and anonymity, and were also informed of their right to withdraw from the study at any stage (Durrheim & Wassenaar, 1999).

Although I developed consent forms (see Appendix 1.C, p.5) so that the learners could append their signature to indicate their voluntary agreement to participate in the study (Cohen et al., 2000), I did not, however, get to use them as the majority of the learners, and even their teachers, did not see a need for signing such forms. This, one may argue, does not conform with accepted research ethics, but I had to comply as most of the learners were beginning to be suspicious of my insistence on their signing an agreement form for activities that were meant to improve and benefit them. The idea of sending consent forms (see Appendix 1.D, p.8) to the parents (or guardians) of 'minors' was turned down by the teachers, as according to them, it would raise unnecessary suspicion from the parents. That notwithstanding, Mr Andile agreed to send written notes to the parents of the participating learners informing them why their children might return home late from school, as all the activities (except classroom videos) were to be done after 'school hours'.

Negotiating access into NS in Nigeria followed the due process as described above, but with less rigour. For example, I did not need to teach the learners in order to familiarise myself with them as many of them already knew me.

I arrived in Nigeria on Friday, October 20, 2006, and on October 25, 2006, I visited the principal of NS. After explaining the purpose of my study to her, she immediately assigned me to Mr Lawal, a grade 12 mathematics teacher, and asked him to assist me in conducting my research in her school. Mr Lawal was my classmate during my master's degree program, and as a result he was very willing to help. He then called two other mathematics teachers, Mr Adeleke (a grade 10 teacher) and Mr Balogun (a grade 11 teacher) and I held a meeting with the three of them. At the end of the meeting, they pledged their support and commitment. Thereafter, Mr Lawal assembled, by turn, the grade 10, grade 11 and grade 12 learners in a large hall, and I explained to them the aim of my study. The learners were very eager to take part in the study and at the end a sample of 24 learners per grade was constituted.

The three mathematics teachers met and drew up a time-table for my research in their school. It was later agreed that I should commence my study on November 6, 2006. As with SAS, no consent forms were involved, but issues relating to the rights of the participants were thoroughly explained..

3.3.4 The Structure

Consistent with the first characteristic of the van Hiele theory (see Chapter 2, section 2.8.3), this study adopts a rather tight but accessible design structure. The aim was to reduce the problem of complexity inherent in a multidimensional approach to the data collection process adopted, and hence to provide an enabling framework for collecting and analyzing requisite data (Schäfer, 2003). In the design of some of the instruments for the study, however, a flexible approach was adopted, so that any emerging process could be captured. As stated in the last paragraph of section 3.2, this study is structured into three interconnected phases. Each of the phases describes the research instruments that were used to generate the data which provided answers to each of the three major research questions posed in the study.

3.3.4.1 Phase 1: Determining van Hiele Geometric Levels

This phase aims to establish the van Hiele levels of geometric thinking of the participating learners. Four different sets of techniques consisting of both pen-and-paper tests and hands-on activity tests were devised and used in this phase to address research question 1. Arksey and Knight (1999, p.21) recommend the combination of "different techniques to explore one set of research questions". The structure, contents and rationale for each of the four sets of tests are briefly discussed below, while details of their construction, administration, and grading are presented in sections 3.4.1.1 through 3.4.1.4.

3.3.4.1.1 Terminology in Plane Geometry Test (TPGT)

This test was presented in the form of a structured questionnaire which was issued to the learners. The questionnaire explored learners' understanding of some key technical terms frequently encountered in the teaching and learning of school geometry. A structured questionnaire allows the researcher to seek answers from the respondents within a given range of responses (Cohen et al., 2000). This means that the respondents are constrained to select an answer or group of answers "from a fixed list of answers provided" (Kanjee, 1999, p.295).

The questionnaire consisted of a sixty-item multiple-choice objective test (Appendix 3.A, p.13). Each question was followed by a list of four options (or foils) from which the learners were expected to choose the correct answer. Choppin (1988, p.354) asserts that the objective test, as a means of data collection in educational research, is structured in such a way that "the testees must choose their answers from a specific list of alternatives rather than creating them for themselves". Anderson (1990), similarly states that distinct choices in a questionnaire eliminate possible ambiguity in the responses of the research subjects and facilitate a very precise form of data analysis.

For the TPGT, two conceptually identical but structurally different sets of questions were drawn up on each terminology. That is, for every question that was presented in verbal form (i.e. without diagrams), there was a corresponding visually presented form of it (i.e. presented diagrammatically). All the items in the test were then juggled such that each member of every homologous (i.e. identical) pair of questions was separated far away from the other. Hence, in all, the TPGT consisted of 30 verbally presented questions and a corresponding 30 visually presented ones. The purpose of the homologous pair of questions in this test was to determine whether a student who can give a correct verbal description of a geometric concept also has the correct visual (or concept) image associated with the concept, and vice versa (see section 4.3.1 of

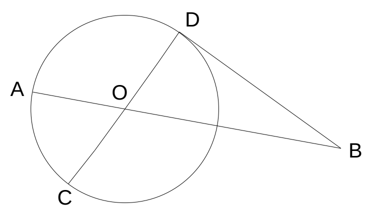
Chapter 4). Questions 5 and 24, for example, are a pair of homologous questions used in the Terminology in Plane Geometry Test (TPGT). They are exemplified as follows:

Question 5 *What is the name of the chord that passes through the centre of a circle?*

A. A tangent
B. A radius
C. An arc
D. A diameter

Its visual image homologous pair (or counterpart) is:

Question 24. *In the diagram, O is the centre of the circle. Which of the following is a diameter?*



A. AOB
B. DB
C. COD
D. OB

For the TPGT, a total of 30 terms pertaining to three different but interrelated concepts in plane geometry were examined. The concepts to which these terms relate are: 1) circles; 2) triangles and quadrilaterals; and 3) angles and lines. Figure 3.1 represents the terminology examined.

Circle	**Triangles and quads.**	**Lines and angles**
Radius	Equilateral triangle	Acute angle
Chord	Isosceles triangle	Right angle
Diameter	Scalene triangle	Obtuse angle
Tangent	Right-angled triangle	Reflex angle
Arc	Similar triangle	Alternate angles
Sector	Altitude of a triangle	Vertically opposite angles
Cyclic quad.	Area of a right-angled triangle	Complementary angles
Concentric circles	Number of sides in a triangle	Supplementary angles
	Number of sides in a quad.	Corresponding angles
	Diagonals of a quad.	Parallel lines
	Lines of symmetry	Perpendicular lines.

Figure 3. 1 Concepts and their associated terminology in the TPGT

Rationale for the TPGT: The rationale for the TPGT is the notion that students' acquisition of the correct terminology in school geometry is important for their success in the subject (see Chapter 2, section 2.7.3.2). The test on geometric terminology was largely aimed, therefore, at determining the relationship that might exist between a student's van Hiele geometric level and his/her knowledge of basic terms in geometry.

3.3.4.1.2 Geometric Items Sorting Test (GIST)

This test was a hands-on activity test that made use of geometric manipulatives. Van Hiele (1999) suggests that giving learners ample opportunity for playful exploration of hands-on manipulatives provides teachers with a chance to observe and assess informally learners' understanding of and thinking about geometric shapes and their properties. Given that this study for the most part explores students' understanding of geometric concepts, the use of hands-on manipulatives allows the learners to demonstrate what they know and think about these concepts. This is supported by Kilpatrick's (1978, p.191) assertion that "we learn by doing…and by thinking about what we do".

The manipulatives were in the form of picture (concept) cards of triangles and quadrilaterals (Appendix 4.B, p.48). The concept cards were made from cardboard cut-outs. Straightedges, protractors and a pair of scissors were used for constructing the cards so as to guarantee accurate side-angle relation properties of the various shapes. In all, there were 30 concept cards of triangles and quadrilaterals numbered 1 to 30. There were 10 triangular cards and 20 cards of various quadrilaterals.

The GIST consisted of a set of structured tasks (Appendix 4.A, p.39) that required individual learners to carry out various operations (identifying, naming, classifying, and defining) on the concept cards. The learners were required to write down their responses as they worked on the various tasks. The questions were structured in a manner that made it possible to decode learners' understanding of and thoughts about the geometric concepts that were presented to them. The question paper for the GIST consisted of five interrelated tasks. Task 1 involved identifying and naming shapes;

Task 2 dealt with the sorting of shapes; Task 3 concerned sorting by class inclusion of shapes; Task 4 was on defining shapes; and Task 5 focused on class inclusion of shapes. Details of these tasks are presented later in this chapter.

The method adopted in the GIST has a wide acceptability in the field of mathematics education among researchers seeking to understand children's thinking about geometric concepts, and it has been used in many earlier studies (Mayberry, 1983; Burger & Shaughnessy, 1986; Fuys & Liebov, 1997; Renne, 2004; Feza & Webb, 2005). In most of these studies, interview schedules, structured or unstructured, were used to tease out students' thought about geometric concepts while the students were engaged in tasks involving the manipulatives (Burger & Shaughnessy, 1986; Renne, 2004; Feza & Webb, 2005). In this study, however, the learners were required to engage with the manipulative tasks, supplying written responses to the questionnaire, before they were interviewed. In essence, these interviews became necessary only during a preliminary on-site analysis of learners' responses (even though it was hoped that the learners would be interviewed at some stage in order to clarify some of their responses). The interviews were, therefore, part of the emerging processes in the study.

Envisaging the difficulty that one might encounter in an attempt to interview all 144 learners, I decided to involve only 36 learners in the Geometric Items Sorting Test (GIST). Selection of the learners involved was based on their performance in the Terminology in Plane Geometry Test (TPGT). In each of the participating schools, 6 learners (2 high, 2 average and 2 low achievers in the TPGT) were selected from each of grades 10, 11 and 12. The purpose was to have a subsample for the GIST that is representative of the entire study sample.

Rationale for the GIST: The rationale for the GIST is based on the notion that many pen-and-paper tests do not necessarily reflect adequately the thinking processes that elicit specific responses from research subjects (see section 3.1). Apart from allowing the learners to articulate their thoughts in writing, the GIST further affords the learners the opportunity to verbalize their thoughts through interactions during the interview sessions. These verbal responses were necessary to enhance the insight afforded by this study into how learners reason about a number of common geometric concepts. The GIST further complements the VHGT (see section 3.3.4.1.4) in providing information about learners' knowledge of school geometry.

3.3.4.1.3 Conjecturing in Plane Geometry Test (CPGT)

The Conjecturing in Plane Geometry Test (CPGT) made use of a constructivist investigative approach (see Chapter 2, section 2.8.4) to explore students' understanding of the properties of simple geometric shapes like circles, triangles, and quadrilaterals (square, rectangle and rhombus). In this approach, Borowski and Borwein's (1989) notion of geometry (see Chapter 2, section 2.2) was employed: learners were required to investigate (through geometrical construction) and discover the properties of these shapes. Such investigation and discovery should lead them to construct their own conjectures about the properties of the shapes and the relationships between these properties.

Worksheets in the form of a semi-structured questionnaire focusing on some grade-specific tasks relating to the selected geometric shapes were developed for the CPGT. The semi-structured questionnaire usually consists of a series of open-ended questions to which participants are expected to respond (Cohen et al., 2000). Because the questions are open-ended, the respondents are allowed, to some degree, to present their answers in their own ways.

Three different sets of worksheets (one for each grade) were developed for the CPGT (see Appendices 5.A.1–5.A.3). The idea was to design and administer questions that were grade-specific, since the sample comprised learners drawn from across three grades with their different and specific curricular focus. The students were expected to follow step-by-step instructions on the worksheets that would lead them to discover the properties of the selected geometric shapes. The worksheets were very well received by the learners and the teachers. The teachers in the participating schools and those in the schools where the instruments for this study were piloted requested copies for use in their geometry classroom instructional design and delivery.

Worksheet 1: This worksheet was designed for the grade 10 learners. The worksheet was developed to explore students' knowledge of the side-angle properties of triangles, rectangles, squares and rhombuses. It required the students to discover, through investigation, the side-angle properties of these shapes and to state conjectures about the relationships between these properties and between the shapes. This worksheet consisted of 6 investigations:

- Investigation 1 was to lead the learners to formulate a conjecture that *the sum of the (interior) angles of a triangle is 180º.*
- Investigation 2 was to lead the learners to formulate a conjecture that *the base angles of an isosceles triangle are equal.*
- Investigation 3 was to lead the learners to formulate a conjecture that *if all the three sides of a triangle are equal, then all the three angles are equal (to one another).* That is, *an equilateral triangle is equiangular.*
- Investigation 4 was to lead the learners to formulate a conjecture that *a parallelogram which has equal diagonals is a rectangle*.
- Investigation 5 was to lead the learners to formulate a conjecture that *a parallelogram which has equal diagonals that bisect each other at right angles is a square*.
- Investigation 6 was to lead the learners to formulate a conjecture that *a parallelogram which has unequal diagonals that bisect each other at right angles is a rhombus*.

Investigations 4 through 6 further required the learners to list as many properties of these shapes as they possibly could, and to formulate a definition of each of the shapes. Details of worksheet 1 are shown in Appendix 5.A.1, p.51.

Worksheet 2: This worksheet was developed for learners in grade 11. The central concept investigated was the similarity properties of triangles. The worksheet was designed for the learners to demonstrate their understanding of:

1. The necessary and sufficient conditions (NASCO) for two triangles to be similar;
2. Proportional division of the sides of a triangle;
3. Similarity of triangles by (corresponding) equal angles.

There were 6 investigations in this worksheet:

- Investigation 1 was to guide the learners to formulate two conjectures: a) *if two triangles are similar, then their corresponding sides are proportional;* b) *if two triangles are similar, then their corresponding angles are equal*. From these conjectures, the learners were required to deduce the necessary and sufficient conditions for two triangles to be similar.
- Investigation 2 was to guide the learners to formulate a conjecture that *if three parallel lines are cut by a pair of transversals, then the corresponding intercepts cut off on each are in the same ratio*.
- Investigation 3 was to guide the learners to formulate a conjecture that *the line drawn parallel to one side of a triangle divides the other two sides proportionally*.
- Investigation 4 was to guide the learners to formulate a conjecture that *the line that joins the midpoints of two sides of a triangle is parallel to the third side and equal to half of it*.
- Investigation 5 was to guide the learners to formulate a conjecture that *if the corresponding angles of two triangles are equal, then their corresponding sides are proportional*.
- Investigation 6 was to guide the learners to formulate a conjecture that *if the corresponding sides of two triangles are proportional, then their corresponding angles are equal*.

Worksheet 3: This worksheet was designed for the grade 12 learners, and it explored learners' mathematical knowledge of circle geometry in the following areas:

1. Chord properties of a circle;
2. Arc-angle properties of a circle;
3. Tangent properties of a circle.

Originally, this worksheet contained 11 investigations. Only 10 of these, however, were carried out and analyzed in this report as an error was detected during the fieldwork in the set of instructions for carrying out investigation 11. The deletion of investigation 11 has no effect on the results of this test since the investigations are mutually exclusive. Investigations 1 through 4 were on the chord properties of a circle; investigations 5, 6, 7 and 8 focused on the arc-angle properties of a circle; and the tangent properties of a circle were explored in investigations 9 and 10. These are briefly explained as follows:

- Investigation 1 was to guide the learners to develop a conjecture that *the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord*.
- Investigation 2 was to guide the learners to develop a conjecture that *the line drawn from the centre of a circle perpendicular to a chord bisects the chord*.
- Investigation 3 was to guide the learners to develop a conjecture that *equal chords are equidistant from the centre of a circle*.
- Investigation 4 was to guide the learners to develop a conjecture that *equal chords subtend equal angles at the centre of a circle*.
- Investigation 5 was to guide the learners to develop a conjecture that *the angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining circumference*.
- Investigation 6 was to guide the learners to develop a conjecture that *the angles in the same segment of a circle are equal*.
- Investigation 7 was to guide the learners to develop a conjecture that *the angle subtended by the diameter of a circle is a right angle*.
- Investigation 8 was to guide the learners to develop a conjecture that *the opposite angles of a cyclic quadrilateral are supplementary*.
- Investigation 9 was to guide the learners to develop a conjecture that *a tangent to a circle is perpendicular to the radius at the point of contact*.
- Investigation 10 was to guide the learners to develop a conjecture that *tangents to a circle from the same external point are equal in length*.

Rationale for the CPGT: The rationale for the CPGT is the notion that making and verifying conjectures is a valuable skill in mathematics generally and geometry in particular (Senk, 1989). Making simple inferences and deductions, and stating definitions, are mathematical abilities associated with levels 3 and 4 reasoning in the van Hiele hierarchy of levels of geometric conceptualization (see Chapter 2, section 2.8). The main focus of the worksheets in the CPGT was to explore students' abilities to formulate conjectures, draw simple inferences and state definitions of simple geometric shapes. The purpose of the CPGT, therefore, was for the most part to determine how students' ability in these cognitive learning activities relates to their van Hiele levels of geometric understanding.

3.3.4.1.4 Van Hiele Geometry Test (VHGT)

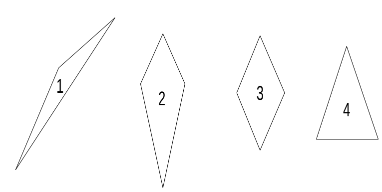
Following the development of the van Hiele theory of the levels of thought in geometry, experts and professional bodies have since developed achievement tests that can be used to measure the attainment of the van Hiele levels among school children (Hoffer, 1983). One such test is the Cognitive Development and Achievement in Secondary School Geometry (CDASSG), which is widely used in the U.S. (Usiskin, 1982). The VHGT was used to classify learners in this study into distinct van Hiele levels of geometric thought.

The van Hiele Geometry Test used in this study is an adapted version of the test constructed by staff of the Cognitive Development and Achievement in Secondary School Geometry (CDASSG), which was originally designed to determine the van Hiele levels of American school children. These children were aged between 11 and 20 years "with 96% of the students being between the age of 14 and 17" (Usiskin, 1982, p.16). The CDASSG test items "were based on direct quotations from the van Hieles' writings and were piloted extensively" (Senk, 1989, p.312). From quotes of the van Hieles regarding what could reasonably be expected of student behaviours at the various levels, questions were written by the CDASSG project personnel for each level that would test students' attainment of specific levels (Usiskin, 1982).

The reason for adapting (rather than adopting) the CDASSG test was that learners do not think at the same van Hiele levels in all areas of geometry contents (Senk, 1989). Therefore, van Hiele (1986) and Senk (1989) suggest that studies that seek understanding of the thinking processes that characterize the van Hiele levels should be content specific. This suggests that as the CDASSG test was constructed, presumably, in accord with the U.S. geometry curriculum, it made sense to adapt the test questions in ways that reflect the Nigerian and South African geometry curricular prescriptions. Nevertheless, 4 questions were adopted from the CDASSG test items for the purpose of comparing the performance of American school children as revealed in Usiskin (1982) with their African (Nigeria and South African) peers in those learning areas. The questions adopted were question numbers 8, 11, 12 and 17 of Part A (see below) of the VHGT. They appeared as question numbers 10, 15, 14 and 20, respectively, in Usiskin (1982).

The van Hiele Geometry Test (VHGT) (Appendices 6.A.1–6.A.3) was made up of two parts – Part A and part B. Part A was a multiple-choice test, and it comprised 4 subtests of the van Hiele Geometry Test. Each subtest consisted of 5 items based on one van Hiele level. That is, there were in all 20 items in Part A , with item numbers 1–5, 6–10, 11–15 and 16–20 testing learners' attainment of van Hiele levels 1, 2, 3 and 4, respectively. Learners' attainment of level 5 was not investigated in this study for reasons explained in section 2.8.5 (paragraphs 4 and 5). Consistent with the CDASSG test, the contents of subtests 1 through 4 were largely the same for all learners across grades 10 through 12. Item numbers 16, 18 and 19 of subtest 4, however, differ across the grades for the purpose of examining some grade-specific concepts. Figures 3.2 through 3.5 are sample items from subtests 1 through 4 of the VHGT.

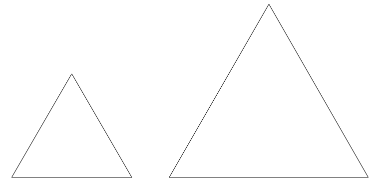
Question 1. Which of these are triangles?



A. All are triangles
B. 4 only
C. 1 and 2 only
D. 3 only
E. 1 and 4 only

Figure 3. 2 Sample item from level 1 subtest

Question 9. An equilateral triangle is a triangle with all the three sides equal in length. Two examples are given below.



Which of (A) – (D) is **true in every** equilateral triangle?

A. Each angle is an acute angle.
B. The measure of each angle must be 60°.
C. Each angle bisector is a line of symmetry.
D. Each angle bisector must also bisect the opposite side perpendicularly.
E. All of (A) – (D) are true.

Figure 3. 3 Sample item from level 2 subtest

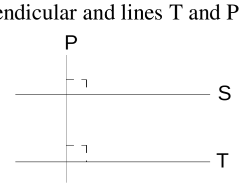
Question 11. What do all rectangles have that some parallelograms do not have?

A. Opposite sides are parallel.
B. Diagonals are equal in length.
C. Opposite sides are equal in length.
D. Opposite angles have equal measure.
E. None of (A) – (D).

Figure 3. 4 Sample item from level 3 subtest

Question 17. Examine these statements.
i). Two lines perpendicular to the same line are parallel.
ii). A line perpendicular to one of two parallel lines is perpendicular to the other.
iii). If two lines are equidistant, then they are parallel.

In the figure below, it is given that lines S and P are perpendicular and lines T and P are perpendicular.



Which of the above statements could be the reason that line S is parallel to line T?

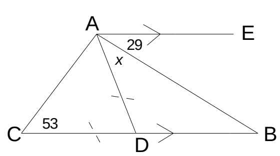
A. (i) only
B. (ii) only
C. (iii) only
D. Either (ii) or (iii)
E. Either (i) or (ii)

Figure 3. 5 Sample item from level 4 subtest

Part B of the van Hiele Geometry Test was an essay test consisting of 3 questions for each grade level, where participants were expected to provide written responses. Although the questions were grade-specific, the structure was similar across all the grades. Question 1 required the learners to calculate a missing value in a given geometrical shape; question 2 required the learners to fill in statements or reasons in an almost-completed geometrical proof; and question 3 was for the learners to write a complete proof of a theorem in geometry.

These questions included some commonly found in texts and examination papers set for these learners. Grade 10 questions were on the side-angle relations of triangles using knowledge of parallel lines. Grade 11 questions were based on the proportion and similarity properties of triangles, while circle geometry was the focus of the grade 12 questions. The focus of each of these questions coincided with the respective areas investigated using the CPGT (section 3.3.4.1.3). An important feature of question 1 in each grade is that it has several solution strategies. Question 1 for grade 10 learners and some of its solution strategies are exemplified in Figures 3.6 and 3.7.

In the diagram, AE//CB and $|\mathrm{AD}| = |\mathrm{CD}|$. $B\hat{A}E = 29^\circ$ and $A\hat{C}D = 53^\circ$. Find the value of *x*. You are to show your workings, giving a reason for each step.



1st Method	2nd Method
$B\hat{A}E = \hat{B} = 29^\circ$ (alt. ∠s, AE // CB)	$\hat{C} = C\hat{A}D = 53^\circ$ (base ∠s of isosc. ΔADC)
$\hat{C} = C\hat{A}D = 53^\circ$ (base ∠s of isosc. ΔADC)	$B\hat{A}E = \hat{B} = 29^\circ$ (alt. ∠s, AE // CB)
Now, $\hat{B} + \hat{C} + C\hat{A}B = 180^\circ$ (sum of ∠s in a Δ)	$A\hat{D}C = x + \hat{B} = x + 29^\circ$ (ext. ∠of ΔABD)
i.e. $29^\circ + 53^\circ + (x + 53^\circ) = 180^\circ$ (same reason)	$\hat{C} = C\hat{A}D = 53^\circ$ (base ∠s of isosc. ΔADC)
$\Rightarrow x + 135^\circ = 180^\circ$ (adding)	$\therefore\ \hat{C} + C\hat{A}D + A\hat{D}C = 180^\circ$ (sum of ∠s in ΔADC)
$\therefore x = 45^\circ$ (Subtracting)	i.e. $53^\circ + 53^\circ + (x + 29^\circ) = 180^\circ$ (substitution)
	$\Rightarrow x + 135^\circ = 180^\circ$ (adding)
	$\therefore x = 45^\circ$ (subtracting)

Figure 3. 6 Exemplifying various solution strategies to a triangle problem

3rd Method	**4th Method**
$\hat{C} = C\hat{A}D = 53^\circ$ (base $\angle$s of isosc. ΔADC)	$B\hat{A}E = 29^\circ$ (alt. $\angle$s, AE // CB)
But, $\hat{C} + C\hat{A}D = A\hat{D}B$ (ext. $\angle$s of ΔADC)	$\hat{C} + C\hat{A}D + A\hat{D}B = 180^\circ$ (sum of $\angle$s in ΔADC)
i.e. $53^\circ + 53^\circ = 106^\circ = A\hat{D}B$ (substitution)	$\Rightarrow 53^\circ + 53^\circ + A\hat{D}C = 180^\circ$ (substitution)
Also, $B\hat{A}E = \hat{B} = 29^\circ$ (alt. $\angle$s, AE // CB)	$\therefore A\hat{D}C = 74^\circ$ (substitution)
$\therefore x + 29^\circ + 106^\circ = 180^\circ$ (sum of $\angle$s in ΔABD)	Now, $B\hat{A}E = \hat{B} = 29^\circ$ (alt. $\angle$s, AE // CB)
$\Rightarrow x + 135^\circ = 180^\circ$ (adding)	But, $x + \hat{B} = x + 29^\circ = A\hat{D}C$ (ext. $\angle$ of ΔABD)
$\therefore x = 45^\circ$ (Subtracting)	$\Rightarrow x + 29^\circ = 74^\circ$ (substitution)
	$\therefore x = 45^\circ$ (Subtracting)

Figure 3. 7 Exemplifying various solution strategies to a triangle problem

During administration of the van Hiele Geometry Test (VHGT), the majority of the grade 10 learners expressed considerable frustration regarding their ability to provide an accurate answer to question 1 of Part B, despite its accessibility via multi-path approaches, as illustrated in Figures 3.6 and 3.7. Driven largely by the curiosity to know what level of questions these learners were capable of answering and by the need to sustain their interest in further activities in this study, I decided to draw up a supplementary van Hiele Geometry Test (SVHGT) soon after they were done with the VHGT. There were only two questions in the SVHGT. Question 1 required only one line of reasoning, and question 2 required two lines of reasoning in determining the value of a missing angle in a triangle. Figure 3.8 shows questions 1 and 2 of the SVHGT.

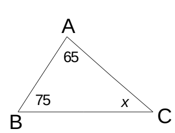
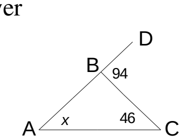
Question 1. Find the value of *x* in ΔABC drawn below. Give a reason for each step in your answer.	**Question 2**. Find the value of *x* in the diagram below. Give a reason for each step in your answer
	

Figure 3. 8 Supplementary van Hiele test for grade 10 learners

Rationale for the VHGT: The rationale for the VHGT is based on the notion that students' understanding of geometry can be described largely by their relative positions in the van Hiele scale of geometric thinking levels. As with the CDASSG van Hiele test (see Usiskin, 1982), the VHGT was designed to determine the van Hiele levels of the participating learners. Since this test was the major instrument that was used to assign learners to various levels of geometric thinking, students' achievement in the VHGT was compared with their achievement in other aspects of geometry (such as those for which the tests described in sections 3.3.4.1.1 through 3.3.4.1.3 were used) so as to determine how achievements in those aspects are related to students' van Hiele levels. Part B of the VHGT was designed further to explore the problem-solving abilities of the learner-participants.

All the tests as described in sections 3.3.4.1.1 through 3.3.4.1.4 were administered in each of the participating schools about two months before the end of the academic session. My belief was that by that time of the school year, the learners would have encountered a significant proportion of the learning experiences intended for them by their respective mathematics curricula. Therefore, students' performances in these tests are to be interpreted as a true reflection of the achieved aspects of the mathematics curricula to which this cohort of learners was exposed. That is, students' achievements in these tests reflected their general abilities in those learning areas.

3.3.4.2 Phase 2: Correlating van Hiele Levels with Achievement in Mathematics

Having assumed the claim that the van Hiele levels correctly describe students' geometric conceptualization (Usiskin, 1982; Hoffer, 1983), it was my concern in this phase to establish among other things whether students' mathematical abilities in general could be described in terms of these levels. Although van Hiele (1986) claims that the levels indeed permeate many other aspects of mathematics other than geometry, there appears to be a dearth of empirical evidence in the literature making this link explicit. The focus of this phase, therefore, was to determine what relationship might exist between the van Hiele levels and the general mathematics achievement of the learners in this study.

The technique used was to correlate the van Hiele test scores of the students with their 2006[4] end-of-year examination scores in mathematics. The end-of-year examination scores of the students were obtained from the archival records of the participating schools with the help of the teachers who also participated in this study. Students' scores in the various tests (TPGT, GIST and CPGT) were also correlated with the van

[4] 2006 was the year in which the learners wrote the VHGT and other tests used in this study.

Hiele test scores for the purpose of determining the relationship between the van Hiele levels of geometric thinking of the learners and their achievement in these other aspects of geometry.

This phase was also concerned with the determination of the relationship between a student's ability in verbal geometry terminology tasks and his/her ability in visual geometry terminology tasks. To do this, students' scores in all 30 verbally presented items and all 30 visually presented items in the TPGT (see section 3.3.4.1.1) were correlated. The visually rendered questions are even-numbered and the verbally presented questions odd-numbered.

In sum, Phase 2 of this study concerned for the most part using the data obtained in Phase 1 to determine the various relationships that might exist between the learners' van Hiele levels and their understanding of geometry in particular, and of mathematics in general. All the correlations were done using StatSoft (2007), version 8.0 of Statistical Data Analysis Software System (SDASS) at the Statistics Department, Rhodes University.

3.3.4.3 Phase 3: Instructional Methods in Geometry Classrooms

Phase 1 of this study concerned the determination of how well students are learning school geometry as revealed by learners' van Hiele levels. Phase 2 interrelated the levels with students' knowledge of other aspects of high school geometry through comparisons with students' achievement (test scores) in those other learning areas. These two phases had been the focus of many earlier van Hiele researchers (see, for example, Burger & Shaughnessy, 1986; Senk, 1989; van der Sandt & Niewoudt, 2003; Feza & Webb, 2005; Siyepu, 2005). As important as it is to know students' van Hiele levels, determinations of these levels alone is, in my view, not sufficient. Also needed is information on the classroom processes – on teaching – that are contributing to the production of the levels among the learners. This is the concern of Phase 3 of this study. Van Hiele himself emphasized the role of instruction in student learning when he proposed, in his learning phases, a specific sequence of instructional activities that could increase students' opportunities to learn in the geometry classroom (see Chapter 2, section 2.8.2). The aim of Phase 3 of this study, therefore, was to provide information on how geometry is taught in Nigerian and South African high schools, and to elucidate what possible learning opportunities the instructional methods observed could offer learners in the subject, in terms of the van Hiele learning phases.

In order to unpack the instructional strategies in geometry classrooms in Nigeria and South Africa that possibly contributed to students' van Hiele levels, I made use of non-participant observational techniques by videotaping classroom processes in six geometry classrooms. A non-participant observer, according to LeCompte and Preissle (1993), assumes a neutral unobtrusive position while observing subjects as they engage in their natural everyday activities. Accordingly, a videotape was used to record on-going instructional activities in three geometry classrooms in Nigeria and three geometry classrooms in South Africa.

The teachers whose classrooms were videotaped were those whose learners participated in this study. In Nigeria, the classes videotaped were those of Mr Adeleke (the grade 10 teacher), Mr Balogun (the grade 11 teacher) and Mr Lawal (the grade 12 teacher). In South Africa, the classes videotaped were those of Mr John, Mr Shlaja and Mr Andile, the grades 10, 11 and 12 teachers, respectively.

Initially, I was hesitant to use video studies of geometry classroom instructions to attempt to explain students' achievement in geometry and to extend insight into the learning opportunities offered by the instructional methods used. This was because there are many other factors that influence learning in a significant way, such as students' home and social life, resources available to the school, and the type of community in which it is situated. Without minimizing the importance of these, there, however, seems to be a consensus in the literature that "much of what our society expects children to learn, they learn at school, and teaching is the activity most clearly responsible for learning" (Stigler & Hiebert, 1999, p.3).

Another challenge that I grappled with was whether it was not too presumptuous to describe observed instructional methods in geometry classrooms in Nigeria and South Africa as 'typical pedagogical patterns' in these countries on account of a video study of only one school in each country. Stigler and Hiebert's (1999) TIMSS video study of instructional methods in mathematics classrooms in Germany, Japan and the U.S., however, gave me much encouragement. In their study, only one eighth-grade mathematics classroom was videotaped in each school across all the three countries (though their study covered 100, 50 and 81 schools in Germany, Japan and the U.S. respectively). The conclusion that Stigler and Hiebert reached was as follows:

> As we looked again and again at the tapes we collected, we were struck by the homogeneity of teaching methods within each culture, compared with the marked differences in methods across cultures. (Stigler and Hiebert, 1999, p.x)

Given Stigler and Hiebert's assertions that there is little variation in teaching methods within cultures, I felt that it was not necessary to observe a large sample in order to capture typical instructional patterns within a culture. Indeed, the view that teaching is culture-based is further supported by Cogan and Schmidt (1999, p.69) when they coin the phrase "characteristic pedagogical flow (CPF)" to refer to the typical distinctive patterns of instructional and learning activities evident in each of the six countries that they studied using classroom videos (France, Japan, Norway, Spain, Switzerland and the United States).

In this study, three geometry classrooms, as against one in some studies (Stigler and Hiebert, 1999), were videotaped in each school in Nigeria and South Africa. Therefore, what the study seemingly loses in breadth (few schools) it gains in depth (several classrooms in one school), which is typical of interpretive qualitative case studies. Although it may be argued that the observed instructional methods in geometry classrooms described in this study are not necessarily typical or representative of geometry classroom instruction in Nigeria and South Africa as a whole, they, however, offer some insight into the face of instruction in geometry classrooms in these countries. Resulting from rich descriptions and in-depth analyses of the classrooms that were observed, insight would also be gained into what opportunity observed instructional methods hold for the learners to learn geometry.

With the potential threats to the validity of this phase of my study thus clarified and addressed, the stage was set for the real task – turning videos into information. Videos by themselves do not contain meaning. It is the responsibility of the researcher to construct meaning out of videos. A common but very useful process of constructing meaning out of classroom videos is for the researcher to watch the lesson videos again and again, so that certain impressions or images of teaching in each lesson begin to stand out gradually (Stigler & Hiebert, 1999). The images thus formed could be used as a common language in terms of which other lessons could be analyzed. But constructing meaning from videos could be highly subjective, as the images produced might well vary according to the individual's construction.

I did not rely on the images alone. It could, in fact, be dangerous to try to construct meaning from classroom videos by merely watching them without some frame of reference – a kind of a system of code developed either from the videos themselves or from a theory on classroom instruction. The frame of reference identifies features of the events in a video objectively so that anyone who watches can agree. Distinct elements in a frame of reference can be used to quantify the events on the video, so that one can know how frequently different categories of activities occur in a lesson (Stigler & Hiebert, 1999).

The frame of reference which gave rise to an objective description of the geometry classroom processes videotaped in this study is the checklist of the van Hiele phase descriptors (see Chapter 2, section 2.8.4.1). The degree of conformity with or deviation from the van Hiele model of the learning phases as exemplified by the checklist of the van Hiele phase descriptors was, therefore, a measure of the learning opportunities that observed instructional methods offer the learners in geometry classrooms in Nigeria and South Africa. In sum, the process of turning videos into information yielded two kinds of results: subjective images of teaching in Nigeria and South Africa, and objectively quantified data that indicates the degree to which observed teaching methods conform to the van Hiele theory on instruction.

3.4 Process

In sections 3.3.4.1.1 through 3.3.4.1.4, I have described the general structure and contents of the various test instruments for data collection used in this study. This section elaborates further on these tests and describes the processes followed in constructing and administering these tests as well as the procedures for grading them.

The analysis procedures and issues relating to validity and reliability are discussed towards the end of this section.

3.4.1 Collection of data

Data were collected mainly through the construction and administration of both pen-and-paper and hands-on activity tests in geometry (see section 3.3.4.1). Information on teaching methods was, however, gathered through the video study of on-going instructional activities in geometry classrooms (see section 3.3.4.3).

3.4.1.1 Construction, administration and grading of TPGT

Test construction: This 60-item multiple-choice test (see Appendix 3.A, p.13) was constructed from scratch, but to a large extent drew for its structure and contents on Usiskin's (1982, p.161) "Entering Geometry Student Test", and to a lesser extent on the TIMSS 1995 test (see Brombacher, 2001 for the TIMSS test). In fact, item 19 of this test was adopted from the 1995 TIMSS test (TIMSS item number O03), while items 36 and 51 were adopted from Usiskin's (1982) test items (numbers 10 and 13, respectively). These items were included in the TPGT, in light of the reportedly poor performance by comparable international students, so as to enable comparison of their scores with those of the learners in this study. Since the TPGT examined students' knowledge of basic terminology frequently encountered in junior and high school geometry, no question was asked that required the learners to calculate the areas and/or perimeters of shapes (except item 51 in which the learners calculated the area of a right-angled triangle).

Test administration: As with all tests used in this study, I personally administered the TPGT with the assistance of the participating teachers. The test was meant to be written by all the participating learners irrespective of their grade levels. As with other tests, the test was written after 'school hours'. Multiple-choice answer sheets (see Appendix 3.B, p.24) were acquired from the Academic Development Centre (ADC) of my university for the purpose of this test. My experience from piloting this test among 12 learners (from a school similar in terms of social and cultural context to the ones involved in this study) necessitated that I demonstrate to the participating learners how to correctly shade their answers on the multiple-choice answer sheets. This precaution was necessary, since scoring of students' responses was done by computer. The pilot study further indicated that the time allocation of 50 minutes was sufficient for the learners to complete the test.

Test grading: Scoring of students' responses was done by staff of the Academic Development Centre (ADC) of my university using 'Scan Tools for Window', version 2.2. All the items in the TPGT carried 1 point each. Hence, students' scores ranged from 0–60 marks. The percentage score was calculated for each student and an item analysis of students' responses was done using Microsoft Excel.

3.4.1.2 Construction, administration and grading of GIST

Test construction: This hands-on activity test (see Appendix 4.A, p.39) made use of geometric manipulatives, and was constructed from scratch by myself. Initially, the set of manipulatives constructed and used by Feza and Webb (2005) were acquired on request for adoption in this study. But, as the acquired set of manipulatives included many polygons other than triangles and quadrilaterals, I had to construct my own set of manipulatives. Feza and Webb's manipulatives nevertheless offered a useful model. The contents and style of questioning in the GIST reflected for the most part those of Mayberry (1983) and Burger and Shaughnessy (1986), both of whose studies were underpinned by the van Hiele theory.

As stated in section 3.3.4.1.2, the manipulatives consisted of numbered concept cards of triangles and quadrilaterals. The cards were cut out of cardboard and numbered 1 to 30. Straightedges, protractors and a pair of scissors were used for constructing the cards so as to guarantee accurate side-angle relation properties of the various shapes. The triangular shapes constructed included isosceles, equilateral, scalene, right-angled triangles, and several combinations of these (see Figure 3.9). The quadrilaterals constructed were squares, rectangles, rhombuses, parallelograms, kites, and trapeziums (see Figure 3.10). There were at least two of each type of shape, differentiated by varying the size, colour or orientation of the number written on the card (see Appendix 4.B, p.48).

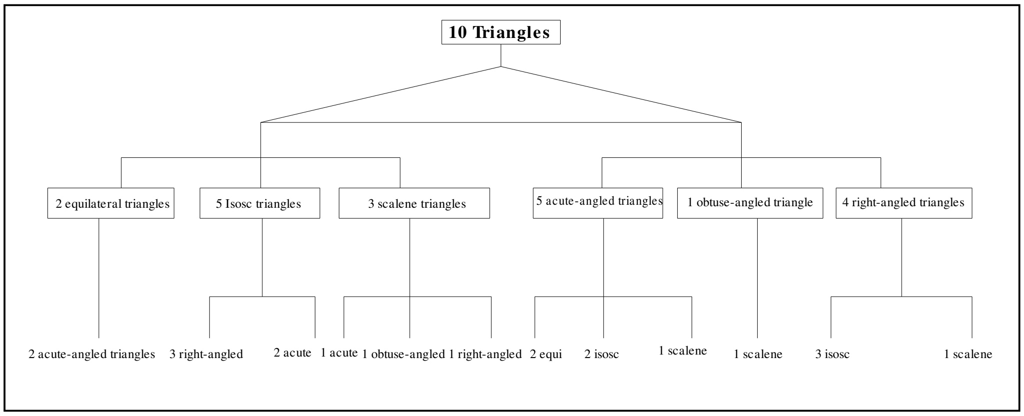


Figure 3. 9 Number and composition of triangles used in the GIST

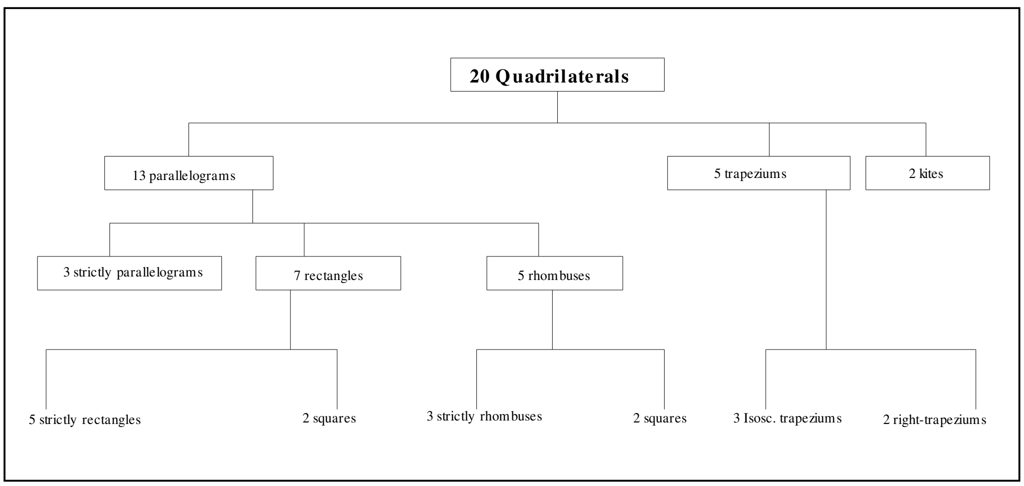


Figure 3. 10 Number and composition of quadrilaterals used in the GIST

As stated in section 3.3.4.1.2, a questionnaire consisting of five interrelated tasks was developed for the GIST. These tasks are briefly explained below.

Task 1 Identifying and naming shapes: This task required the learners to identify each shape by stating the correct name of the shapes. Each learner was requested to justify his/her naming.

Task 2 Sorting of shapes: This task required the learners to sort all 30 shapes into two groups – groups of triangles and quadrilaterals. The students were required to state the criterion for their grouping and also to state the general/common or collective name of the shapes in either group.

Task 3 Sorting by class inclusion of shapes: This task required the learners to make a further sorting of the shapes in either group into smaller subgroups of shapes that are alike in some way. The learners were requested to state how the shapes in each subgroup were alike. This task, therefore, explores the extent of students' knowledge of the class inclusion of shapes.

Task 4 Defining shape: This task required the learners either to state a definition of a shape or to list the distinctive properties of a shape. A sample question from this task is as follows:

> *What would you tell someone to look for in order to pick out all the parallelograms from among these shapes*?

This question was repeated for rectangles, rhombuses, squares, trapeziums and isosceles triangles.

Task 5 Class inclusions of shapes: Students were required to state with justification whether a shape belonged to a class of shapes with some more general properties. A sample question from this task is as follows:

> *Is shape No. 23 a rectangle? How do you know?*

Shape No. 23 was a concept card of a square. Similar questions were asked for the other shapes.

Test administration: This test was administered to only 36 learners (see section 3.3.4.1.2, para.5 for reasons and selection criterion). During the testing time, each student was given a questionnaire consisting of five interrelated tasks. A pack of the concept cards constructed for the GIST was also given to each student. Straightedges and protractors were provided to each student. The students were then required to work through all five tasks in the questionnaires following detailed instructions for each task. The participating mathematics teachers and I were frequently called upon by the learners while they were working on the various tasks to demonstrate how to determine the size of an angle using a protractor.

The pilot study indicated that 60 minutes was ample time for this test, with a further 5 minutes set aside for interviewing each learner. These interviews were unstructured as the questions asked were based on the individual learner's written responses. For example, a learner who had correctly named a shape as a "*square*", but stated: "*it has 4 equal sides*" as the only reason, would be asked whether all shapes having 4 equal sides were (necessarily) squares. These interviews helped to clarify some of the learners' written responses.

Test grading: The fact of subjectivity in the scoring of responses to essay questions is not unfamiliar to anyone in the education community. Different examiners often arrive at different scores for the same student script in an essay test (Choppin, 1988). It was therefore imperative for me to develop a clear 'marking scheme' or memo that would enable me to assign marks to students' responses with a reasonable degree of objectivity. Accordingly, I formulated a 'marking scheme' (see Appendix 4.C, p.49) with some general criteria for grading the responses of the learners based on the work of Senk (1985). In terms of these criteria, predetermined marks were assigned to specific elements in students' responses that reflected the correct answer. For example, marks were awarded to the learners for their responses to question 1 (Task 1) of the GIST as follows:

0 – Student does not name the shape at all, or names the shape incorrectly.

1 – Student names the shape correctly, but gives wrong/inadequate reason. For example, the student names a 'square' correctly, but offers "*all the sides are equal*" as the only reason.

2 – Student names the shape correctly and gives reasons that are both correct and adequate. For example, the student names a 'square' correctly and gives such reasons as: "*it is a parallelogram; all 4 sides are equal; it has 4 right angles*". Or the student gives a correct definition of a square as the reason, e.g. "*it is a rectangle with all the 4 sides equal*".

3.4.1.3 Construction, administration and grading of CPGT

Construction: This grade-specific test was developed in three worksheets, one worksheet for each grade (see section 3.3.4.1.3). Although the test is originally mine, important ideas from the interview schedules of Mayberry (1983) and Burger and Shaughnessy (1986), as well as from the work of Siyepu (2005), were incorporated into its general format and method of questioning. In addition, the respective geometry curricula of the learners greatly influenced the choice of shapes and the properties that were investigated in this test. The learners were required to follow detailed step-by-step instructions that would lead them to discover the properties of the selected geometric shapes and to formulate conjectures about the shapes.

Test administration: At the time of testing, each student was provided with a worksheet and a set of mathematical instruments including straightedges, protractors, compasses, setsquares and dividers. The learners were expected to enter their responses in spaces provided for that purpose on the worksheets. Whenever the learners encountered difficulties concerning the procedures detailed for constructing any shape, the participating mathematics teachers or I would explain the procedures to them. In some instances, I had to demonstrate to the learners how to determine the measure of an angle using a protractor. The time given for the completion of this test was 90 minutes.

Test grading: The CPGT was graded in a manner similar to the method used for grading the GIST, as described in section 3.4.1.2. The percentage score for each student was calculated. See Appendix 5.B, p.77, for the 'marking schemes'.

3.4.1.4 Construction, administration and grading of VHGT

Construction: This test was adapted from the CDASSG tests used by Usiskin (1982). Part A of the VHGT was designed to reflect the CDASSG van Hiele test, while Part B was designed to reflect, to a lesser extent, the CDASSG proof-writing test (see Usiskin, 1982). Hence, in general, the structure and method of questioning adopted in the VHGT are largely consistent with the CDASSG van Hiele tests. The composition of the VHGT was described in some detail in section 3.3.4.1.4. See Appendix 6.A.1–3, pp. 84, 94 and 104 for the design and contents of the VHGT.

Test administration: This test was meant to be written by all 144 learners who participated in this study. However, 139 learners wrote the test as 5 of them were absent from school on the day of testing. The learners provided their answers to Part A of the VHGT on multiple-choice answer sheets (see Appendix 6.B, p.114) which I acquired from the Academic Development Centre (ADC) of my university. The learners were required to write their responses to Part B in the spaces provided in the question booklet. Since both parts of the test were included in the same question booklet, the learners were not expected to begin Part B until they had finished Part A.. Pilot testing indicated that Part A could be completed in 30 minutes, and Part B in 20 minutes. Therefore, a total of 50 minutes was allowed the learners to complete the VHGT.

Test grading (Part A): As with the TPGT, scoring of students' responses in the VHGT was done by staff of the ADC of my university using 'Scan Tools for Window' version 2.2. Two methods of grading were used to assign marks to the learners.

First grading method: Each correct response to the 20-item multiple-choice test was assigned 1 point. Hence, each student's score ranged from 0–20 marks. The percentage score was calculated for each student and an item analysis of students' responses was done using Microsoft Excel.

Second grading method: The second method of grading the VHGT (Part A) was based on the "3 of 5 correct" success criterion suggested by Usiskin (1982, p.33). By this criterion, if a student answered correctly at least 3 out of the 5 items in any of the 4 subtests within the VHGT, the student was considered to have mastered that level. Using this grading system developed by Usiskin (1982), the learners were assigned weighted sum scores in the following manner:

1 point for meeting criterion on items 1–5 (Level 1)
2 points for meeting criterion on items 6–10 (Level 2)
4 points for meeting criterion on items 11–15 (Level 3)
8 points for meeting criterion on items 16–20 (Level 4)

Thus, the maximum point obtainable by any student was $1+2+4+8=15$ points. The method of calculating the weighted sum makes it possible for a person to determine upon which van Hiele levels the criterion has been met from the weighted sum alone. For example, a score of 11 indicates that the learner met the criterion at levels 1, 2 and 4 (i.e. $1+2+8=11$). The second grading system served the purpose of assigning the learners into various van Hiele levels based on their responses. Working with the modified[5] van Hiele levels, the weighted sums and their corresponding van Hiele levels are as shown in Table 3.7.

Table 3. 7 Modified van Hiele levels and their weighted sums

Levels	Corresponding weighted sum
0	0
1	1
2	3
3	7
4	15

Test grading (Part B): The general criteria for grading Part B of the VHGT were adapted from Senk's (1985) proof-writing grading scheme which is described above, in section 3.4.1.2. Each of the 3 items was assigned 4 points. Thus, students' scores ranged between 0 and 12 marks. The percentage score was calculated for each student.

3.4.2 Analysis

> Qualitative researchers study meaning. The quality of research into meanings and interpretative processes can not be assured simply through following correct procedures. Interpretations and meanings are situated…The quality of …data analysis depends on following well-thought-out procedures, and on ensuring that these procedures reveal the structures of understanding of participants.
>
> Ezzy (2002, p.81)

[5] Modified theory (as against the classical theory) refers to the van Hiele theory with level 5 deleted (see Usiskin, 1982, p.42). Whether one uses the classical theory or the modified theory, the assigning of levels requires that the student at level *n* satisfies the classification criterion not only at level *n*, but also at all levels preceding *n*. This study focuses only on the first 4 van Hiele levels. The modified van Hiele levels referred to in this study, however, apply only to the extent that they are consistent with the requirement for the assigning of levels just described above, and not in the sense of deleting yet another level (in this case, level 4) from the van Hiele theory. Hence, the modified levels are used in the sense of the classical theory.

I can identify with the point made in the above quotation as it implies that the process of data analysis can be both innovative and unique. The quotation further suggests that the process of data analysis followed by the researcher should be the one that best elicits participants' understanding of the concept or phenomenon studied, rather than one adopted simply because it is in some sense accepted as standard. This does not, however, mean that data analysis should be unsystematic. Of course, issues relating to "validity and reliability checks cannot [ordinarily] be ignored" (Schäfer, 2003, 66). Hence, Berg (2004, p.266) states that a researcher with an interpretive bent is likely to start the data analysis process by organizing or reducing the data into categories that "uncover patterns of human activity, action, and meaning". Ezzy (2002, p.83) similarly expresses the view that content analysis begins with predefined categories developed through logical deductions from a pre-existing theory, and that "this way the pre-existing theory is tested against empirical data". Accordingly, the theory that shapes my study is the van Hiele theory (already explicated in Chapter 2), and the categories into which "human activity, action and meaning" (in this case, learners' demonstrated conceptual understanding of geometry) are sorted are the van Hiele levels.

As stated in section 3.2, this study is located in the interpretive paradigm. The view has been expressed that in an interpretive study, it is difficult to clearly separate the stage at which data is collected from the one at which data is analysed (Terre Blanche & Kelly, 1999). This view resonated with my study as the data analysis procedure oscillated between the stage at which data was collected and the stage at which an in-depth data analysis was necessitated. For example, preliminary analysis of participants' achievement scores on TPGT informed the choice of the 36 learners who partook in the GIST (see section 3.3.4.1.2). Evidence in support of this process (preliminary data analysis) can be found in Ezzy (2002, p.63), who states that "data gathered early in a research project guide both the formulation of concepts and the sampling process" – a point corroborated by Corbin and Strauss (1990, p.6) who suggest that "in order not to miss anything that may be salient [to the study], the investigator must analyse the first bits of data for cues". Seidman (1991), however, suggests that any in-depth analysis should be avoided until the data collection process is completed.

Despite the many instruments used for data collection, the desire to keep a close focus on the data and the findings has meant that the data analysis, results and discussion are in this study all organized into a single process. That is, for each set of data, the analysis, results and discussion are presented concurrently in the same chapter (see Chapters 4 through to 9).

3.4.2.1 Quantitative analysis

It was stated in section 3.3.4.1 that Phase 1 of this study aims to determine the van Hiele geometric thinking levels of the participating learners. Consistent with the practice and results of many earlier van Hiele researchers (e.g. Usiskin, 1982; Mayberry, 1983; Senk, 1989), this phase generated mainly quantitative numerical data in the form of the test and examination scores of the learner-participants. Therefore, the use of statistical procedures for data analysis was considered appropriate to this study. Durrheim (1999b, p.96) asserts that "statistical procedures are used to analyse quantitative data". Basically, statistical analysis in educational research is of two types: descriptive data analysis and inferential data analysis (Daramola, 1998; Durrheim, 1999b). Descriptive analysis seeks to organise and describe the data by investigating how the scores are distributed on each construct, and by determining whether the scores on different constructs are related to each other (Durrheim, 1999b). It does not allow the researcher to extend conclusions beyond the sample data. Inferential data analysis, by contrast, allows the researcher to extend knowledge obtained from a sample data to the whole population.

Given that this study is a case study (see section 3.3), I employed largely descriptive data analysis in my attempt to understand, interpret and describe the experiences of the research participants in terms of their levels of geometric conceptualization. In specific terms, various descriptive statistics such as frequency distribution, charts, measures of central tendency, and correlation coefficients were used to analyse, describe and compare separate sets of quantitative data in this study. For example, although the rationale for the TPGT was basically to determine the relationship that might exist between a student's van Hiele geometric level and his/her knowledge of common geometric terminology (see section 3.3.4.1.1), many other analytic computations (such as frequency distribution and learners' mean scores) were carried out so as to determine how the scores are distributed according to the major categories of terminology relating to circle, triangles and quadrilaterals, and lines and angles.

Correlation coefficients were calculated in order to determine the relationships among different constructs in this study. For example, the relationship between students' knowledge of common geometric terminology and the van Hiele levels was determined through correlational formula. According to Durrheim (1999b), the correlation coefficient is a more exact way of representing relationships between constructs. The Pearson's product-moment correlation was used for all the correlational computations in this study. The correlation between a student's van Hiele level of geometric thinking and his/her achievement in 'general' mathematics, for example, was determined using the Pearson's product-moment correlation formula. Durrheim (1999b) asserts that the Pearson's product-moment correlation is the most commonly used of the correlation coefficients.

3.4.2.2 Qualitative analysis

Cohen et al. (2000, p.282) state that "in qualitative data the data analysis [process] is almost inevitably interpretive". The video data from this study yielded categories of instructional process collectively referred to as images of teaching. "Images of teaching" is a phrase coined by Stigler and Hiebert (1999, p.25) to describe qualitatively mathematics classroom teaching processes in Germany, Japan and the U.S. As used in this study, the phrase refers to a description that captures the dominant and distinctive activity in geometry classroom instructional processes in the participating schools.

It was stated in the last paragraph of section 3.3.4.3 that the process of turning videos into information yielded two kinds of results: subjective images of teaching and objectively quantified data based on the van Hiele learning phases descriptors that indicates the degree to which observed teaching methods conform with (or deviate from) the van Hiele theory on instruction. The concept of images of teaching served the important purpose of extending our knowledge about the nature of instruction in geometry classrooms in Nigeria and South Africa beyond the understanding that observed instructional patterns did (or did not) conform with the van Hiele model of instruction.

Developing images of teaching from the video study of classroom processes was an iterative process that involved watching the videotaped classes again and again, with my trying on each occasion to make more sense and arrive at deeper interpretations of the teaching activities. Terre Blanche and Kelly (1999, p.139) opine that "a key principle of interpretive analysis is to stay close to the data, to interpret it from a position of empathic understanding". After 'staying close' to the classroom videos, questions about themes like concept development, lesson coherence, making connections within the lesson, and the type of task given by the teacher began to emerge as I watched the videos repeatedly. But, as was remarked in section 3.3.4.3, constructing meaning out of videos can be highly subjective. In order to reduce this subjectivity, I did not rely upon my judgment alone but invited an additional three independent observers to join me in a consultative panel. Two of these people were colleagues in the final stages of their PhD study, and the third was my supervisor, who had a wealth of experience in video study of classroom processes. Each of the observers watched the videos individually and wrote an outline of the images of teaching observed in each lesson studied. The images of teaching described in this study are the outcome of the consensus reached by the consultative panel of observers. Concerning objectively quantified data, each observer was guided by the checklist of van Hiele phase descriptors (see Chapter 2, section 2.8.4.1). In applying the checklist to the lessons, each observer first wrote a definition of what "counts as" evidence of each criterion on the checklist. The panel of observers then met and after careful deliberation adopted a definition for each criterion on the checklist. These definitions are stated in section 9.2 of Chapter 9.

Classroom videos were not the only data that lent themselves to qualitative interpretive analysis in this study. As a way of integrating both aspects[6] of my study, I looked beyond students' achievement scores in the various tests. I tried to give qualitative descriptions of these scores by looking closely at how the scores are

[6] It was stated in section (3.1, para. 2) that there are two aspects to this study: The first aspect concerned students' levels of geometric understanding, while the second aspect dealt with geometry classroom instructional practices.

distributed among the major concepts embodying the test items. In particular, qualitative analyses were applied to learners' written responses to the hands-on activity test (i.e. GIST) and Part B of the VHGT. The purpose was to attempt to isolate elements in learners' response patterns that could possibly be explained or described in terms of the type of instruction that they had received.

3.4.2.3 Integration of qualitative and quantitative data

Creswell (2003) suggests that integration of two types of data might occur at several stages in the research process. It could occur during data collection, analysis, interpretation, or in some combination of these stages. In this study, integration of qualitative and quantitative data occurred largely at the interpretation stage and to a lesser extent at the data collection stage. For example, during the data collection stage, open-ended questions (e.g. GIST) were combined with closed-ended questions (e.g. TPGT). Both data sets were aimed at achieving the same goal – an understanding of students' geometric thinking levels. Creswell (2003) believes that 'mixing' the data at the collection stage enables the researcher to gather a richer and more comprehensive data set, making possible more detailed description and a deeper understanding of the phenomenon being studied.

Integration at the interpretation stage involved interpreting qualitative and quantitative data separately. Attempt was then made to 'see' how the two data sets converge or diverge in terms of the construct both sought to describe. For example, learners' geometric understandings were interpreted in terms of their numerical achievement scores in the closed-ended questions (TPGT, VHGT) and in terms of their response patterns in the open-ended questions (GIST, CPGT and Part B of VHGT).

As was stated in the last section, an attempt was also made to interpret students' response patterns in relation to the type of classroom instruction they had received. Creswell (2003) and Brannen (2004) suggest that qualitative and quantitative data may be combined and interpreted to corroborate, cross-validate or complement results from either data source. In this study, qualitative and quantitative data were combined to achieve a combination of these elements. For example, the images of teaching in geometry classrooms served both to cross-validate and complement objectively quantified data from the classroom video studies.

3.4.3 Validity

My answer to whether qualitative and quantitative methods require different approaches to validity is a clear "no".

Tschudi (1989, p.130)

The above quotation reflects a general orientation in the literature on the issue of validity, which is that different research traditions hold different positions on how to ensure validity in the research process. Traditionally, the notion of validity "[has] been based on positivist standards of objectivity and neutrality" (Southwood, as cited in Schäfer, 2003, p.69). In this tradition, the issue of validity has hinged on an emphasis on the appropriate use of data to conduct analysis, test hypothesis, make inferences and draw generalizable conclusions (Schäfer, 2003). Typically, the concept of validity in the qualitative-interpretive research tradition concerns issues about procedures for establishing the trustworthiness and authenticity of a piece of research (Lincoln & Guba, 1985). It is around this issue (trustworthiness) that the debate on paradigmatic preference – qualitative or quantitative – as a research approach appears to be fiercest (Lincoln & Guba, 2000). Schäfer (2003, p.70), following Lincoln and Guba (1985, p.289) and Kvale (1989, p.73), states that a body of "critical literature [exists] that questions qualitative methodologies and accuses them as being 'soft' [Kvale uses 'unreliable'] options in terms of their validity processes and lack of generalizabilty". On the other hand, much has been written in defence of the qualitative approach, arguing for its status as a-most-preferred method of inquiry (Lincoln & Guba, 1985; Tschudi, 1989; Lincoln & Guba, 2000). Given the integrative approach adopted in my study, I consider it more useful to explicate the validity measures taken in the study rather than to engage in this methodological warfare. What is important, according to Tschudi (1989, p.109), is that "whether research is carried out under (predominantly) qualitative or quantitative 'tribal banners', interpretations and conclusions must be justified".

In its broadest sense, validity refers to the extent to which the "research conclusions are authentic" (Durrheim & Wassenaar, 1999, p.61). It is a demonstration that a particular research instrument in fact measures what it purports to measure (Durrheim, 1999a). Validity is a measure "of the extent to which research conclusions effectively represent empirical reality and … [of] whether constructs devised by researchers accurately represent or measure categories of human experience" (LeCompte & Preissle, 1993, p.323). The validity measures taken in this study are based on these conceptions of the notion of validity, and are discussed in the next section.

3.4.3.1 Ensuring validity in my study

To validate my measurement instruments, I consulted the geometry curriculum as well as the textbooks for the learner-participants. The purpose was to gain insight into what the learners were expected to learn so that I could develop my instruments accordingly. As stated in section 3.1, para.2, the main focus of this study was to explore and explicate the van Hiele levels of geometric understanding of the learners. Thus, only questions on students' understanding of geometry were asked. Zeller (1988, p.324) states that establishing *content validity* "involves specifying the domain of content for the concept and selecting indicants that represent that domain of content". After constructing the test items, I consulted two experts – one in geometry and the other (my supervisor) in geometry education – to crosscheck them. (Durrheim [1999a] suggests that the researcher approach others in the academic community to check the appropriateness of his or her measurement tools.)

To further ensure that the contents chosen were within the prescribed domain of study for the learners concerned, I administered a teacher questionnaire (see Appendix 2, p.11) which gave the teachers the chance to crosscheck and contribute to the geometry content areas that were tested in this study. Their responses indicated that the contents examined in this study reflected the prescribed geometry contents for the learners. Piloting the test instruments also helped to refine them.

While drawing up the test items, I constantly referred to the van Hiele (1986) readings as a workable guide. The models of the van Hiele levels developed by Hoffer (1981), Usiskin (1982), Mayberry (1983), Burger and Shaughnessy (1986) and Senk (1989) all guided the design of my questions. In particular, Usiskin's (1982) CDASSG tests were adapted for the VHGT used in this study. This latter concern for validity is what

Durrheim (1999a, p.87) calls *construct validity* and interprets as "the extent to which a measure of a construct is empirically related to other measures with which it is theoretically associated".

Since research interpretation and conclusions that are built upon *triangulation* (i.e. evidence from several sources) are claimed to be stronger and more believable (or simply, more valid) than those that rest primarily on the narrow framework of a single method (Denzin, 1988), I strengthened the results of this study by using data from different sources. The hands-on activity test and the many pen-and-paper tests described in sections (3.3.4.1.1 through 3.3.4.4) were different methods of gathering data that helped to explain students' van Hiele levels of geometric understanding. This is what Cohen et al. (2000) refer to as *methodological triangulation* and explain as a researcher's use of different methods to gather data about the same object of a study to ensure validity.

After all the tests had been written and the grading had been completed, I returned to the schools (as I had earlier been asked to do by both the teachers and the learners) to show the participating learners their scores in the various tests. I did not stop at showing them their scores, but also discussed the solutions to some of the tests with them. By engaging in this activity, I was making sure that the learners (and their teachers too) were persuaded that the scores assigned to them accurately represented their abilities in these learning areas. The process of validity just described is what Lincoln and Guba (1985, p.314) refer to as *member checking*, a process that has the advantage of "put[ting] the respondent on record as having said [or done] certain things and having agreed to the correctness of the investigator's records of them".

The process of turning classroom videos into information involving a consultative panel of four independent observers (as described in section 3.4.2.2) was a validity measure in my treatment of classroom instructional practices. As a further validity measure, to ensure that the teachers did not prepare a special stand-alone lesson for the classroom videos, I requested them to make available to me copies of the previous and next day's lessons and checked that the videotaped lessons fitted into an on-going sequence. It was not a surprise that the taped lessons fitted into an on-going sequence of lessons as I had earlier on obtained from the teachers a time-table indicating when the geometry aspects of the curriculum would be taught in each of the participating schools.

3.4.4 Reliability

> Since there can be no validity without reliability (and thus no credibility without dependability [in qualitative parlance]), a demonstration of the former is sufficient to establish the latter.
>
> Lincoln and Guba (1985, p.316)

The above excerpt suggests that many of the validity measures taken in a qualitative study implicitly guarantee the reliability/dependability of the research. Lewis' (1967, p.190) assertion that "a test cannot have a high validity without a corresponding high reliability [or that] a high reliability is not in itself a guarantee of high validity" corroborates Lincoln and Guba's claim. Accordingly, having discussed some of the validity/credibility measures taken in this study in the preceding section, my concern in this section is to focus on some of the more conventional indexes or measures of reliability employed in this study.

Conventionally, reliability refers to the extent to which a measurement instrument (a questionnaire, a test) yields the same results on repeated applications (Durrheim, 1999a). It means the degree of dependability of a measurement instrument.

3.4.4.1 Ensuring reliability in my study

There are many different ways of determining the reliability of a measuring instrument in educational research. These include test-retest reliability, parallel forms, the split-half method, and internal consistency (Durrheim, 1999a).

In this study, the split-half method was used to check the reliability of the test instruments, because it is a "more efficient way of testing reliability" and it is less time consuming (Durrheim, 1999a, p.90). The split-half method requires the construction of a single test consisting of a number of items. These items are then divided (or split) into two parallel halves (usually, making use of the even-odd item criterion). Students' scores from these halves are then correlated using the Spearman-Brown formula. The value of the reliability coefficient ranges between -1 and 1.

All the tests used in this study were piloted among students from schools of socio-cultural contexts equivalent to those involved in the study. Initially, my aim was to determine and report the reliability coefficients of all four test instruments (TPGT, GIST, CPGT and VHGT) used in this study. However, due to certain constraints, which I will return to presently, the reliability coefficients of only two of the tests (that of TPGT and VHGT) were determined and reported. The Spearman-Brown reliability coefficient (r) calculated for the TPGT and VHGT were $r = 0.87$ and $r = 0.25$, respectively. The comparatively low reliability coefficient calculated for the VHGT is a result of the fewness of the number of items in the test. Usiskin (1982) similarly obtains a low reliability coefficient for the van Hiele geometry test and suggests that increasing the number of items in the test would improve the reliability – a suggestion that this study could not accommodate because of the many other instruments being used.

The reliability coefficients of the GIST and CPGT were not determined in this study as a result of the following constraints inherent in the tests: First, these were essay tests with very few items, which made it problematic to split them into two halves in order to employ the split-half method. Second, even the option of using the test-retest method was constrained by time as it was difficult to assemble the same set of students for retesting. Despite these constraints, the methods of validity explicated in section 3.4.3.1 confer reliability on these test instruments.

3.5 Chapter conclusion

The intention of this chapter has been to describe the research methodology. The methodology was articulated in terms of the research orientation, design and process. It was explained that the study is oriented largely within the interpretive research paradigm and employs both qualitative and quantitative methods of data collection and analysis. The design of the study is a collective case study focusing on a total of 144 mathematics learners drawn from two high schools in Nigeria and South Africa. The overall sample and the sampling procedures were described alongside the research ethics. The research process was explicated with a focus on procedures for data collection, analysis and validity measures. The data gathering tools included both traditional pen-and-paper tests and hands-on activity tests, as well as video recordings of geometry classroom instructional processes. Issues concerning the reliability of the research instruments were discussed.

The results of this study, together with their analysis and discussion, constitute the focus of each of Chapters 4 through 9. In the chapter that follows, the analysis of learners' performance on the TPGT is presented and the results are discussed.

CHAPTER FOUR

DATA ANALYSIS, RESULTS AND DISCUSSIONT 1: THE TPGT

4.1 Introduction

The main focus of this study was to determine the van Hiele levels of geometric understanding of the participating learners and to explicate such instructional practices as may have contributed to these levels of geometric conceptualization. In pursuance of this broad goal, the study adopted a multidimensional approach to the data collection process which was explained in Chapter 3. In order to achieve depth in data analysis and yet maintain coherence in interpreting the results, an organized and systematic process of data analysis was undertaken. Accordingly, separate chapters are devoted to the simultaneous analysis, results and discussion of each set of data that contributed to our understanding of students' geometric knowledge in the overall data analysis process. Chapter 8 integrates and synthesizes the different results from this study by correlating learners' scores in the other tests (the TPGT, GIST and CPGT) and their scores in their school examination in mathematics (SEM) with their scores in the VHGT. In Chapter 9, an attempt is made to relate learners' van Hiele levels to their geometry instructional experiences by analysing and discussing the data from the classroom videotaped lessons. In this particular chapter, the data from the TPGT (Terminology in Plane Geometry Test) are analysed and the results interpreted.

The first part of this chapter provides information on participant students' understanding of basic geometric terminology through the analysis of their mean scores in the TPGT. The next part provides information on students' knowledge of common geometric terminology through analyses of correlation coefficients. The third part focuses on analyses of students' mean scores in the TPGT according to the three major concepts on which these terminologies were drawn up (see Chapter 3, section 3.3.4.1.1). The last part of this chapter presents ***other results*** that allowed for comparison of participants' knowledge of geometric terminology pertaining to selected items from the TPGT with their international peers (see Chapter Three, section 3.4.1.1).

4.2 Students' Knowledge of Geometric Terminology

Information about students' knowledge of basic geometric terminology is provided by an in-depth analysis and interpretation of participants' performance in the TPGT used in this study.

4.2.1 Overall participants' performance in the TPGT

Students' general performance in the TPGT was described in terms of the overall participants' percentage mean score obtained in this test. Table 4.1 summarizes participants' performance in the TPGT.

Table 4. 1 Percentage mean score of all participants in the TPGT

School	N	Mean score	Std Dev.	Min score	Max score
NS	69	40.49	16.78	17	90
SAS	72	47.85	13.82	27	87

As evident from Table 4.1, the percentage average score obtained by learners from the NS (Nigerian subsample) in the TPGT was 40.49% and that of the learners from the SAS (South African subsample) was 47.85%. Given their respective standard deviations as indicated in the Table, one may question whether these averages adequately represent the individual ability of the participating learners. In order to clarify this later concern, the convention in educational measurement and statistics is to determine whether or not the set of students are homogeneous in relation to their scores on the TPGT (Daramola, 1998). To establish this, the tradition has been to obtain a range denoted by R within a closed interval given by *mean – standard deviation* $\leq R \leq$ *mean + standard deviation*, such that if at least ⅔ of the learners' scores lie within the range R, then the set of scores are believed to be homogeneous, and thus the mean is representative of the group's scores; otherwise the scores are heterogeneous and the mean is not representative of the group's scores (Daramola, 1998).

Following the above convention, participants' scores in the TPGT (and indeed all other tests used in this study) were found to be homogeneous as the range R calculated for the NS subsample was R = [23; 58] and 57 (i.e. 83% of the) learners obtained scores that lie within this range. Since 57 > 46 (which is ⅔ of 69), it follows that the scores obtained by the Nigerian subsample on the TPGT are homogeneous, and hence the mean score (40.49) is representative of the group's performance. By a similar calculation, 55 > 48 (which is ⅔ of 72) representing 76% of the South African subsample obtained scores in the TPGT that lie within the range R = [34; 62] calculated for this group of learners. Hence, as in the case of their Nigerian counterparts, the scores obtained by South African learners in the TPGT are homogeneous, and thus the mean score (47.85) calculated for the group is representative of the group's performance.

It is important to make the above initial clarifications concerning the use of mean scores as the analytic tool with which to interpret students' performance in a learning area. The reason is that the value of the mean could be affected by extreme scores (Bennie, Blake & Fitton, 2006). Hence it is useful to ascertain the homogeneity of scores and determine how many learners obtained scores that lie within the acceptable range of scores when using the mean to describe and interpret learners' performance in a given test. As stated earlier, students' scores for the various tests used in this study were found to be homogeneous. Consequently, subsequent references to participants' mean scores in each of the tests are made on the understanding that the mean scores as stated adequately represent the group's performance.

A simple calculation from Table 4.1 indicates that the percentage mean score obtained by all the participating learners in the TPGT was 44.17%. Given that the items that made up the TPGT were largely of van Hiele level 1 in nature, and that the TPGT as a whole was a simple test of learners' knowledge of the simplest and most common geometric terminology frequently encountered in junior and high school geometry, this rather low percentage mean score is an indication that this cohort of high school learners had a low level of knowledge in this learning area. That is, learners in this study had a weak understanding of basic terminology associated with high school geometry. Individual learners' performance in the TPGT is presented in Appendix 3.C.1–6, pp.25–30, and item analyses of participants' responses are discussed later in this chapter.

4.2.2 Performance of Nigerian and South African learners in the TPGT

The mean score of learners from NS was compared with that of learners from SAS. The aim was to determine how Nigerian high school children compare with their South African peers in the TPGT. The results are summarized in Table 4.2.

Table 4. 2 School percentage means for learners in the TPGT

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	69	40.49	16.78	- 2.85	139	0.0051
SAS	72	47.85	13.82			

Table 4.2 indicates that the mean score of the SAS learners on the TPGT was higher than the mean score obtained by learners from the NS. A test of significance revealed that the difference between the means of the NS and the SAS learners in the TPGT was significant at the confidence level of $p < 0.05$, i.e. ($t = -2.85$, 139*df*, $p < 0.05$) in favour of the SAS learners. What this result has shown is that on average, participants from South Africa performed significantly better than their Nigerian peers in the TPGT; or in other words, that the Nigerian subsample in this study had a somewhat weaker understanding of basic geometric terminology than its South African counterpart.

4.2.3 Grade level performance in the TPGT

Grade level analysis of learners' performance in the TPGT focused on the relative performance of grade 10, 11 and 12 learners in the Nigerian and South African subsamples. These results are represented in Chart 4.1.

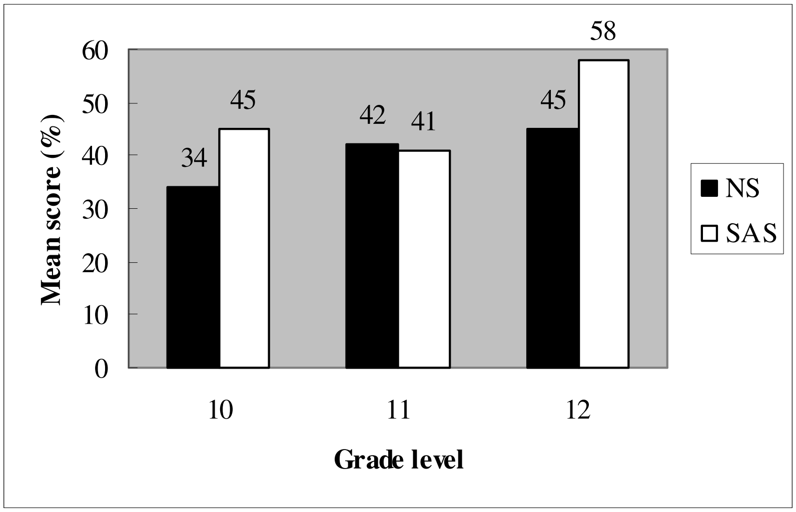


Chart 4. 1 Grade level performance of learners in the TPGT

Chart 4.1 reveals a marginal progressive increase in performance along the grade levels for the Nigerian subsample. From the Nigerian participants, the percentage mean score (45%) obtained by grade 12 learners was marginally higher than that of grade 11 learners (42%), which was in turn marginally higher than the mean score of the grade 10 learners. Given these little differences that occurred in the mean scores of learners from across grades 10–12 of the Nigerian subsample, it could be hypothesized that the Nigerian high school learners in this study add only a little to their repertoire of geometric terminology as they progress from grade 10 through 12.

An interesting revelation in Chart 4.1 about grade level performance of South African participants is that grade 10 learners outperformed grade 11 learners in the TPGT. As the chart illustrates, the mean score (45%) of South African grade 10 learners was marginally greater than the mean score (41%) obtained by grade 11 learners. South African grade 12 learners, however, obtained a higher mean score (58%) than both grade 10 and 11 learners. What these results show is that South African grade 11 learners involved in this study had a weaker understanding of basic geometric terminology than their peers in grades 10 and 12. That grade 10 learners from the South African subsample outperformed their grade 11 peers in the TPGT turned out not to be a fluke, since for all other tests used in this study (as will be revealed in due course), grade 10 South African learners consistently obtained higher mean scores than grade 11 learners. Several reasons (e.g. learners' prior knowledge and attitude, teachers' classroom instruction etc.) could be advanced to explain why this is so. However, one important observation made in this study that could possibly account for this situation in the SAS was that the grade 11 teacher, more than his participating colleagues (as would be revealed during analysis of classroom video studies), engages in code-switching (Xhosa ↔ English) during his instructional delivery. Whether this code-switching ought to (or could) enhance learners' mathematical understanding is beyond the scope of this study (see Marawu, 1997; Simon, 2001). However, it would seem at first glance that code-switching possibly limited grade 11 learners' acquisition of the requisite mathematical vocabulary in the SAS.

The grade level analysis of learners' performance in the TPGT further indicated that South African learners, with the exception of grade 11 learners, obtained higher mean scores than their comparative Nigerian peers. In fact, the Nigerian grade 12 learners obtained a mean score (45%) equal to that of the South African grade 10 learners. In grade 11, the mean score (42%) of the Nigerian learners was marginally greater than that of their South African counterparts, which was 41%.

4.2.4 Grade level comparison of mean scores in the TPGT

Further analysis was done to determine whether or not the differences in the mean scores of the Nigerian and the South African participants in the TPGT reported in the preceding section at each grade level are significant. The results of this analysis are presented in Table 4.3.

Grade level differences in the mean scores of the Nigerian and the South African learners in the TPGT were tested for significance. The results which are represented in Table 4.3 indicated the following: There was a statistically significant difference in the mean score of Nigerian grade 10 learners and South African grade 10 learners in favour of the latter at the 0.001 level (t = - 4.23, 43*df*, $p < 0.001$). That is, South African grade 10 learners performed significantly better than their Nigerian peers on the TPGT. The test of significance also revealed that although Nigerian grade 11 learners obtained a marginally higher mean score on the TPGT than their South African counterparts, the difference in the mean scores of these two groups was not statistically significant (t = 0.27, 46*df*, $p > 0.05$). This means that Nigerian grade 11 learners did not achieve significantly better results than their South African grade 11 counterparts in the TPGT.

Table 4. 3 Grade level mean scores in the TPGT

Grade	NS			SAS			*t-value*	*df*	*p-value*
	N	**Mean**	**Std Dev.**	**N**	**Mean**	**Std Dev.**			
10	21	33.62	10.15	24	44.63	7.22	- 4.23	43	0.0001
11	24	41.67	19.51	24	40.50	8.16	0.27	46	0.7882
12	24	45.33	17.13	24	58.42	16.94	- 2.66	46	0.0107

The *t-test* further revealed that there was a significant difference between the mean score of Nigerian grade 12 learners and that of their South African peers in favour of South African learners (t = - 2.66, 46df, $p < 0.05$). That is, South African grade 12 learners performed significantly better than Nigerian grade 12 learners in the TPGT. These results further buttress the claim in section 4.2.2 that on the average, South African learners involved in this study have a better knowledge of basic geometric terminology than their Nigerian peers.

4.2.5 Gender differences in performance in the TPGT

Analysis of students' performance in the TPGT according to gender was done by comparing:

- The male mean score with the female mean score of all the participants.
- The male mean score with the female mean score of the Nigerian subsample.
- The male mean score with female mean score of the South African subsample.
- The Nigerian male mean score with the South African male mean score.
- The Nigerian female mean score with the South African female mean score.

4.2.5.1 Mean scores in the TPGT of all participants by gender

As Chart 4.2 illustrates, this study identified a gender difference in performance in the TPGT in favour of male learners. On average, male learners obtained higher scores, with a mean score of 48%, than female learners, who obtained a mean score of 41%.

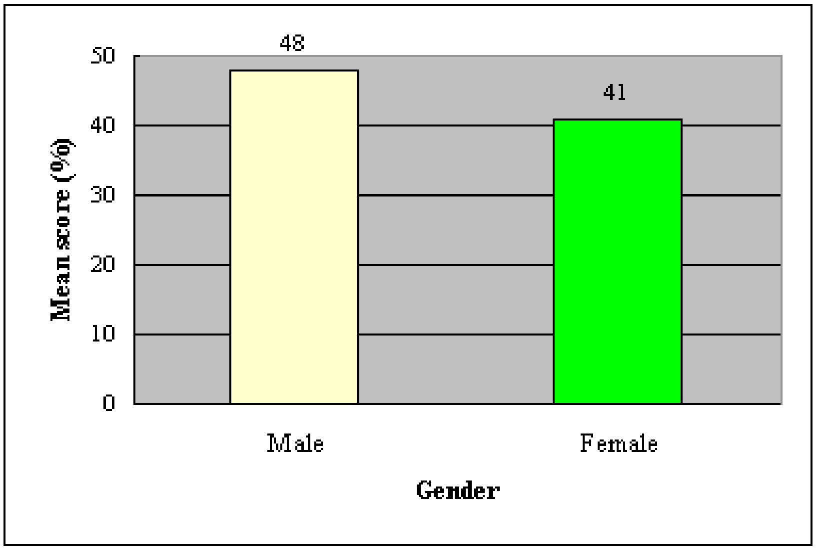


Chart 4. 2 Gender difference in mean scores in TPGT

A test of significance conducted indicated that the difference between the male and female mean scores was statistically significant at the 0.01 level as shown in Table 4.4.

Table 4. 4 Mean scores in the TPGT by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	68	48.04	16.49	- 2.84	139	0.0053
Female	73	40.71	14.19			

These results were consistent with those of Usiskin's (1982, p.84) study in which he reported that in the comparative and similar "Entering Geometry test (EG)", the mean score of American high school male learners was significantly greater than that of their female counterparts. The results were also consistent with those of Barnard and

Cronjé's (1996, p.1) study in which "differential gender performance was in favour of most males" in South Africa in a 20-item multiple-choice Euclidean geometry test. While it might seem bold to conclude that the Nigerian and South African high school male learners had a better grasp of basic geometric terminology than their female counterparts, the results from this study cannot suggest anything to the contrary, since male learners (as will become evident in due course) consistently obtained higher mean scores than female learners in nearly all the tests used in this study. Given that this study involved learners from two separate but similar school and social contexts (Nigeria and South Africa), it was deemed necessary to consider these gender differences separately for each school.

4.2.5.2 Mean scores in the TPGT of the Nigerian subsample by gender

Analysis of the scores of learners from the Nigerian subsample in the TPGT revealed that there was a difference between the male mean score and the female mean score in favour of the male learners. A *t-test* analysis indicated that the difference between the male mean score of 45.76% and the female mean score of 34.41% was statistically significant at the 0.005 level. This means that the female learners from the Nigerian subsample were conceptually poorer than their male counterparts in their knowledge of basic geometric terminology. Table 4.5 summarizes these results.

Table 4. 5 Mean scores in the TPGT of Nigerian participants by gender

Gender	**N**	**Mean score**	**Std Dev.**	***t-value***	***df***	***p-value***
Male	37	45.76	17.01	- 2.96	67	0.0043
Female	32	34.41	14.52			

4.2.5.3 Mean scores in the TPGT of South African subsample by gender

As revealed in Table 4.6, although South African male learners obtained a higher mean score (50.77%) than their female peers who obtained a mean score of 45.63% in the TPGT, the test of significance indicated that the difference between the means is not statistically significant (t = - 1.58, 70*df*, $p > 0.05$). That is, South African male learners' knowledge of common geometric terminology was not significantly better than that of their female counterparts involved in this study. These results further indicate that the gender difference in the mean scores reported in section 4.2.5.1 was due more to the differences that occurred between Nigerian male and female scores than it was due to differences in the scores of South African male and female learners.

Table 4. 6 Mean scores in the TPGT of South African participants by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	31	50.77	15.70	- 1.58	70	0.1188
Female	41	45.63	11.93			

4.2.5.4 Mean scores of Nigerian and South African male learners in the TPGT

The mean score of Nigerian male learners in the TPGT was 45.8%, while that of their South African peers was 50.8% (Table 4.7). The difference in these means was found not to be statistically significant (t = -1.25, 66*df*, p = 0.2141). These results indicate that South African male learners involved in this study were not significantly better than their counterparts from the Nigerian subsample in terms of their knowledge of basic terminology in high school geometry.

Table 4. 7 Mean scores of Nigerian and South African male learners in the TPGT

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	37	45.76	17.01	- 1.25	66	0.2141
SAS	31	50.77	15.70			

4.2.5.5 Mean scores of Nigerian and South African female learners on the TPGT

In this analysis, the mean score of Nigerian female learners was compared with the mean score of South African female learners. As indicated in Table 4.8, there was a statistically significant difference between the mean score (34.41%) of the Nigerian female learners and that of the South African female learners (45.6%) at the 0.001 confidence level in favour of the South African female learners (t = - 3.63, 71*df*, p < 0.001). These results indicated that South African female learners involved in this study had a better knowledge of basic geometric terminology compared with their Nigerian international peers.

Table 4. 8 Mean scores of Nigerian and South African female learners in the TPGT

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	32	34.41	14.52	- 3.63	71	0.0005
SAS	41	45.63	11.93			

4.2.5.6 Mean scores of grade 10, 11 and 12 learners in the TPGT compared

The analysis in this section focuses on a much broader picture of grade level differences in students' performance in the TPGT than that presented in sections 4.2.3 and 4.2.4, which discussed these differences only at each school level. Although there was a difference in the mean score of South African grade 10 and grade 11 learners in favour of the former (see section 4.2.3), this difference diminished in importance when the mean score for all the grade 10 learners (from NS and SAS) was computed and compared with the mean score for all the grade 11 participants. Thus, in general, there was a marginal progressive increase in the achievement of these learners along the grade levels. That is, grade 12 learners achieved marginally better results than grade 11 learners, whose achievement was likewise marginally higher than that of grade 10 learners in the TPGT.

As evident in Table 4.9, there was a difference between the mean scores of grade 12 learners (51.88%), grade 11 learners (41.08%) and grade 10 learners (39.49%) across the entire sample. A one-way analysis of variance (ANOVA) indicated that these differences in mean score are significant (F = 9.77, (2, 138)*df*, p < 0.001).

Table 4. 9 Grade level differences in mean scores in the TPGT

Grade	N	Mean	Std Dev.
10	45	39.49	10.24
11	48	41.08	14.81
12	48	51.88	18.11
All grades	141	44.15	15.73

In order to determine between which grade levels the differences in mean score were significant, a *Scheffe post-hoc test* was conducted. The result of the *post-hoc* comparison (Table 4.10) indicated that the difference in the mean scores of grade 10 and 11 learners was not statistically significant ($p > 0.05$). However, the difference in the mean scores of grade 10 and 12 learners was found to be statistically significant at the 0.001 level. Also, the difference in the mean scores of grade 11 and 12 learners was statistically significant ($p < 0.05$).

Table 4. 10 Scheffe post-hoc test for the TPGT

Grade	10	11	12
10		0.874410	0.000469
11	0.874410		0.002282
12	0.000469	0.002282	

What is evident from the *post-hoc test* is that there is only a little improvement in students' acquisition of geometric terminology as they progress from grade 10 to grade 11. This claim is supported by the non-significance of the difference in the mean scores of learners from these grades as revealed by the *post-hoc test*. Hence, in terms of students' knowledge of basic terminology in high school geometry, grade 11 constitutes a problem for the participating schools. While this is so for the entire study sample, the situation in the SAS deserves special comment.

Grade 10 South African learners obtained a higher mean score than their grade 11 peers not only in the TPGT, but also in all the other tests used in this study, as will be revealed in due course. Although this study did not formally interrogate why this was so in the SAS, an informal interview that I held with a grade 10 mathematics teacher, who incidentally, was the HOD of mathematics in the SAS, nevertheless introduced a reason that could possibly explain this situation in the SAS. According to the HOD, the current crop of grade 11 learners involved in this study learned little mathematics the previous year because *the teacher*[7] *who taught them in grade 10 was very uncommitted.* Whether this was a tenable explanation or not was not interrogated further in this study. However, it is noteworthy to mention that this study, as was stated in the last paragraph of section 3.3.4.1.4 of Chapter Three, was conducted at the SAS towards the end of the academic session in 2006, which means that at that time of the school year, the learners would have presumably been taught a significant part of their mathematics curriculum for that year.

4.3 Correlation analysis between Students' Verbal and Visual Abilities in the TPGT

It was stated earlier that the principal rationale for the TPGT was to determine what relationship might exist between a student's van Hiele geometric level and his/her knowledge of basic terminology in school geometry (see Chapter Three, section 3.3.4.1.1, last para.). A second purpose served by the TPGT was to determine whether a student who knows a correct verbal description of a geometric concept also has the correct visual (or concept) image associated with the concept, and conversely (see Chapter three, section 3.3.4.1.1). As was stated in the second last paragraph of section 3.3.4.2 in Chapter Three, the method used was to correlate students' scores in all 30 verbally presented items with their scores in the 30 visually presented items in the TPGT.

While the analysis of the correlations between students' van Hiele levels and concurrent knowledge of geometric terminology is presented in a later chapter, the analysis of the correlations between a student's ability in verbal geometry terminology tasks and his/her ability in visual geometry terminology tasks as exemplified by students' scores in the TPGT is the focus in this section of this chapter. In the analysis

[7] The teacher referred to was no longer teaching at SAS at the time of this study.

that follows, the verbally presented questions are the odd-numbered items and the visually presented questions the even-numbered items in the TPGT.

4.3.1 Correlation between verbal and visual abilities of learners in the TPGT

In this section, the correlation between students' ability in verbal and visual geometry terminology tasks was determined. By correlating students' scores from the verbally presented items in the TPGT, correlation coefficients were obtained separately for the Nigerian and South African subsamples. The correlation coefficient calculated for the Nigerian subsample (NS learners) was r = 0.83, and the one calculated for the South African subsample (SAS learners) was r = 0.63. Both correlations were found to be significant at the 0.001 level as indicated in Table 4.11.

Table 4. 11 Correlation coefficients for the TPGT by school

School	Odd (verbally presented items)		Even (visually presented items)		N	*r-value*	*p-value*
	Mean	Std Dev	Mean	Std Dev			
NS	11.16	5.12	13.13	5.45	69	0.83	0.00001
SAS	12.68	4.61	15.99	4.58	72	0.63	0.00001

As evident in Table 4.11, the values of the correlation coefficients calculated for both the Nigerian and South African subsamples are positive. This means that a student who correctly answered a verbally presented question in the TPGT also answered its visually presented identical counterpart correctly, and vice versa. That the values of the coefficients are fairly large does not necessarily indicate that the learners have an impressive grasp of geometric terminology. The coefficients only give information about the level of consistency with which participants responded to homologous pairs of questions in the TPGT.

The correlation coefficient calculated for the Nigerian subsample (0.83) was greater than that calculated for the South African subsample (0.63).This again does not imply superiority in performance on the part of the Nigerian subsample over that of the South African. What it does mean is that Nigerian learners were more consistent in their responses to the items in the TPGT than their South African peers. That is, more Nigerians passed or failed identical pairs of questions in the TPGT than South Africans. The higher mean score obtained by the South African subsample (see section 4.2.2) in the TPGT compared with that of the Nigerian participants supports this disclaimer.

Even with the interpretation given in the preceding paragraph, there could be many other dimensions to what these correlation coefficients tell us about students' conceptual understanding of geometric terminology. That the correlation coefficient for the Nigerian subsample was greater than that of the South African participants, means that more Nigerian learners than South Africans who knew a correct verbal description of a geometric concept also had the correct visual/concept image associated with the concept. In terms of the conceptual understanding of basic terminology in geometry, this would mean that South African participants were less conceptually grounded than their Nigerian peers; it may be that the SAS learners engaged in random guessing that resulted in the correlation coefficient being lower than that of the Nigerian participants.

Another interpretation, and perhaps one more tenable, is that South African learners had a more comprehensive understanding of basic geometric terminology than their Nigerian counterparts and hence obtained a higher mean score in the TPGT (see section 4.2.2). But this understanding is less conceptual, as the SAS learners tended to understand the terminology better only in one form of presentation, namely the visual form. Simple calculations from Table 4.11 reveal that there is a wider difference between the mean scores of South African learners for the verbally presented items (odd) and the visually presented items (even) than there is for learners from the Nigerian subsample. This indicates that the South African learners were less successful with geometry terminology tasks that were presented in verbal form than the Nigerian participants.

To conclude, Table 4.11 further indicates that learners from both the Nigerian and the South African subsamples had a better understanding of geometric terminology presented in terms of visual tasks than those that were presented in verbal form, and hence, obtained higher mean scores in the former than in the latter.

4.3.2 Grade level correlation between verbal and visual abilities of NS learners in the TPGT

Further analysis of students' scores was done at each grade level in each of the participating schools in order to see how the scores are distributed between the verbally and visually presented items on the TPGT. In this section, grade level analysis of the correlation coefficients calculated for learners from NS are presented.

The correlation coefficients calculated for grade 10, 11 and 12 learners from the NS were all positive and statistically significant (see Table 4.12). For grade 10 learners, the correlation coefficient was moderately high ($r = 0.53$, $n = 21$, $p < 0.05$); for grade 11 learners, it was very high ($r = 0.90$, $n = 24$, $p < 0.0001$; and for grade 12 learners, it was also very high ($r = 0.84$, $n = 24$, $p < 0.0001$). These high correlation coefficients at each grade level indicated that a learner who did well on the verbally presented items in the TPGT equally did well on their corresponding visually presented items, and vice versa. Given the low mean scores obtained by these learners in the TPGT (see section 4.2.4), these high correlation coefficients suggest that although the learners demonstrated their possession of conceptual knowledge of basic geometric terminology, this knowledge lacks breadth in terms of number of concepts. That is, a student who knew a correct verbal description of a geometric concept also had the correct visual image associated with it, but the number of concepts in which these learners had this conceptual knowledge turned out to be very few. This later point accounts for the very low mean scores that are associated with the high correlation coefficients obtained by these learners in the TPGT.

Table 4. 12 Correlation coefficients at grade level in NS in the TPGT

Grade	Odd		Even		N	*r-value*	*p-value*
	Mean	Std Dev.	Mean	Std Dev.			
10	9.43	2.94	10.76	4.09	21	0.53	0.01370
11	11.67	6.12	13.33	5.87	24	0.90	0.00001
12	12.17	5.34	15.00	5.48	24	0.84	0.00001

As indicated in Table 4.12, the mean scores[8] obtained (out of a maximum of 30 points) by learners at each grade level for the visually presented items on the TPGT were higher than those obtained for the corresponding verbally presented items. Hence, in relative terms, these learners had a better grasp of geometric terminology presented in visual form as against that presented verbally.

4.3.3 Grade level correlations between verbal and visual abilities of SAS learners in the TPGT

Table 4.13 illustrates the correlations between students' scores for the verbally presented items and their scores for the visually presented items on the TPGT across grades 10–12 in SAS. Unlike in the case of the Nigerian subsample, where the correlation coefficients were all positive and significant at each grade level, for the South African subsample, the correlation coefficients were positive only for grade 11 and 12 learners, with that of grade 12 learners being the only one that was statistically significant. For grade 10 learners, the correlation coefficient was negative ($r = -0.06$, $n = 24$, $p = 0.7786$); for grade 11 learners, it was low positive but not significant ($r = 0.37$, $n = 24$, $p = 0.0745$); and for grade 12 learners it was positively moderate and significant ($r = 0.74$, $n = 24$, $p < 0.0001$). These results are summarized in Table 4.13.

[8] These are not percentage mean scores. For the percentage mean scores, simple calculations could be done. For example, in grade 12, the percentage mean score for the even-numbered items would be $\frac{15}{30} \times 100 = 50\,\%$, and for the odd-numbered items it would be $\frac{12.17}{30} \times 100 \approx 40.6\%$. Averaging these gives a percentage mean score of 45.3% for Nigerian grade 12 learners. Compare this figure with that of Table 4.3.

Table 4. 13 Correlation coefficients at grade level in SAS in the TPGT

Grade	Odd		Even		N	*r-value*	*p-value*
	Mean	Std Dev.	Mean	Std Dev.			
10	11.54	3.26	15.21	3.02	24	- 0.06	0.77860
11	10.67	2.90	13.58	3.02	24	0.37	0.07434
12	15.83	5.52	19.17	5.40	24	0.74	0.00003

As with the Nigerian learners, the mean scores obtained (out of a maximum of 30 points) by the SAS learners at each grade level for the visually presented items in the TPGT were higher than that obtained for their corresponding verbally presented items. Hence, these learners, like their Nigerian counterparts, had a better understanding of geometric terms that were presented in visual form than ones that were represented verbally. Again, since grade 10 learners in the SAS had a higher mean score in the TPGT than grade 11 learners (see Table 4.3), the implication is that grade 11 learners correctly answered a higher proportion of both members of a pair of verbally and visually presented questions than grade 10 learners. This would mean that in terms of comprehensiveness (i.e. number of concepts), grade 10 learners in the SAS had a better knowledge of geometric terminology than their grade 11 peers. However, in terms of conceptual understanding, that is, in terms of demonstrating that a geometric concept understood verbally is also understood visually, grade 11 learners tended to display a deeper knowledge of geometric terminology than their grade 10 counterparts.

From the analysis of the correlations between participants' verbal and visual abilities in the TPGT, it is evident that on the whole, there was a positive relationship between students' ability in verbal geometry terminology tasks and their ability in visual geometry terminology tasks. That is, a student who knew a correct verbal description of a geometric concept was very likely to have the correct visual image associated with the concept. Hence, for this crop of learners, verbal ability in geometric terminology implied ability in visual tasks. However, these learners were marginally more successful with visually presented geometry terminology tasks than verbally presented ones. These results were found to be consistent with those of Clements and

Battista (1992, p.421), who reported that in general "students can handle some [geometry] problems much better if the problem is presented visually rather than verbally".

4.4 Students' Knowledge of the Concepts of Circles, Triangles and Quadrilaterals, and Lines and Angles

The analysis that follows focuses on how students' scores in the TPGT were distributed among the three major concepts in terms of which the TPGT was drawn up (see Chapter Three, section 3.3.4.1.1).

4.4.1 Mean scores of all participants in the TPGT by concept

For the analysis in this section, students' mean scores in the TPGT were calculated separately for items on geometric terminology associated with the concepts of circles, triangles and quadrilaterals, and lines and angles. The results are as shown on Chart 4.3.

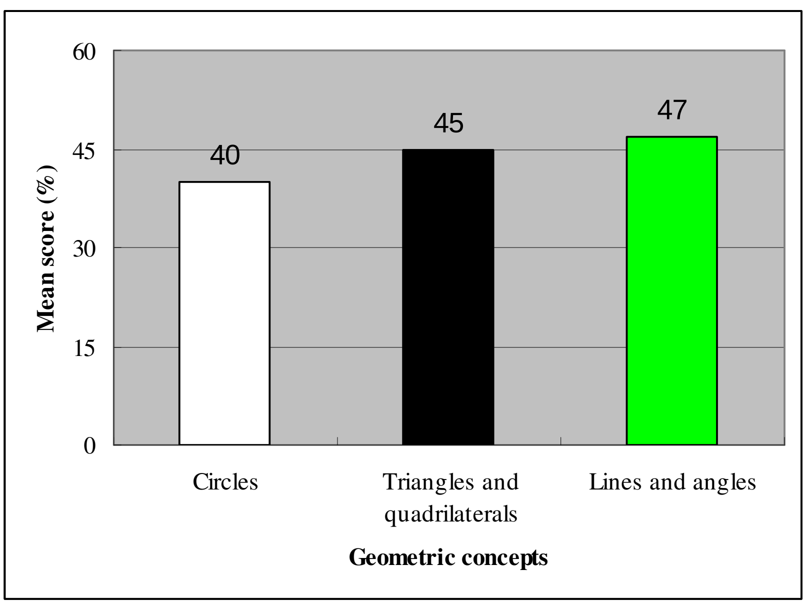


Chart 4. 3 Mean scores of learners in the TPGT by concept

As Chart 4.3 illustrates, the participating learners in this study showed a rather weak knowledge of geometric terminology across all three geometric concepts according to which items in the TPGT were drawn up. In relative terms, however, these learners demonstrated a better knowledge of terms in geometry that are associated with the concept of lines and angles than of those associated with circles, and triangles and quadrilaterals. The higher mean score of 47% indicated that these learners were more comfortable dealing with geometric terminology associated with lines and angles than that which dealt with the concepts of circles, and triangles and quadrilaterals, for which they obtained mean scores of 40% and 45% respectively. These results were found to be partly consistent with those of Kouba et al. (as cited in Clements & Battista, 1992, p.421) when they reported that in America, "students' performance at identifying common geometric figures, such as parallel lines…[was] acceptable", but that students' knowledge of some basic geometric terms associated with the concept of the circle was deficient.

In order to see how the scores of learners from the Nigerian subsample (NS learners) compare with those of the learners from the South African subsample (SAS learners), mean scores for each of the three concepts were calculated separately for the Nigerian participants and the South African participants. The results are illustrated on Chart 4.4.

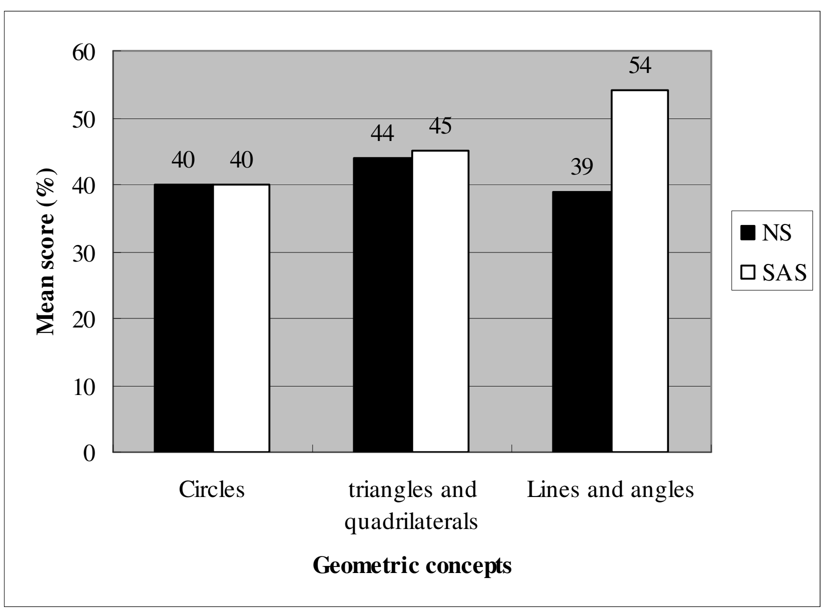


Chart 4. 4 Mean scores of NS and SAS learners in the TPGT by concept

As revealed on Chart 4.4, there was no difference in the mean scores of learners from the Nigerian and South African subsamples with regard to their knowledge of terminology associated with the concept of the circle. There was also only a marginal difference between the mean scores of Nigerian and South African learners concerning the geometric concept of triangles and quadrilaterals in favour of the South African learners. There was, however, a wide difference in the mean scores of Nigerian learners (39%) and South African learners (54%) on terminology associated with the concept of lines and angles. Importantly, while learners from the South African subsample obtained their highest mean score from the geometric terminology associated with the concept of lines and angles, this is the area where Nigerian learners obtained their lowest mean score in the TPGT. Given that the mean scores of Nigerian and South African learners for the geometric concepts of circles, and triangles and quadrilaterals are equal (or nearly so), then the difference in the mean scores of these two groups of learners on the TPGT as a whole reported in section 4.2.2 can be attributed almost entirely to the difference in their mean scores on the geometric terminology associated with the concept of lines and angles.

4.4.2 Mean scores of learners in the TPGT for the concept of circle by grade per school

Mean scores of learners for items (16 of them) in the TPGT that tested students' knowledge of basic terminology associated with the geometric concept of circles were calculated at each grade level in each of the participating schools. The results are as shown on Chart 4.5.

Chart 4.5 indicates that in the NS, there was a marginal progressive increase in the mean scores of learners across the grade levels for the concept of circle in the TPGT. Grade 12 learners from the NS obtained a marginally higher mean score (44%) than grade 11 learners whose mean score (40%) was in turn marginally higher than that of grade 10 learners (34%). These results, which are consistent with those reported in section 4.2.3, indicate that students from the NS add only a little to their repertoire of geometric terminology associated with the geometric concept of the circle as they progress from grades 10 through 12.

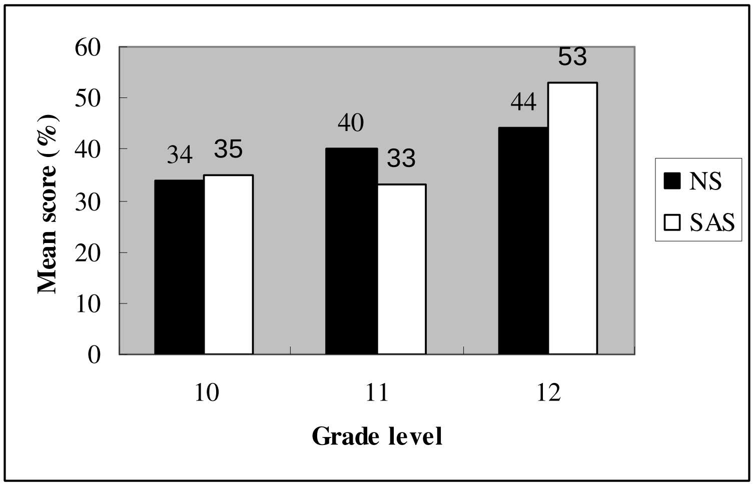


Chart 4. 5 Mean scores of learners in the TPGT for the concept of circles by grade per school

In the SAS, there was a marginal difference in the mean scores of grade 10 learners (35%) and that of grade 11 learners (33%) on the concept of the circle. There was, however, a huge difference in the mean scores of grade 12 learners (53%) and those of the grade 10 and 11 learners from the SAS. As with the mean scores calculated for learners from the SAS at grade level in the TPGT as a whole (see section 4.2.3), these results indicated that in the SAS, grade 10 learners had a better knowledge of geometric terminology associated with the concept of circles than their grade 11 peers.

As indicated on Chart 4.5, at each grade level (with the exception of grade 11), participants from the SAS obtained a marginally higher mean score than their comparative international peers from the NS. This means that on the average, learners from the Nigerian subsample had a poorer knowledge of geometric terminology associated with the concept of the circle than their South African counterparts involved in this study. These results corroborated the findings reported in sections 4.2.2 and 4.2.4.

Further analysis of students' scores was done in order to provide insight into students' knowledge of particular terms associated with the components of a circle tested in the TPGT. For this analysis, the mean scores of learners from each of the participating schools were obtained for each component of the circle included in the TPGT. The results are represented in Table 4.14.

Table 4. 14 Mean scores of learners[9] on the TPGT per school for terminology associated with a circle

Component of circle	NS (n = 138)	SAS (n = 144)
	Mean score (%)	Mean score (%)
Chord	38	31
Radius	30	33
Diameter	54	63
Tangent	31	38
Arc	33	31
Sector	56	41
Cyclic quadrilateral	28	49
Concentric circles	46	38

As indicated in Table 4.14, of all the terminology associated with the concept of a circle tested in the TPGT, participants demonstrated a better knowledge of diameter than of other circle components. Of the items on the components of a circle, learners from the SAS obtained their highest mean score (63%) on items that tested learners' knowledge of the diameter of a circle. It was also on these items that the Nigerian subsample obtained their second highest mean score (54%).

As further revealed in Table 4.14, the radius of a circle was among the least familiar terms associated with the concept of a circle for the participants, since learners from the Nigerian and South African subsamples could only manage to obtain a mean score of 30% and 33%, respectively, for items pertaining to this term. These results were found to be consistent with those of Kouba et al. (as cited in Clements & Battista, 1992, p.421) when they reported that in American high schools, "students' performance at identifying common geometric figures, such as…the diameter of a circle [was] acceptable, but students' performance with figures …such as the radius of a circle, [was] deficient".

[9] In the NS, 69 learners wrote the TPGT. Hence, there were a total of 69 X 2 = 138 responses for each circle component as there were two items for each of the circle components. This was the sense of usage of n = 138. Similarly, in the SAS, n = 72 X 2.

Another interesting result concerning participants' knowledge of terminology associated with the concept of the circle tested in the TPGT was that these learners obtained very low mean scores for nearly all the terminology, despite the very elementary nature of the TPGT. With the exception of items on the diameter of a circle in which SAS learners obtained a mean score of 63%, these (SAS) learners scored below 50% on average for all the other terminology associated with a circle. The Nigerian subsample was similarly only able to obtain mean scores of over 50% on two conceptual terms pertaining to a circle – 54% for diameter and 56% for a sector. These low mean scores indicated that the learners had very little knowledge of terminology associated with the geometric concept of circles. These results are consistent with those of Siyepu (2005, pp.77–78), in which of the 21 grade 11 learners that made up his study sample in South Africa, "only eight participants (38%) managed to identify a chord among other parts or components of a circle…, [and] only seven participants (33%) managed to recognise and identify a radius…".

4.4.3 Mean scores of learners in the TPGT for the concept of triangles and quadrilaterals by grade per school

Mean scores of learners for items (18 of them) in the TPGT that tested participants' knowledge of terminology associated with the concept of triangles and quadrilaterals were computed at each grade level in each participating school. The results which are illustrated on Chart 4.6 indicated that grade 10, 11 and 12 learners from the NS obtained mean scores of 40%, 45% and 46%, respectively, for items on the TPGT that examined learners' knowledge of basic terminology associated with the concept of triangles and quadrilaterals. As with the NS learners' knowledge of the geometric concept of circles (see Chart 4.5), the results on Chart 4.6 indicated only a marginal progressive increase in the mean scores across the grade levels. That is, grade 12 learners in the NS obtained a marginally higher mean score than the grade 11 learners whose mean score was in turn marginally higher than that of grade 10 learners. In the SAS, the mean score (43%) of grade 10 learners was slightly higher than that of grade 11 learners (39%). Grade 12 learners from the SAS clearly obtained a much higher mean score (54%) than their grade 10 and 11 counterparts, in the same way that they did on the concept of circle (see Chart 4.5).

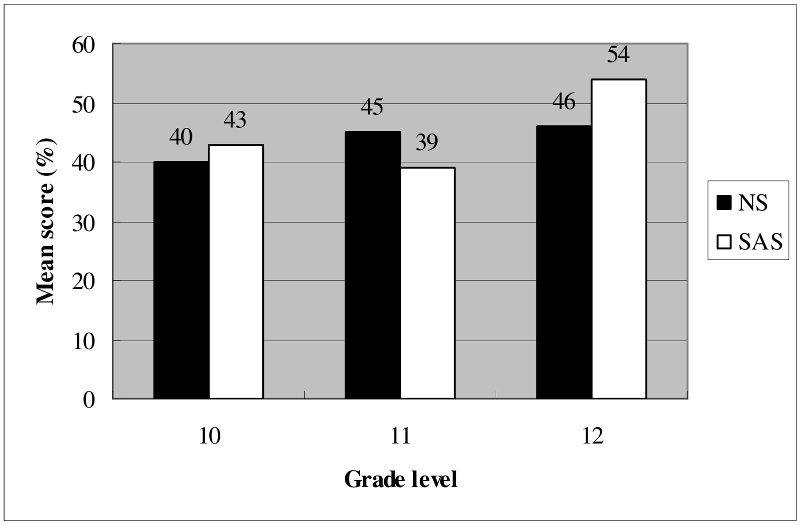


Chart 4. 6 Mean scores of learners in the TPGT for the concept of triangles and quadrilaterals by grade per school

A perusal of the mean scores on Charts 4.5 and 4.6 indicates that learners at each grade level and in each school obtained higher mean scores for items on the geometric concepts of triangles and quadrilaterals (Chart 4.6) than they did for items on the geometric concept of circles (Chart 4.5). That is, the learners at each grade level had a better understanding of the terminology associated with the concept of triangles and quadrilaterals than that associated with the concept of a circle. These results provide support for and confirm the results reported in section 4.4.1 (see also Chart 4.4).

Furthermore, although the mean scores of learners for the geometric concept of triangles and quadrilaterals were higher than their mean scores for the geometric concept of circles, these scores, as shown on Chart 4.6, were still very low, given the fact that the items in the TPGT were almost entirely of van Hiele levels 1 and 2 types (see Chapter Two, section 2.8). With the exception of grade 12 learners from the SAS who obtained a mean score of 54%, the mean score of all other learners in grades 10 and 11 in both the NS and the SAS were below 50% (Chart 4.6). These results indicate that participants in this study had a weak knowledge of the terminology associated with the geometric concepts of triangles and quadrilaterals.

The results as stated in the preceding paragraph were found to be consistent with research from both international studies and those from the Nigerian and South

African contexts (see Chapter Two, sections 2.7.2 and 2.7.3.3). According to Clements and Battista (1992, p.421), in the United States, for example, "only 63% [of high school learners] were able to correctly identify triangles that were presented along with distractors" and "only 64% of the 17-year–olds knew that a rectangle is a parallelogram". In Nigeria, the WAEC Chief Examiner's Report (WAEC, 2003, p.175) indicates that "questions on the angle properties of a triangle were also unpopular" with the majority of the candidates. In South Africa, Roux (2003, p.362) states that many "secondary school learners cannot identify and name shapes like kite, rhombus, trapezium, parallelogram and triangles".

Students' scores were further analysed in order to determine how the scores were distributed among items in the TPGT that tested participants' knowledge of the basic terminology associated with various concepts or sub-concepts that embodied the geometric concept of triangles and quadrilaterals. For this analysis, mean scores of learners from each of the participating schools were calculated for each type of geometric concept of triangles and quadrilaterals included in the TPGT. The results are as shown in Table 4.15.

Table 4.15 indicates that participants from both NS and SAS obtained their three highest mean scores in the TPGT for the same geometric concepts of similar triangles, diagonals and sides, and the altitude of a triangle. Learners from the Nigerian subsample obtained different mean scores for these concepts: 55% for the altitude of a triangle; 49% for similar triangles; and 47% for diagonals and sides. Participants from the South Africa subsample, however, obtained an equal mean score (51%) for these three concepts. The least mean score obtained by learners from the NS was 35% and it was for items in the TPGT that examined learners' knowledge of the geometric concept of symmetry in various quadrilaterals. It was also on these items that South African learners obtained their second lowest mean score (40%) after items on the geometric concept of scalene triangles in which they obtained a mean score of 33%.

Table 4. 15 Mean scores of learners in the TPGT per school for terminology associated with triangles and quadrilaterals

Geometric concept	NS (n = 138)	SAS (n = 144)
	Mean score (%)	Mean score (%)
Scalene triangle	45	33
Isosceles triangle	39	41
Equilateral triangle	43	47
Right-angled triangle	38	49
Similar triangles	49	51
Diagonals and sides	47	51
Altitude of a triangle	55	51
Line of symmetry	35	40
Triangle and quad.	43	43

These results indicate that learners in this study had a better knowledge of the geometric terminology associated with the concepts of the altitude of a triangle, similar triangles, and diagonals and sides, than of the terminology associated with other geometric concepts of triangles and quadrilaterals examined in the TPGT. The results further indicate that line of symmetry and right-angled triangle were among the most unfamiliar terms or concepts for the Nigerian subsample. For the South African subsample, line of symmetry and scalene triangle were among the least familiar concepts. These results echo those of Clements and Battista (1992), in which difficulty with identifying figures having lines of symmetry was evident among 11th-grade U.S learners.

Table 4.15 revealed that the Nigerian subsample obtained mean scores that were lower than 50% on all the concepts except on the concept of the altitude of a triangle, in which they obtained a mean score of 55%. Similarly, South African learners obtained mean scores that were below 50% on all the concepts with the exception of three concepts in which the mean score obtained was 51%. Given the very elementary nature of the TPGT, these results, as illustrated in Table 4.15 portray an unimpressive performance on the part of the participants concerning the geometric terminology associated with the concept of triangles and quadrilaterals.

On a comparative level, Nigerian and South African learners obtained equal mean scores (43%) for items in the TPGT that tested learners' knowledge of the concept of a triangle and quadrilateral. Except for concepts relating to scalene triangles and the altitude of a triangle in which Nigerian learners had higher mean scores (45% and 55%, respectively), South African learners outperformed their Nigerian peers on all the other concepts in the TPGT associated with the concept of triangles and quadrilaterals.

4.4.4 Mean scores of learners in the TPGT for the concept of lines and angles by grade per school

There were 24 items on the TPGT that tested learners' knowledge of the terminology associated with the geometric concept of lines and angles. The mean scores obtained by learners on these items at each grade level in each of the participating schools were computed. The results are illustrated on Chart 4.7.

As indicated on Chart 4.7, the mean score (47%) obtained by Nigerian grade 12 learners for items in the TPGT that examined participants' knowledge of terminology associated with the concept of lines and angles was higher than the mean score (41%) of grade 11 Nigerian learners, which was in turn higher than that of their grade 10 peers (29%). In terms of knowledge of terminology relating to the concept of lines and angles, these results showed that grade 12 learners from the Nigerian subsample surpassed grade 11 learners, who in turn, surpassed grade 10 learners in this group.

As with the results in sections 4.4.2 and 4.4.3, grade 10 South African learners obtained a higher mean score (51%) than their grade 11 peers who had a mean score of 47% for items on the geometric concept of lines and angles. The mean score of 65% obtained by grade 12 South African learners was much higher than those of grades 10 and 11 learners. These results indicated that grade 10 South African learners had a better understanding of geometric concept of lines and angles than their grade 11 counterparts. Grade 12 learners' knowledge of terminology associated with the geometric concept of lines and angles was much better than that of grade 10 and grade 11 learners in the SAS.

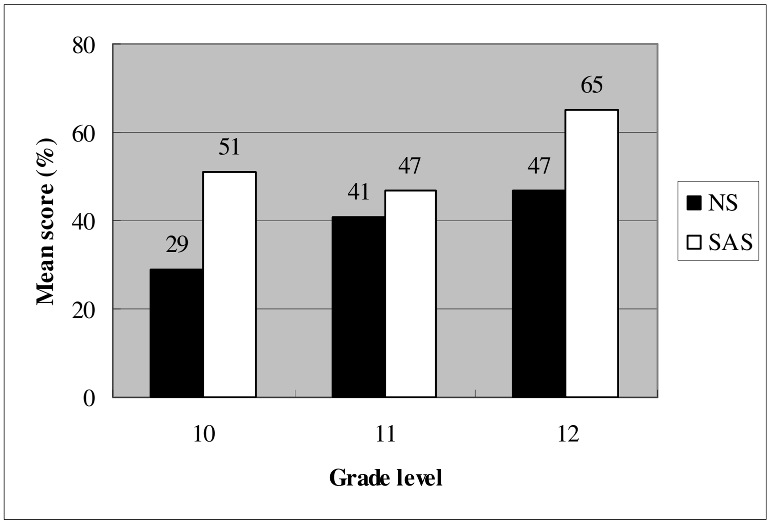


Chart 4. 7 Mean scores of learners in the TPGT for the concept of lines and angles by grade per school

Chart 4.7 further revealed that at each grade level, South African learners obtained a higher mean score than their Nigerian peers. In fact, the lowest mean score (47%) obtained by grade 11 learners from the SAS was equal to the highest mean score (47%) obtained by grade 12 learners from the NS. Therefore, South African learners demonstrated a better knowledge of the geometric concept of lines and angles than their Nigerian peers.

A point that deserves separate mention is that grade 10 learners from the South African subsample consistently obtained a higher mean score than the grade 11s for all the three major concepts (circles, triangles and quadrilaterals, and lines and angles) featured in the TPGT (see Charts 4.5 through 4.7).This observation reinforces the point made earlier in section 4.2.3, which was that in general, grade 10 learners from the South African subsample outperformed their grade 11 peers in the TPGT. In Nigeria, however, there was a consistent marginal progressive increase in the mean scores of learners across the grade levels for each of the three major concepts that embodied the TPGT.

The distribution of participants' scores among concepts or terminology associated with the concept of lines and angles in the TPGT was analysed. Mean scores of participants in each school were computed for each concept. The results are presented in Table 4.16.

As indicated in Table 4.16, the highest mean score obtained by the Nigerian subsample from items in the TPGT that tested participants' knowledge of terminology or concepts associated with the geometric concept of lines and angles was 75%, and the second highest mean score obtained by these learners was 54%. These learners obtained these mean scores from the geometric terminological concepts of right angles and acute angles, respectively. It was on these same concepts that South African learners obtained their highest mean score (88%) and their second highest mean score (76%), respectively.

The geometric concepts of angle complementarity and the perpendicularity of lines were the most unfamiliar terms for learners from both the Nigerian and South African subsamples, since they obtained their lowest mean scores for items on these concepts. Participants from Nigeria obtained mean scores of 20% and 25% for items in the TPGT that examined learners' knowledge of complementary angles and perpendicular lines, respectively. The respective mean scores obtained by South African learners for items on these concepts were 21% and 35%.

The analysis presented in the two preceding paragraphs indicates that participants in this study had a relatively acceptable knowledge of the terminological concepts of right angles and acute angles. As is evident in Table 4.16, although students' performance in respect of the other terminology (corresponding angles, supplementary angles, alternate angles, etc.) associated with the geometric concept of lines and angles was generally less impressive, their performance with regard to the concepts of complementary angles and perpendicular lines was the most disheartening. These results are consonant with those of Clements and Battista (1992) in which in the U.S, students' knowledge of perpendicular lines was reported to be deficient (see Chapter Two, section 2.7.3.6).

Table 4. 16 Mean scores of learners in the TPGT per school for terminology associated with lines and angles

Geometric concept	NS (n = 138)	SAS (n = 144)
	Mean score (%)	Mean score (%)
Acute angles	54	76
Alternate angles	38	43
Complementary angles	20	21
Corresponding angles	41	46
Obtuse angles	38	63
Reflex angles	47	67
Right angles	75	88
Supplementary angles	29	46
Vertically opposite angles	38	68
Parallel lines	39	55
Perpendicular lines	25	35

As further revealed in Table 4.16, learners from the Nigerian subsample obtained lower mean scores than their South African peers in respect of each of the terminological concepts associated with the geometric concepts of lines and angles in the TPGT. This means that South African learners had better knowledge of the terminology associated with the geometric concept of lines and angles than their Nigerian counterparts.

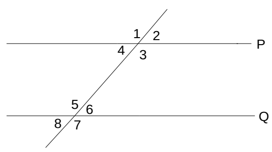
In the next section, 'other results' from selected items in the TPGT that allowed for the comparison of participants' knowledge of basic geometric terminology with their international peers are presented. Also included in the 'other results' is the item analysis of students' responses to the TPGT.

4.5 Other Results from the TPGT

As was stated in section 3.4.1.1 of Chapter Three, item 19 on the TPGT was adopted from the 1995 TIMSS' test item number O03, while items 36 and 51 were adopted from Usiskin's (1982) entering geometry test item numbers 10 and 13, respectively.

The reason for adopting these items was also explained in that section. Since the original tests from which these items were adopted provided options with five foils (A – E), minor changes, such as juggling the options and reducing the number of foils, were made so as to conform to the four-foil format used in the TPGT. Although the entire TPGT test is presented in Appendix 3.A, p.13, the three items adopted are reproduced here in Figure 4.1 for ease of reference.

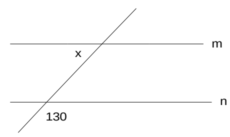
Question 19
In this figure, lines P and Q are parallel. Two angles whose measures must add up to 180° are



A. 2 and 4
B. 4 and 5
C. 5 and 7
D. 2 and 8

(a)

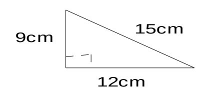
Question 36
Lines m and n are parallel. What is the measure of the angle marked x?



A. 30°
B. 40°
C. 50°
D. 130°

(b)

Question 51
What is the area of the triangle shown below?



A. 36 cm sq.
B. 54 cm sq.
C. 108 cm sq.
D. 1620 cm sq.

(c)

Figure 4. 1 Selected items in the TPGT for international comparison of students' scores

4.5.1 Performance in the TPGT for selected items

The percentage of learners who chose the correct answers to items 19, 36 and 51 in the TPGT in each of the participating schools was calculated and the results compared with those of the original studies (see section 4.5). For items 36 and 51, an item analysis was conducted in order to determine possible misconceptions in participants' responses. The source of information about students' performance on item 19 did not include an item analysis of participants' responses (see Brombacher, 2001). Consequently, only the overall percentage scores of participants were compared in the present study.

Performance on item 19: In the original TIMSS study conducted in 1995, the international figure for the percentage number of participants who answered this question (Figure 4.1a) correctly was 44.5%, while 20.5% of the South African subsample in that study answered it correctly. Nigeria has never participated in any TIMSS. In the present study, 27% of the Nigerian subsample and 51% of the South

African subsample answered this question correctly. Table 4.17 summarizes these results.

Table 4. 17 Percentage correct for item 19 in the TPGT adopted from TIMSS 1995

1995 TIMSS study		**Current study**	
% correct international	% correct South Africa	% correct NS (Nigeria)	% correct SAS (South Africa)
44.5	20.5	27	51

As indicated in Table 4.17, although the Nigerian subsample in the present study performed 6.5% above the South African 1995 TIMSS subsample, it nevertheless performed 17.5% below the international figure for the 1995 TIMSS study for item 19 in the TPGT. Although it is difficult to determine whether the samples (that of TIMSS and those of the current study) are comparable, Howie's (2001) analysis of the TIMSS study's populations, however, indicates that this question (Figure 4.1 a) was answered by a cohort of eighth-graders in the TIMSS study. This implies that at an international level, learners from across grades 10–12 of the Nigerian subsample in the current study performed below grade 8 learners in the 1995 TIMSS study on item 19 of the TPGT.

Although for item 19 the South African subsample in the current study performed 30.5% and 6.5% above the South African TIMSS subsample and the international figure for the 1995 TIMSS study respectively, this figure (51%) does not necessarily indicate an impressive performance, given the fact that this question can be correctly answered by about 44.5% of eighth-graders. Focusing on the current study, more South African learners (51%) than Nigerian learners (27%) correctly answered item 19 in the TPGT. These results only reiterated once more the recurrent theme in this chapter, which is that, even although both the Nigerian and South African subsamples in this study had a rather weak knowledge of basic geometric terminology, the performance of the Nigerian subsample was consistently inferior to that of their South African peers in this learning area.

Performance on items 36 and 51: For ease of reference in this section, Usiskin's (1982, p.150) sample for the "Entering Geometry Student Test", which was made up of high school U.S learners, is referred to as the EG sample, while the sample in the present study is referred to as the TPGT sample. The analysis that follows does not necessarily assume the comparability of these samples beyond the simple facts that the majority of the learners in both samples were roughly within the same age brackets (14–17 years for the EG sample and 15–19 years for the TPGT sample), and that many of them had been taught high school geometry at the senior phase of secondary education (see Usiskin, 1982).

In Table 4.18, the percentage of students from the EG and the TPGT samples who chose a particular option (A, B, C or D) are indicated for both items 36 and 51. The percentage with the correct choice is made bold and underlined.

As is evident in Table 4.18, the percentage correct for item 36 was 30% in the EG sample, while it was 35% and 58% in the Nigerian and South African TPGT subsamples, respectively. This indicated that the EG sample performed 5% below the Nigerian TPGT subsample and 28% below the South African TPGT sample. Table 4.18 further indicates that many students in each of the sample groups chose option D as the correct answer to item 36. Given that by applying the concept of a straight angle, the value of x (Figure 4.1b) could be determined by using any of a series of terms such as corresponding angles, vertically opposite angles, alternate angles and co-interior angles, students' poor response to this question indicated that these learners either lacked conceptual understanding of these terms or simply could not operate beyond one line of reasoning.

As with the EG sample, learners in the present study performed rather poorly on item 51 of the TPGT. Table 4.18 indicated that the percentage correct for item 51 was 24% in the EG sample, 23% in the Nigerian TPGT subsample and 25% in the South African TPGT subsample. Although the percentage performance on item 51 was generally poor for all the sample groups, the South African TPGT subsample performed 1% above the EG sample and 2% above the Nigerian TPGT subsample. As indicated in Table 4.18, the majority of the learners in each sample group chose option A as the correct answer to item 51. This would seem for me a case of misconception about the geometric concepts of area and perimeter – most of these learners simply summed up the given values for the sides of the triangle (Figure 4.1c) as the area of the right-angled triangle.

Table 4. 18 Item analysis for items 36 and 51 in the PTGT

Item	**Choice**	**Percentage with choice**		
		EG sample	TPGT sample (NS)	TPGT sample (SAS)
36	A	4	22	10
	B	4	16	7
	C	**30**	**35**	**58**
	D	37	27	25
	Blank	0	0	0
51	A	43	42	57
	B	**24**	**21**	**25**
	C	11	20	8
	D	15	13	10
	Blank	0	1	0

While the two questions (items 36 and 51) might be considered too few for any meaningful conclusion to be drawn about how learners in this study compared with their international peers in terms of their knowledge of basic terminology/concepts in high school geometry, what emerged from the above analysis was that, as with the EG sample, the participants' performance was not satisfactory. The results of this study in respect of these two items were, therefore, consistent with those of Usiskin (1982, p.69) when he concluded that "on an absolute scale, the performance [of U.S learners on the EG test] cannot be considered satisfactory". In the next subsection, an item analysis of participants' responses to the entire TPGT is provided.

4.5.2 Item analysis of participants' responses to the TPGT

Although an item analysis was done for participants' responses to the TPGT at each grade level in each of the participating schools, only an overview of the results of this analysis is presented in this section. For a more comprehensive view of the item analysis, reference should be made to Appendix 3.D.1–8, pp.31–38. The first part of this analysis provides information about the number of students within given percentage ranges of scores in the TPGT.

Students' unimpressive performance in the TPGT becomes even more evident as one looks at the number of students within percentage ranges of scores. Simple calculations from Table 4.19 indicate that of the Nigerian subsample, only 13 learners (19%) scored 51% or over, while 56 learners (81%) scored 50% or less, with the majority, 24 learners (35%), scoring between 31% and 40%. In relative terms, South African learners performed better than their Nigerian counterparts in the TPGT. By similar calculations, Table 4.19 indicates that of the South African participants, 22 learners (31%) scored 51% or over, while 50 learners (69%) scored 50% or less, with the majority, 26 learners (36%) scoring between 41% and 50% in the TPGT.

At grade levels, Table 4.19 shows that no learner from grade 10 of the Nigerian subsample scored over 60%, and that 4 learners each from grades 11 and 12 scored over 60% in the TPGT. For the South African subsample, no learner in grades 10 and 11 scored over 60%, while 10 learners in grade 12 scored over 60% in the TPGT. No learner in either subsample scored 91% or over.

Given that all of the items in the TPGT deal with the most basic of geometry facts and concepts requiring only straightforward applications of terminology, these are disturbing results. Usiskin (1982, p.87) draws a similar conclusion about American school children's knowledge of geometry when he writes that "many students leave the geometry course not versed in basic terminology and ideas of geometry".

Table 4. 19 Number of students within percentage range of score in the TPGT

Score (%)	Number scoring							
	Grade 10		Grade 11		Grade 12		All learners	
	NS	SAS	NS	SAS	NS	SAS	NS	SAS
1 – 10	0	0	0	0	0	0	0	0
11 – 20	1	0	0	0	1	0	2	0
21 – 30	8	0	8	2	2	1	18	3
31 – 40	7	7	7	11	10	3	24	21
41 – 50	4	13	5	8	3	5	12	26
51 – 60	1	4	0	3	4	5	5	12
61 – 70	0	0	1	0	1	3	2	3
71 – 80	0	0	1	0	2	5	3	5
81 – 90	0	0	2	0	1	2	3	2
91 – 100	0	0	0	0	0	0	0	0
Total	**21**	**24**	**24**	**24**	**24**	**24**	**69**	**72**

Item analysis: Although the learners' performance in the TPGT was, on average, not impressive, an item analysis of their responses was considered necessary to reveal their strengths or weaknesses, and in some cases, possible misconceptions in geometry. As was stated earlier, an item analysis was provided for only a few selected items in this data narrative – the complete item analysis for the TPGT being in Appendix 3.D.1–8, pp.31–38. There was no special criterion for selecting these items other than the simple fact that participants performed either very well or very poorly at them. Some of the items that elucidated possible students' misconceptions in geometry were also accorded a special mention.

No item in the TPGT was correctly answered by over 80% of the learners from the Nigerian subsample (see Appendix 3.D.1, p.31). Of the Nigerian participants, three items (22, 41 and 44) were correctly answered by 80% of the learners; one item (32) was correctly answered by 75% of the learners, and one other item (27) was correctly answered by 70%. No other item was correctly answered by 70% of the learners from the Nigerian subsample. Although the learners from the Nigerian subsample tended to perform relatively better on items dealing with terminology associated with the concept of triangles and quadrilaterals (see Chart 4.4 in section 4.4.1), items dealing with the geometric concept of a right angle (22 and 27) proved to be among the simplest for the majority of the learners. Items 1, 18, 26, 31 and 54 in the TPGT were among the least correctly answered by a large majority of learners from the Nigerian subsample. Of the Nigerian participants, only 26%, 32%, 14%, 19% and 24% of the learners, respectively, correctly answered these items. Students' choices on these items revealed some possible misconceptions about certain geometric concepts in both the Nigerian and South African subsamples. I will return to this issue shortly.

Although the performance of the South African subsample in the TPGT was, like that of their Nigerian counterparts, generally speaking not impressive (see Table 4.1 in section 4.2.1), their performance on some of the items was, however, quite encouraging. For the South African participants, there were in all 10 items that were correctly answered by over 70% of the learners. In fact, one of the items (22) was correctly answered by 96% of the learners and two other items (27 and 32) were correctly answered by 89% of the learners. Other items that were correctly answered (item number followed by percentage of learners) included items 41 (86%), 44 (81%), 8 (79%), 4 and 16 (76%), 25 (74%), and item 28 (71%). No other item was correctly answered by 70% or more of this group of learners. The results as reported here support those reported in section 4.4.1, that is, that South African learners in this study had a better knowledge of the geometric terminology associated with the concept of lines and angles than that associated with the concepts of circles, triangles and quadrilaterals (see Chart 4.4 in section 4.4.1). Their performance on items dealing with the geometric concept of angles, particularly right angles, was better than their performance on items that dealt with the concept of lines, especially perpendicular lines (see their performance on items 22, 18 and 31, Appendix 3.D.5, p.35). As with the Nigerian subsample, items 1, 18, 26, 31 and 54 were among those to be least correctly answered by a large percentage of South African learners. Among the South African learners, only 19%, 39%, 15%, 31% and 29%, respectively, correctly answered these questions. An illustrative analysis of students' responses to these five items was done so as to elucidate their possible misconceptions about the geometric ideas informing the questions.

Illustrative analysis of items 1, 18, 26, 31 and 54

Item number 1: This question required the learners to state the name of the straight line that joins any two points on the circumference of a circle. 42% of the learners from the Nigerian subsample answered that it was a diameter (option B), and another 19% said it was a radius (option C). (See Appendix 3.D.1, p.31 for percentages of learners choosing other options.) Within the South African subsample, 33% of the learners said it was a diameter (option B) and another 47% said it was a radius (option C) (see Appendix 3.D.5, p.35). The evidence that these learners hold the misconception that a line that joins any two points on the circumference of a circle is called a radius was revealed when their responses to item 10 – the visually presented version of item 1 – was analysed. 32 % of Nigerian learners and 44% of South African learners responded that line OD (the radius of the circle), represents a chord (see item 10 of the TPGT). These patterns of response could also be interpreted to mean that the learners did not understand what is meant by the circumference of a circle.

Item numbers 18 and 31: These were a pair of visually and verbally presented items in the TPGT that tested learners' knowledge of perpendicular lines (see Appendix 3.A, p.13 for the contents of these items). Item 18 was a visually presented item, while item 31 was a verbally presented one. For item 18, 36% of the Nigerian subsample and 31% of the South African subsample chose option B – a pair of parallel lines – as two lines that are perpendicular. For item 31, 49% each of the Nigerian and South African learners answered that two lines that are perpendicular to each other are also parallel to each other (option A). This is a clear case of conceptual misunderstanding of the concept of perpendicularity. There were several other cases of this nature.

Item number 26: This was the item that was incorrectly answered by the greatest percentage of learners in both the Nigerian and South African subsamples. This question required the learners to indicate the number of sides and diagonals in a triangle. 77% of Nigerian learners and 71% of South African learners answered that there are 3 sides and 3 diagonals (option A). This indicated that these learners lacked a conceptual knowledge of the diagonal. Similar findings were earlier reported by French (2004), when he stated that many students hold the misconception that a triangle has 3 sides and 3 diagonals, while a rectangle has 4 sides and 0 diagonal – the learners apparently counting the number of sloping edges as diagonals (see Chapter Two, section 2.7.3.1).

Item number 54: This item required the learners to identify a scalene triangle among four triangles in which the measures of the angles were indicated. 48% of Nigerian learners and 36% of South African learners answered that the triangle whose angle measures were 30º, 120º and 30º is a scalene triangle. These responses indicated not only that these learners were unfamiliar with the term 'scalene triangle', but also that they lacked a conceptual understanding of the isosceles triangle.

The foregoing analysis has revealed that participants in this study did not only have inadequate knowledge of many basic geometric terms, but also held a number of misconceptions about the correct usage of these terms. In particular, responses of the type just explained indicate that these learners reasoned about many geometric shapes as a whole without attending to their properties, which is typical of van Hiele level 1 thinking.

4.6 Chapter conclusion

In this chapter, students' responses to the TPGT were analysed and the results interpreted. The major findings include the following.

- The overall percentage mean score obtained by all the learners on the TPGT was 44.17%. This mean was considered unsatisfactory, given that the items that constituted the TPGT were largely of a van Hiele level 1 nature. These items dealt with the simplest of geometry facts and concepts that required only straightforward applications of terminology. The conclusion was reached that learners in this study had an inadequate knowledge of basic geometric terminology.

- Learners from the South African subsample performed significantly better than their Nigerian peers on the TPGT. There was a statistically significant difference in the mean scores of South African learners (47.85%) and the Nigerian subsample (40.49%) in favour of the former at the 0.05 confidence level. With the difference in the mean scores, the conclusion was reached that the Nigerian subsample in this study had a weaker understanding of basic geometric terminology than their South African counterparts.

- At grade level, learners from the Nigerian subsample appear to add little to their repertoire of geometric terminology as they progress from grade 10 through 12. Evidence in support of this claim came from the closeness of the mean scores obtained by grade 10, 11 and 12 learners in the TPGT.

- South African grade 10 learners obtained a higher mean score than their grade 11 peers in the TPGT. Grade 12 South African learners clearly outperformed learners from both grade 10 and 11. The conclusion was reached that grade 10 South African learners had a better knowledge of basic geometric terminology than their grade 11 counterparts in the study.

- South African grade 10 and 12 learners performed significantly better than Nigerian grade 10 and 12 learners in the TPGT. The difference in mean scores of Nigerian grade 11 learners and South African grade 11 learners in favour of the former was not statistically significant. The conclusion was reached that South African learners in this study had a better knowledge of basic geometric terminology than their Nigerian peers at each grade level.

- There was a significant difference between the mean scores of the male learners (48%) and that of the female learners (41%) at the 0.01 level in favour of the former in the TPGT. It was concluded that in this study, male learners had a better knowledge of basic geometric terminology than their female counterparts.

- At school level, male learners from the Nigerian subsample performed significantly better than their female counterparts as the male mean score (45.8%) was significantly greater than the female mean score (34.4%) at the 0.005 level.

The conclusion drawn was that Nigerian female learners were conceptually poorer in their knowledge of basic geometric terminology than their male counterparts.

- At school level, the difference between the mean score of South African male learners (50.8%) and that of their female peers (45.6%) was not statistically significant ($p > 0.05$). It was concluded that South African male learners' knowledge of basic geometric terminology was not significantly better than that of their female counterparts.

- In this study, South African female learners performed significantly better than comparable Nigerian female learners in the TPGT. Although South African male learners obtained a higher mean score (50.8%) than the Nigerian male learners (45.8%) in the TPGT, the difference in the mean scores was not statistically significant ($p > 0.05$). The conclusion was reached that male learners from the South African subsample were not significantly better than their Nigerian peers in terms of their knowledge of basic geometric terminology.

- There were high positive correlations between participants' ability in verbal geometry terminology tasks and their ability in visual geometry terminology tasks. For the Nigerian subsample, the correlation coefficient calculated for these two forms of geometry terminology tasks was $r = 0.83$, and for the South African subsample, it was $r = 0.63$. Both correlations were found to be significant at the 0.001 level. The conclusion reached was that for participants in this study, learners who knew the correct verbal description of a geometric concept also had the correct visual image associated with the concept, and vice versa. That is, success in verbal geometry terminology tasks implied success in visual geometry terminology tasks for the cohort of learners in this study.

- In this study, learners had a better knowledge of geometric terminology associated with the geometric concept of lines and angles than of terminological concepts of circles, and triangles and quadrilaterals. Of the components of a circle, learners were most familiar with the concept of a diameter, while the concept of a radius was the least known. Of the items in the TPGT that tested learners' knowledge of sub-concepts of triangles and quadrilaterals, the altitude of a triangle was among the concepts in which participants obtained their highest mean scores. Line of symmetry in triangles was an unfamiliar term to the majority of the learners. Of the terminology associated with the geometric concept of lines and angles, the one most popular with the learners in this study was the concept of a right angle, while perpendicular lines and complementary angles were among those least correctly identified.

- For terminology associated with the three concepts of circles, triangles and quadrilaterals, and lines and angles, the Nigerian subsample obtained a mean score (40%) equal to that of their South African counterparts on the concept of circles, but obtained marginally lower mean scores on the other two concepts. The conclusion was reached that learners from the Nigerian subsample performed marginally lower than their South African peers on these three concepts.

- Two of the several misconceptions held by learners in this study as revealed by their responses to the TPGT were that the radius of a circle is called a chord, and that two lines that are perpendicular to each other are also parallel. The latter was a clear case of a basic misunderstanding of the concepts of perpendicularity and parallelism.

These results as reported above were found to be consistent with those of many earlier studies. In the next chapter, participants' responses to the GIST are analysed and the results interpreted and discussed.

CHAPTER FIVE

DATA ANALYSIS, RESULTS AND DISCUSSION 2: THE GIST

5.1 Introduction

In this chapter, the Geometric Item Sorting Test (GIST) is examined separately with minimal reference to its possible connections with van Hiele levels, for the purpose of making an in-depth analysis of students' performance in the test. In Chapter 8, learners' performance in the GIST and other tests is correlated with their performance in the VHGT in order to determine a possible relationship. The first part of this chapter provides a quantitative analysis of students' performance in the GIST by examining their mean scores. The second part provides a more incisive qualitative analysis of learners' responses to the GIST through an item-by-item exploratory analysis. Recall that 36 learners participated in the GIST. The reason for involving only 36 learners and the selection criterion were explained in Chapter Three, section 3.3.4.1.2, paragraph 5. The analyses of students' performance on the GIST were based on the responses of these 36 learners.

5.2 Overall Participants' Performance in the GIST

Students' general performance in the GIST was described in terms of the overall participants' percentage mean score obtained for the test. The overall percentage mean score obtained by the participating learners for the GIST was 40.19%.

Item 1 of the GIST was allocated 60 points out of a total of 105 points (see Appendix 4.C, p.49), and it required participants to give the correct names of the geometric shapes represented by the 30 concept cards of various triangles and quadrilaterals, and to state their properties (see section 3.4.1.2 of Chapter Three). Given the high points allocated to item 1 of the GIST and the fact that even learners who reason entirely at van Hiele levels 1 and 2 could be successful in this task, the participants' mean score as stated above was considered unsatisfactory. That is, on the whole, learners' performance in the GIST was poor.

5.2.1 Performance of Nigerian and South African learners in the GIST

The mean scores obtained by learners from the Nigerian subsample (i.e. NS) and the South African subsample (i.e. SAS) in the GIST were calculated separately for each of the participating schools. The results are summarized in Table 5.1.

Table 5. 1 School percentage means for learners in the GIST

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	18	36.94	14.45	- 1.33	34	0.1932
SAS	18	43.44	14.92			

As indicated in Table 5.1, the mean score (43.44%) of the SAS learners in the GIST was higher than the mean score (36.94%) obtained by learners from NS. A test of significance indicated that the difference between the mean scores of these two groups of learners was not statistically significant (t = -1.33, 34df, p > 0.05). That is, South African learners in this study did not perform significantly better than their Nigerian peers in the GIST.

As indicated in the preceding section, a learner who responded correctly to item 1 of the GIST should have scored at least 57% (i.e. $\frac{60}{105} \times 100$) in the test as a whole. This being the case, the mean scores obtained by learners from each of the participating schools in the GIST were considered evidence of their weak knowledge of school geometry. That is, the majority of these learners were not even able to correctly identify each of the 30 concept cards of triangles and quadrilaterals and list their properties. This is an indication that a large proportion of the learners were only able to reason about the geometric shapes in the GIST at van Hiele levels lower than level 2.

5.2.2 Gender differences in performance in the GIST

Students' performance on the GIST was further examined for possible gender differences. This was done by comparing:

- The male mean score with the female mean score of all the participants.
- The male mean score with the female mean score of the Nigerian subsample.
- The male mean score with the female mean score of the South African subsample.
- The Nigerian male mean score with the South African male mean score.
- The Nigerian female mean score with the South African female mean score.

5.2.2.1 Mean scores in the GIST of all participants by gender

As indicated in Chart 5.1, there was a differential gender performance in the GIST in favour of male learners. On average, male learners obtained marginally higher scores, achieving a mean score of 44%, than female learners, who had a mean score of approximately 37%. These mean scores reinforce the point made earlier on that the male learners in this study consistently outperformed their female counterparts (see Chapter Four, section 4.2.5.1, last para.).

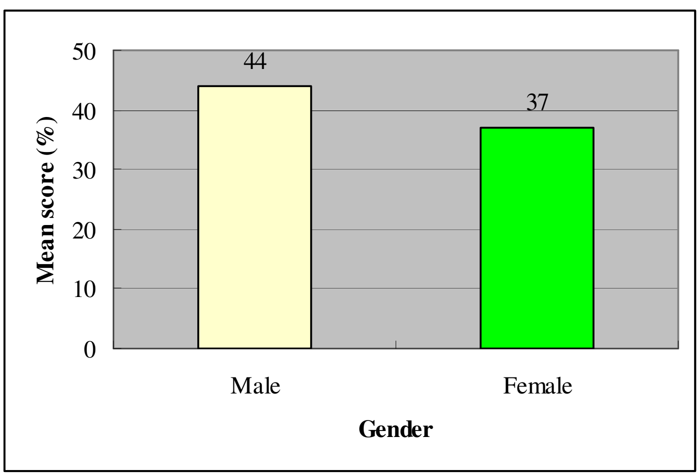


Chart 5. 1 Gender difference in mean scores in the GIST

A test of significance indicated that the difference between the male and female mean scores was not statistically significant (t = 1.54, 34df, p > 0.05), as shown in Table 5.2.

Table 5. 2 Mean scores of learners in the GIST by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	16	44.38	16.22	1.54	34	0.1327
Female	20	36.85	13.11			

Table 5.2 reveals that although the male learners obtained a higher mean score for the GIST than the female learners, the performance of the male learners was in general not significantly better than the performance of the female learners. These results were found to be consistent with those of Schäfer's (2003) study in South Africa, in which in a Hands-on Activity test (HAT) similar to the GIST, "the males [in Schäfer's sample] performed better than the females in every task" (p.179), but "the statistics available do not reveal meaningful gender differences" (p.153). In fact, as stated earlier, the male learners outperformed their female peers in all four tests (VHGT, TPGT, GIST and CPGT) used in this study, as well as in their school mathematics examination for the study year (i.e. 2006).

5.2.2.2 Mean scores in the GIST of the Nigerian subsample by gender

Analysis of the scores of learners from the Nigerian subsample in the GIST showed that there was a difference between the male mean score (40.67%) and the female mean score (33.22%) in favour of the former. A *t-test*, however, indicated that the difference between the mean scores was not statistically significant ($p > 0.05$). The results are summarized in Table 5.3.

Table 5. 3 Mean scores in the GIST of Nigerian learners by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	9	40.67	16.16	1.09	16	0.2877
Female	9	33.22	12.31			

As evident in Table 5.3, the Nigerian male learners did not obtain a significantly higher mean score than their female peers in the GIST. It is noteworthy to observe that Nigerian male and female learners obtained mean scores on the GIST that were respectively lower than the male and female mean scores for the study sample (see Table 5.2). This provides support for the claim that South African learners performed (marginally) better than their Nigerian counterparts reported earlier in section 5.2.1.

5.2.2.3 Mean scores in the GIST of the South African subsample by gender

As with the Nigerian subsample, the male learners from the South African subsample (i.e. SAS) obtained a marginally higher mean score (49.14%) than their female peers whose mean score on the GIST was approximately 39.82%. The difference between the mean scores was, however, not statistically significant ($p > 0.05$), as indicated in Table 5.4. This means that South African male learners, like their Nigerian peers, did not perform significantly better that their female counterparts in the GIST.

Table 5. 4 Mean scores in the GIST of South African learners by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	7	49.14	16.20	1.32	16	0.2053
Female	11	39.82	13.56			

Comparing male and female mean scores in Table 5.4 with the respective male and female mean scores in Table 5.2, one observes that the South African male and female mean scores were higher than those of the study sample. This further underlines the point made earlier in sections 5.2.1 and 5.2.2.2 that South African learners performed marginally better than the Nigerian subsample in the GIST.

5.2.2.4 Mean scores in the GIST by male gender

The mean scores obtained by the Nigerian and South African male learners on the GIST were calculated for learners in each subsample (i.e. NS and SAS). The mean score calculated for the male learners in the Nigerian subsample was 40.67%, while the mean score calculated for the South African male learners was 49.14%. A *t-test* (Table 5.5) indicated that the difference between the mean scores was not statistically significant ($p > 0.05$). This means that South African male learners did not perform significantly better than their Nigerian international peers in the GIST.

Table 5. 5 Mean scores in the GIST by male gender

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	9	40.67	16.16	1.04	14	0.316
SAS	7	49.14	16.20			

5.2.2.5 Mean scores on the GIST by female gender

As with the male learners, South African female learners obtained a marginally higher mean score (39.82%) than the Nigerian female learners, who obtained a mean score of 33.22% in the GIST. A *t-test* analysis, however, indicated that the difference between the mean scores of these two groups of learners was not statistically significant ($p > 0.05$). This means that female learners from SAS did not perform significantly better than female learners from NS in the GIST. The results are represented in Table 5.6.

Table 5. 6 Mean scores in the GIST by female gender

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	9	33.22	12.31	1.13	18	0.274
SAS	11	39.82	13.56			

5.3 Exploratory Qualitative Analysis of Learners' Responses to the GIST

In the preceding sections of this chapter, information about participants' knowledge of school geometry was provided through the analysis of learners' mean scores for the GIST. As was stated in section 3.4.1.2 of Chapter Three, the GIST consisted of five interrelated geometry tasks. Each of these tasks was designed in such a way that learners' responses could be interpreted in relation to one or more of the van Hiele levels. Consequently, a more incisive exploratory item-by-item analysis of learners' responses was necessitated, and this is the focus of the next few sections in this chapter.

5.3.1 Learners' responses to Task 1 of the GIST

As stated in section 3.4.1.2 of Chapter Three, this task required the learners to identify each of the 30 geometric shapes by stating the correct names of the shapes. Each learner was also requested to justify his/her naming by stating a reason for the name given to each shape. Table 5.7 represents the results.

As evident in Table 5.7, there were a lot of inconsistencies in students' ability to correctly name the various shapes in this task. Lack of conceptual understanding in geometry became evident as many of the participants from both the Nigerian and the South African subsamples seemed able to recognise shapes only in some basic or standard orientations. As indicated in Table 5.7, three of the shapes in the park of shapes used for the GIST were rhombuses (shape Nos. 1, 9 and 13). With reference to the number written on the cards, shape No. 1 was presented in the more basic orientation than shape Nos. 9 and 13, as illustrated in Figure 5.1.

Although 8 learners (44%) from the Nigerian subsample were able to identify shape No. 1 as a rhombus, only 4 (22%) of them were successful in identifying No. 9 as a rhombus. Similarly, 12 learners (67%) from the South African subsample were able to identify shape No. 1 as a rhombus, but only 8 (44%) of them succeeded in recognizing shape No. 9 as a rhombus. This lack of conceptual ability to recognize rhombuses in different orientations was also evident in the learners' identification and naming of other shapes, as can be seen from Table 5.7. Similar findings had earlier been reported by Mayberry (1983, p.64), where in the U.S. some pre-service elementary teachers "had difficulty in recognizing a square with a nonstandard orientation".

Table 5. 7 Students who named shapes correctly and stated the correct reason

Shape No.	Name of shape	No. correctly naming shape		No. stating correct reason	
		Nigeria (n=18)	**S/Africa (n=18)**	**Nigeria (n=18)**	**S/Africa (n=18)**
1	Rhombus	8	12	0	1
2	Isosceles trapezium	10	14	0	0
3	Rectangle	16	17	2	6
4	Obtuse-angled scalene triangle	15	16	4	6
5	Rectangle	6	9	0	6
6	Square	16	18	0	3
7	Isosceles trapezium	12	13	0	1
8	Kite	10	10	2	0
9	Rhombus	4	8	0	2
10	Isosceles triangle	16	16	11	12
11	Parallelogram	14	12	1	3
12	Equilateral triangle	17	17	7	9
13	Rhombus	5	11	0	2
14	Isosceles triangle	17	14	5	9
15	Rectangle	17	18	3	9
16	Isosceles trapezium	11	13	0	0
17	Rectangle	15	16	1	6
18	Equilateral triangle	18	15	8	5
19	Rectangle	9	10	1	5
20	Right-angled trapezium	9	9	0	0
21	Right-angled isosceles triangle	17	16	7	7
22	Right-angled isosceles triangle	16	15	8	7
23	Square	12	17	1	4
24	Right-angled isosceles triangle	16	16	9	10
25	Parallelogram	12	11	1	6
26	Right-angled trapezium	5	6	0	0
27	Scalene triangle	13	13	9	9
28	Kite	10	10	2	1
29	Parallelogram	12	12	0	6
30	Right-angled scalene triangle	16	13	7	8

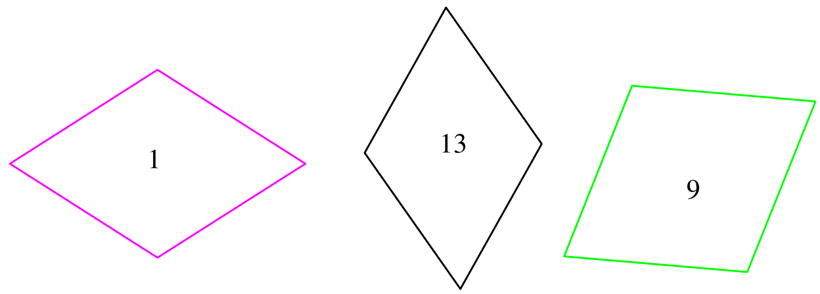


Figure 5. 1 Rhombuses in different orientations

Table 5.7 further indicates that more students can name shapes than state the properties of shapes. For example, although 8 learners from the Nigerian subsample were able to name shape No. 1 as a rhombus, none of them could state its distinguishing properties. Similarly, only 1 out of the 12 South African learners who correctly named shape No. 1 as a rhombus was able to correctly state its properties. This seems to link up with the hierarchical property of the van Hiele levels (see Chapter Two, 2.8.1).

The use of imprecise properties for describing the shapes was common among the learners. The majority of the learners described the shapes entirely by the property of sides while neglecting their angle properties. For example, all 8 of the Nigerian learners who correctly identified shape No. 1 as a rhombus used only the side property to justify their naming: “It is a polygon having four equal sides”; “It has four equal sides”; “Four sides are equal” and so forth. Among the South African learners, only 1 out of the 12 that correctly identified shape No. 1 as a rhombus made use of both side and angle properties to justify his naming: “It is a quad with 4 equal sides, angles are not right angles”. The rest, like their Nigerian counterparts, focussed exclusively on the property of sides: “It has 4 equal sides, stude (for skewed) square”; “4 equal sides are equal and it is the same as square but its squde” (for skewed); “It has four equal sides”; “It is like a square (a square has 4 equal sides) but skewed”, and so forth. In fact, students’ inadequate knowledge of geometric terminology reported in Chapter Four resonated with some of the learners’ responses to this task. As with item 26 in the TPGT, in which the learners’ lack of conceptual knowledge of the diagonal was evident (see Chapter Four, section 4.5.2, 3rd last para.), two grade 10 learners (both from SAS) who had correctly named shape No. 1 as a rhombus gave such reasons as

“It is a polygon having four diagonal equal sides” and “it has four diagonal sides equal”. Responses such as these, for me, are clear cases of either the imprecise use of geometric terminology or a lack of conceptual understanding of the concept of the diagonal – with the learners thinking that every slanting edge of a shape is a diagonal (see French 2004).

There was no student (Nigerian and South African learners alike) who used more than one attribute of a shape in naming the shape. For instance, right-angled isosceles triangles (shape Nos. 21, 22 and 24) were either named as “isosceles triangle” or “right-angled triangle” by learners who named them correctly, with the majority showing preference for the former name. In short, only 2 learners (1 Nigerian and 1 South African) named these shapes as “right-angled triangle”. About half of the learners simply referred to these shapes (and other different triangles) as simply “triangles”.

This manner of naming shapes by reference to an attribute of the shape was absent with regard to quadrilaterals. None of the learners used such words as “right-angled trapezium” (shape Nos. 20 and 26) or “isosceles trapezium” (shape Nos. 2, 7 and 16) even when they used straightedges and protractors to establish these attributes. The learners simply called them “trapezium”.

Even given the common patterns in participants’ responses described above, there were some elements in their responses to Task 1 of the GIST that allowed for comparison of the two subsamples. At a comparative level, more of the South African learners were successful in this task (identifying and naming shapes) than learners from the Nigerian subsample. As evident in Table 5.7, there were only 7 shapes (shape Nos. 11, 14, 18, 21, 22, 25 and 30) that more Nigerians named correctly than South African learners. More South African learners correctly named 15 of the shapes (shape Nos. 1, 2, 3, 4, 5, 6, 7, 9, 13, 15, 16, 17, 19, 23 and 26) than Nigerian learners. An equal number of Nigerian and South African learners correctly named 8 of the shapes (the shaded ones in Table 5.7). It should be noted, however, that for each shape named correctly by more South African learners than Nigerian learners, the difference in the number of learners from both subsamples who correctly named the shape was negligible. For example, in the case of the five rectangles (shape Nos. 3, 5, 15, 17 and

19), the number of South African learners who correctly named four of these shapes exceeded that of their Nigerian counterparts by only one (shape Nos. 3, 15, 17 and 19), but they (the South African learners) did better by three learners for shape No. 5. The analysis just given provides support for the results of the quantitative analysis presented earlier in section 5.2.1, which indicated that on the whole, South African learners performed only marginally better than Nigerian learners in the GIST.

Initially I did not intend to interview the learners in this study (see Chapter 3, section 3.3.4.2, third last para.). However, during a preliminary on-site analysis of learners' responses to the GIST, some of the inconsistencies that were noticed in learners' responses prompted me to interview them. As was stated in section 3.4.1.2 (under the subtitle "Test Grading") of Chapter Three, these interviews were unstructured as the questions asked were based on the individual learner's written responses. These interviews helped to tease out more information about students' misconceptions and ideas about geometry. Although all 36 learners involved in the GIST were interviewed, a sample of only 4 learners interviews was selected for reporting in this narrative. This sample was chosen for the variety of responses the learners concerned exhibited during the interviews, a variety that seemed representative of the entire subsample for the GIST. These interviews were very short (5 minutes per learner) and are labelled 'Interview episode' 1 through 4.

Interview episode 1: The case of misconception

In the identifying and naming of shapes task (i.e. Task 1 of the GIST), Vusumzi, a grade 12 learner from the South African subsample, named a rhombus (shape No. 1) as a "square" and gave as reason that "it has four equal sides". The following interview took place:

Researcher: Do you mean that all shapes having four equal sides are squares?

Vusumzi: Yes.

Researcher: If a shape is a square, what other property would it have apart from four equal sides?

Vusumzi: All four angles measure 90° each.

Researcher: Did you measure the angles of shape No. 1?

Vusumsi: No.

Researcher: Why?

Vusumzi: I know that as soon as the four sides are all equal, then the angles must be 90° each.

This line of reasoning was common among the majority of the learners, while many others made reference to a visual prototype, as was evident in interview episode 2.

Interview episode 2: Reference to a visual prototype

Asisat, a grade 11 learner from the Nigerian subsample, had correctly named shape No. 1 as a rhombus and shape No. 6 as a square, but stated that "it has 4 equal sides" as the only reason for both shapes. I interviewed her as follows:

Researcher: Do you mean that if a shape has four equal sides, then it is a rhombus?

Asisat: No. The shape has to have four equal sides and look like a kite.

Researcher: You gave the same reason [four equal sides] for the rhombus and the square. How is a square different from a rhombus?

Asisat: A square is like a carpet, a rhombus is like a kite.

Researcher: Is there anything else that you can tell me about the properties of a square apart from having four equal sides?

Asisat: [Prolonged silence, then shook her head slowly] No.

Researcher: Ok. But is there anything that you can remember about the angles of a square and a rhombus?

Asisat: Em …em… am not sure.

Researcher: That's fine. Now tell me, how were you sure that the two shapes were not both either squares or rhombus?

Asisat: You see, how I used to know them is that the one that is like a carpet is the square, and that one [pointing at the rhombus] that is like a kite I know that it is a rhombus.

Clearly, Asisat, like many other learners, was not attending to the properties of the shapes. A shape was what it is called because it looked like some known shape or object – typical of van Hiele level 1 reasoning. There were other students for whom the orientation of a shape was an important attribute in identifying it.

Interview episode 3: Reference to orientation of shape

Kolela was a grade 10 learner from the South African subsample. This learner correctly named shape No. 1 as a rhombus, giving as a reason: "two opposite sides are equal". He also correctly named shape No. 6 as a square, stating that "all four sides are equal". The following interview took place:

Researcher:	You said shape No. 1 is a rhombus because two opposite sides are equal. What about the other two opposite sides?
Kolela:	They are also equal.
Researcher:	But are they equal to the first two sides that you referred to?
Kolela:	Yes, all of them are equal.
Researcher:	How do you know?
Kolela:	I checked them with this [touching a ruler].
Researcher:	So, is it correct for me to say that if a shape has four equal sides, then it is a rhombus?
Kolela:	Yes…but…em…yeah, I think so.
Researcher:	But you called shape No. 6 a square because "all four sides are equal". Can I call it a rhombus?
Kolela:	Maybe, but I am not sure [silent]. No, because a square is straight, but a rhombus is not.
Researcher:	Can you explain what you mean by a square being straight?
Kolela:	All the four sides are equal, two pointing up and the other two pointing this way [uses his hands to indicate horizontal parallel lines].
Researcher:	Can I then call a rhombus a square?
Kolela:	I…I… [Prolonged silence and uneasy expressions on his face], am not sure. But I think there is something about their angles that I can't remember.

Researcher: Yes, talking about their angles, did you measure the angles of these two shapes?

Kolela: No.

Researcher: Why?

Kolela: I recognize them when I see them. The square is like this [places the square such that two of its sides are horizontal to the page and the other two sides are pointing vertically upwards], but the rhombus is like this [places it so as to rest obliquely on the page of his worksheet].

Researcher: Fine. Now look at this. If I place this square this way [placing it on the page so that one of its angles points downwards with the other angle opposite this one pointing upwards], can I still call it a square?

Kolela: [Laughs] No, a square will not look like that.

As revealed in the preceding interview with Kolela, some of the learners, even when they identified these shapes explicitly by properties of sides, still allowed the orientation of the shape (as dictated by the number written on the concept card) to influence their naming. This explains the variations in the number of learners who correctly named similar shapes (e.g. the three rhombuses in Figure 5.1) in different orientations. The results here generally link up with those of Mayberry (1983), in which the difficulty experienced by learners in recognising a square with a nonstandard orientation was evident (see Chapter Two, section 2.7.3.3).

Interview episode 4: Inadequate terminology

Bulelwa, a grade 12 learner from the South African subsample, correctly named shape No. 6 as a square, stating that "it has four equal sides and four angles which are equal to 90°". She, however, incorrectly named shape No. 9 (a rhombus) as a square stating such reason as "it has four equal sides, and two angles more than 90° and two angles are less than 90°". As was the case with many other learners, Bulelwa had a problem with certain terms, and in some cases a conceptual misunderstanding of some geometric shapes, as was revealed in the ensuing interview.

Researcher: You called both shape Nos. 6 and 9 squares. But you said No. 6 has four equal sides and angles of 90° each, while No. 9 has four equal sides with two angles greater than 90° and two other angles less than 90°. Now, tell me, do different sizes of squares have different angles?

Bulelwa: No. But the square having 90° angles is straight, and the one having two angles greater than 90° and two angles less than 90° looks like this [uses her hands to demonstrate skewness].

Researcher: Can we find a name for this second type of square [the one with two angles greater than 90° and two angles less than 90°] that we can call it?

Bulelwa: I think it has its own name. But I can't remember it.

Researcher: So, why do you call it a square? Why not some other names like kite, rectangle, etc?

Bulelwa: No. You know, all the four sides are the same like that of a square. So, I think I can call it a square.

There was a rather strange conceptual misunderstanding of the concepts of rhombus and trapezium by Bulelwa. Looking at her written responses, I discovered that she consistently named each of the isosceles trapeziums (shape Nos. 2, 7 and 16) a rhombus, and the two other rhombuses (shape Nos. 1 and 13) she simply named parallelograms. This, for me, was a clear case of conceptual misunderstanding, even though she attempted naming the shapes using side and angle properties.

With interviews like the ones exemplified in the preceding episodes, together with participants' written responses, it was possible to assign levels to the students in accordance with the descriptions of the van Hiele levels in section 2.8 of Chapter Two, while also making use of Burger and Shaughnessy's (1986) descriptors of the van Hiele levels. This task (Task 1 of the GIST) was the only task in the GIST (as with all other tests but the VHGT) for which participants were assigned van Hiele levels because of the markedly visible evidence of the levels in students' responses. Note, however, that the overall assignment of levels for the entire study sample (which is discussed later in Chapter 7) was based on results from the VHGT.

On the assignment of levels, learners who failed to correctly name the shapes were generally assigned level 0. Those who correctly named the shapes but could not state the correct reason, as Asisat did, were assigned van Hiele level 1. Learners who succeeded in naming the shapes correctly and also succeeded in stating the correct reason were assigned level 2. Learners who correctly named the shapes and stated minimal and sufficient properties as the reason were to be assigned level 3, but no learner in the sample met this condition. This task, as Burger and Shaughnessy (1986, p.43) put it, could not "elicit reasoning beyond Level 2" (level 3 for the numbering scheme used in this study). The number of learners in each van Hiele level for this task is shown on Chart 5.2.

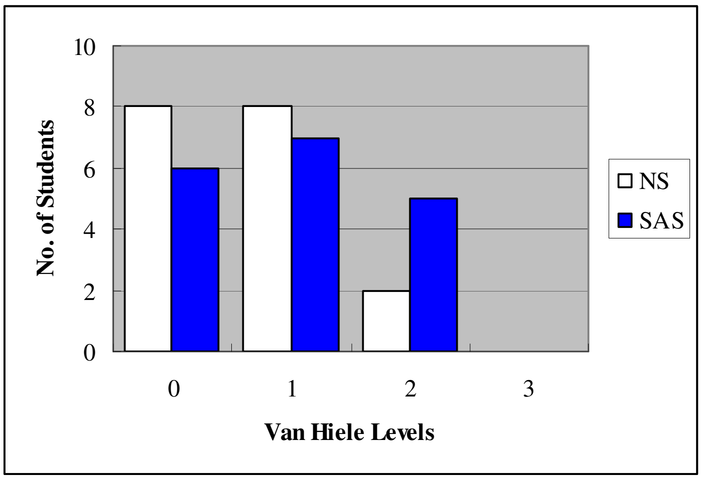


Chart 5. 2 Learners' van Hiele levels for Task 1 in the GIST

Chart 5.2 indicates that many students could not distinguish between shapes belonging to the same class of shapes, typical of level 0 reasoning (see Chapter Two, section 2.8, last para.). Rhombuses in different orientations, for example, meant different shapes to the majority of the learners. The task of recognizing and stating the name of a shape (van Hiele level 1) was easier than that of listing the discerning properties of the shape (van Hiele level 2) for many learners. Learners' inadequate knowledge of basic geometric terminology, which was reported in Chapter Four, resonated with these results. As indicated on Chart 5.2, only 15 learners (at level 1) and 7 learners (at level 2) out of the 36 participants for the GIST possessed the correct terminology to name the shapes and explicitly state their properties. There was no learner in the group who evinced understanding indicative of van Hiele level 3 reasoning in Task 1 of the GIST, as shown on Chart 5.2.

5.3.2 Learners' responses to Task 2 of the GIST

The sorting shapes task (i.e. Task 2) consisted of three subtasks that the learners were required to perform (see Chapter Three, section 3.4.1.2). Many learners – 25 out of 36, representing 69% of the sample group for the GIST – were able to sort all 30 shapes into two distinct groups – one of triangles and the other of quadrilaterals – using the property of sides. For example, Kolela (our grade 10 learner for interview episode 3), would tell someone that "they [the shapes] all have three sides" in order for the person to pick out a shape that belongs to Group A (his group of triangles) and that "they [the shapes] all represent four sides" in order to pick out a shape belonging to Group B (his group of quadrilaterals). Even though many students sorted the shapes into two groups of 3-sided shapes (triangles) and 4-sided shapes (quadrilaterals), only a few of them, 5 learners each from the Nigerian and the South African subsamples, were able to use the correct terminology "quadrilateral" for the 4-sided shapes. The results are as summarized in Table 5.8.

Table 5. 8 Learners who correctly sorted shapes into groups of triangles and quadrilaterals

<table>
<tr><th></th><th colspan="2">NS (n = 18)</th><th colspan="2">SAS (n = 18)</th></tr>
<tr><td>No. correctly sorting shapes into 3-sided and 4-sided shapes</td><td colspan="2">10</td><td colspan="2">15</td></tr>
<tr><td>No. sorting shapes by property of sides</td><td colspan="2">9</td><td colspan="2">14</td></tr>
<tr><td rowspan="2">No. stating the correct reason for the group of shapes</td><td>Triangles</td><td>Quadrilaterals</td><td>Triangles</td><td>Quadrilaterals</td></tr>
<tr><td>9</td><td>5</td><td>14</td><td>5</td></tr>
</table>

At a comparative level, more South African learners (15) were successful in the task of sorting shapes into two distinct groups of triangles and quadrilaterals than Nigerian learners (10), as is revealed in Table 5.8. Also, more South African learners (14) than Nigerian learners (9) were able to state the correct criterion (3 sides for triangles and 4 sides for quadrilaterals) for grouping the shapes. There were more learners from the South African subsample (14) who used the correct terminology "triangle" for the group of 3-sided shapes than there were from the Nigerian subsample (9). These results offer further support to the findings in Chapter Four, which were that, on the whole, learners from the South African subsample had a better knowledge of geometric terminology than their Nigerian counterparts in this study.

The sorting shapes task further revealed some important misconceptions about geometric concepts among the learners. There were 8 learners (2 Nigerians and 6 South Africans) in the GIST sample who reasoned that all 4-sided shapes were called "square". There were 4 other learners (3 Nigerians and 1 South African) who reasoned that all 4-sided shapes were called "rectangle". There was yet another learner (a Nigerian) who thought that all 4-sided shapes are called "parallelogram".

Given the very elementary nature of this task (i.e. Task 2 of the GIST) and the grade levels of the learners, their performance in this task was considered unsatisfactory. Even though learners' inability to use the correct terminology for the group names for 3-sided and 4-sided shapes may be condoned – perhaps they had forgotten them – their inability to correctly sort these shapes into distinct groups of triangles and quadrilaterals is in my view inexcusable for a cohort of senior phase high school learners.

5.3.3 Learners' responses to Task 3 of the GIST

As stated in section 3.4.1.2 of Chapter Three, Task 3 (sorting by class inclusion of shapes) required the learners to make a further sorting of the shapes in within the groups (groups of triangles and quadrilaterals) into smaller subgroups of shapes that are alike in some way. None of the 36 participants was successful in this task. All the students sorted the shapes so as to prohibit class inclusions. Rectangles, squares and rhombuses, for example, were all excluded from the class of parallelograms by all the students who also failed to perceive squares to be rectangles (or rhombuses). Right-angled isosceles triangles were excluded either from the class of right-angled triangles or from that of isosceles triangles, the students focusing on a single attribute. None of the students sorted the shapes so as to reflect the pattern indicated in Figure 3.9 and Figure 3.10 of section 3.4.1.2 in Chapter 3. These results as reported here are consistent with those of De Villiers (1994, p.17), in which in South Africa, many students in his study showed a preference for a "partition classification" of quadrilaterals as against a hierarchical classification.

Even with prohibition of class inclusion, the majority of the learners could not sort the shapes into distinct classes of triangles and quadrilaterals, as many learners either omitted members from or included non-members in a class. Forming distinct classes of triangle (with class exclusion) proved easier for many learners than the same task using quadrilaterals, as indicated in Table 5.9 and Table 5.10. With class exclusion, Tables 5.9 and 5.10 seem to be indicating that learners from the Nigerian subsample were more successful in this task than learners from the South African subsample. However, as will be made evident towards the end of this chapter, South African learners, in general, performed marginally better than their Nigerian counterparts on this task, as on the other four tasks of the GIST.

Table 5. 9 Learners correctly grouping triangles with class exclusion

Shape name (shape No.)	**No. correctly grouping shape**	
	NS (n = 18)	SAS (n = 18)
Equilateral triangles (12 & 18)	5	1
Isosceles triangle (10, 14, 21, 22 & 24)	6	0
Right-angled triangles (21, 22, 24 & 30)	0	0
Right-angled isosceles triangles (21, 22 & 24)	0	0
Scalene triangles (4, 27 & 30)	4	2
Acute-angled triangles (10, 12, 14, 18 & 27)	0	0
Obtuse-angled triangle (4)	0	0

Table 5. 10 Learners correctly grouping quadrilaterals with class exclusion

Shape name (shape No.)	No. correctly grouping shape	
	NS (n = 18)	SAS (n = 18)
Parallelograms (11, 25 & 29)	2	0
Rectangles (3, 5, 15, 17 & 19)	3	1
Rhombuses (1, 9 & 13)	1	0
Squares (6 & 23)	3	2
Kites (8 & 28)	5	5
Trapeziums (2, 7, 16, 20 & 26)	1	0

As with task 1, the results from Task 3 of the GIST indicated that the majority of the learners were yet to attain level 3 in the van Hiele geometric thinking hierarchy. That is, the learners were not yet able to perceive the relationships between the properties of a shape and between different shapes (see Chapter Two, section 2.8). More importantly, these results provided further evidence that learners' knowledge of school geometry was far from meeting the objectives of geometry teaching in Nigeria and South Africa as articulated in sections 2.5.2.3 through 2.5.2.5 of Chapter Two. These findings are consistent with those of earlier studies (see Chapter Two, section 2.7.3.5).

5.3.4 Learners' responses to Task 4 of the GIST

This task revealed a number of imprecise visual qualities that many learners used in describing the shapes. Reference to visual prototypes was common in learners' definitions of the shapes: "Rhombuses look like squares, but if you look carefully it's sides are slanting/elevational and all equal". Many students defined the shapes so as to prohibit class inclusions, while some others simply gave a litany of their properties in defining them – far from stating minimal properties, consistent with De Villiers (1998). The concept of the isosceles triangle appeared to be widely understood, as about 56% of the learners (10 each from the Nigerian and the South African subsamples) correctly answered that they would tell someone to "look for a triangle with two equal sides" in order to pick out the isosceles triangles from among the shapes.

A number of misconceptions were also noticed in the students' definitions. Bulelwa (our grade 12 learner for interview episode 4), for example, would tell someone to look for a shape that has "four unequal sides" in order to pick out all the rhombuses from a set of quadrilaterals. As stated in the paragraph just after interview episode 4, Bulelwa had named each of the isosceles trapeziums as rhombuses, and provided as a reason "it has four unequal sides" for Task 1 of the GIST. Apparently, Bulelwa had carried her conceptual misunderstanding in Task 1 over to the defining shapes task (i.e. Task 4). The results presented here supply further evidence that the majority of the learners were not yet at van Hiele level 3.

5.3.5 Learners' responses to Task 5 of the GIST

For this task, the learners were required to state with justification whether a shape belonged to a class of shapes with some more general properties (see Chapter Three, section 3.4.1.2). Thus, Task 5 investigated learners' knowledge of class inclusions of shapes. In the analysis that follows, a student was considered to have answered correctly if he/she responded in the affirmative and gave a correct reason to justify his/her answer.

The class exclusion that dominated learners' reasoning about geometric shapes in the previous tasks became more evident in this task. As indicated in Table 5.11, only 1 learner (a South African) perceived a square (shape Nos. 6 and 23) as belonging to the class of rectangles and of rhombuses. No learner from the Nigerian subsample perceived a square as belonging to either class of shapes. Students' denial of a shape with the more specific properties as not belonging to the class of the one with the more general properties was usually accomplished by the listing of a few properties of the special case not shared by the more inclusive shape. For example, some of the learners reasoned that a square is not a rectangle "because all the sides [of a square] are equal", just as some others said that a rhombus is not a parallelogram because "all four sides are equal". A summary of learners' responses is presented in Table 5.11.

Table 5. 11 Students who correctly responded to Task 5 (class inclusions task)

Question posed	No. with correct response	
	NS (n = 18)	SAS (n = 18)
Is shape No. 23 a rectangle?	0	1
Is shape No. 17 a parallelogram?	0	2
Is shape No. 6 a rhombus?	0	1
Is shape No. 1 a parallelogram?	3	1
Is shape No. 30 a scalene triangle?	8	7

Although learners from the South African subsample generally outperformed learners from the Nigerian subsample on Task 5 of the GIST, Nigerian learners seemed to have a better knowledge of the geometric concept of a scalene triangle than South African learners. This is evident in Tables 5.9 and 5.11. Taken as a whole, Table 5.11 indicates that knowledge of class inclusions of shapes was lacking with the majority of the learners. This shows that many of the learners were yet to perceive the relationships between the properties of shapes and between different shapes; that is, that they had not attained level 3 reasoning on the van Hiele scale of geometric conceptualization. Given that task 5 of the GIST required a simple “Yes” or “No” answer with justifications, the learners’ performance of this task was unimpressive. In the analysis that follows, the learners’ performance of each task in the GIST is examined in terms of their percentage mean scores on each task.

5.3.6 Percentage mean scores of learners on each task of the GIST

The aim in this section is to provide information on participants’ strengths and weaknesses with respect to the five tasks comprising the GIST through analysis of their percentage mean scores for each task. Chart 5.3 summarizes the findings.

As evident in Chart 5.3, of the five tasks of the GIST, Task 2 (sorting of shapes into two distinct groups of triangles and quadrilaterals) was the easiest for the majority of learners from both the Nigerian and the South African subsamples. It was for this task that learners obtained their highest percentage mean score – 52.8% for the Nigerian subsample and 71.3% for the South African subsample. The task of sorting shapes into smaller subgroups of triangles and quadrilaterals by class inclusions (Task 3) was the most difficult for learners from both subsamples. Nigerian and South African learners could only manage to obtain a percentage mean score of 17.8% and 19.2%, respectively, on this task.

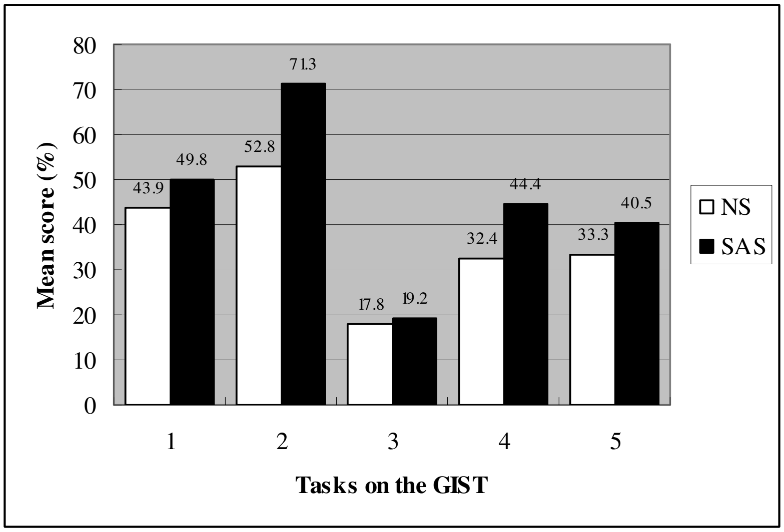


Chart 5. 3 Learners' percentage scores for each task of the GIST

Task 3 and Task 5 were similar tasks that were presented differently in the GIST. Students did better on Task 5 (class inclusions of shapes) than on Task 3 possibly because the former required only simple "Yes" or "No" responses with justifications from the learners, while the latter required the learners to form subclasses of shapes by themselves using the notion of class inclusions of shapes. The differential performance by learners on these two tasks could be interpreted as evidence of a lack of conceptual understanding in this learning area.

On a comparative level, Chart 5.3 indicates that learners from the South African subsample performed marginally better than learners from the Nigerian subsample on each task of the GIST. This is consistent with the results reported in section 5.2.1, above, which suggested that South African learners in this study performed only marginally better than their Nigerian peers in the GIST.

In sum, although learners' performance of each of the five tasks of the GIST was not satisfactory, their performance on Task 3 was particularly interesting as it has important implications for the teaching and learning of high school geometry. These learners were able neither to interrelate properties of shapes nor to perceive the relationships between shapes. Their knowledge of geometry appeared to be grossly inadequate for success in high school geometry. There was little evidence from the learners' performance on the GIST that they had had sufficient experience with basic geometric concepts at earlier stages (primary and middle school) of their education. These were among the concerns raised by earlier studies (see Chapter Two, section 2.7.4.1).

5.3.7 Students' misconceptions and imprecise terminology from the GIST

As was stated in section 2.7.3 of Chapter Three, learners' conceptual difficulties in geometry come in many different forms. Two of the several forms identified were students' use of imprecise geometric terminology, and their having misconceptions about geometric concepts. As with the TPGT, students' responses to the GIST revealed several misconceptions in geometry and these are examined in this section.

Although some instances of the students' use of imprecise terminology could be due to language problems (as suggested by various spelling errors), in this study the majority of instances appeared to be due to conceptual misunderstandings. The importance of teachers' knowledge of learners' misconceptions and use of imprecise geometric terminology, with regard to the teaching and learning of school geometry, was explained in sections 2.7.3.1 and 2.7.3.2 of Chapter Two. Figure 5.2 represents some of the imprecise terminology (taking the form of spelling errors) used by many learners in their descriptions of various geometric shapes in the GIST. In Figure 5.3, some of students' misconceptions about geometric concepts in the GIST were highlighted.

An additional fact that emerged from the qualitative analysis of learners' responses to the GIST was that learners who were minimally prone to these (spelling) errors (Figure 5.2) proved to have a deeper understanding of geometry than learners who committed several of these errors. Therefore, lack of precision in the use of terminology by students should not be taken for granted or tolerated on the ground that "these are mere spelling mistakes". As was the case in several earlier studies (see

Chapter Two, section 2.7.3.2), there were many learners in this study with spelling problems who were generally less successful in the GIST.

Equalateral triangle, equadilateral triangle, equadrilateral triangle, regular triangle – all for equilateral triangle. Isosceless, Iscocelene, Isoscele, Isoscelist, Isosscilice – all for isosceles triangle. Scarlene, Scaline, Scalelan, Irregular – all for scalene triangle. Robus, rombus – for rhombus. Parallelogramme – for parallelogram. Traipyzium – for trapezium

Figure 5. 2 Some of the students' imprecise terminology in the GIST

As the GIST gave the learners the opportunity to freely express themselves both in written and verbal forms, it revealed a far greater number of misconceptions about geometry on their part than the TPGT discussed in Chapter Four. Some of the several misconceptions evident in learners' responses to the GIST were exemplified in Figure 5.3.

- Regular rectangle – it is a rectangle having four equal side.
- Small rectangle – It is a small rectangle that has no equal side.
- A square is not a rectangle because it have the same size.
- A square is not a rhombus because it has all it shapes equal.
- As soon as the four sides [of a geometric figure] are all equal, then the angles must be 90° each.
- If the opposite sides of a shape are equal, the shape is a rectangle.
- A rectangle is not a parallelogram because it is not a quadrilateral.
- All quadrilaterals have four equal shapes [not sides].
- A rectangle is not a parallelogram because it is a straight shape.
- If a shape has four equal sides, then it is a square.
- A square is like a carpet, a rhombus is like a kite.
- A rhombus is not a parallelogram because a parallelogram has 2 pairs of opposite sides equal, not 4 sides equal.
- Isosceles [triangle] has 3 sides, 2 equal sides and 1 horizontal line.

Figure 5. 3 Some of students' misconceptions in geometry

Misconceptions in geometry, such as those quoted above, are unfortunate because they cause learning difficulties for the majority of students. The cases of Bulelwa referred to in section 5.3.4 and Vusumzi (interview episode 1) show the extent to which misconceptions could hamper learners' progress in geometry. Teachers should pay attention to students' imprecise use of and misconceptions about geometric concepts and correct them early in their geometry course. Similar students' misconceptions in geometry had been reported in earlier studies (see Chapter Two, section 2.7.3.1).

5.4 Chapter conclusion

In this chapter, learners' responses to the GIST were analysed both quantitatively and qualitatively and the results interpreted. Some of the major findings are the following.

- On the whole, participants' performance in the GIST was considered unsatisfactory as learners were only able to obtain an overall percentage mean score of 40.19%.

- Although learners from the South African subsample obtained a marginally higher mean score (43.44%) than their international peers from the Nigerian subsample whose mean score was 36.94%, the difference in the mean scores was not statistically significant. The conclusion reached was that, in this study, South African learners did not perform significantly better than their Nigerian counterparts on the GIST.

- There was in general no significant difference in the male mean score (44.38%) and the female mean score (36.85%) on the GIST. Even at each school level, there were also no significant gender differences in the mean scores, as the male mean scores were only slightly higher than the female mean scores on the GIST.

An exploratory qualitative analysis of learners' responses to the GIST further yielded the following results:

- On the identifying and naming shapes task (i.e. Task 1 of the GIST), the majority of the learners made many imprecise visual identifications involving shape and orientation in their attempts to describe the shapes. On this task, 14 learners, 15 learners and 7 learners were respectively at van Hiele geometric thinking levels 0, 1 and 2. No leaner in the sample group for the GIST was at van Hiele level 3.

- Although the performance of task 1 of the GIST by learners was generally poor, their responses indicated that more students can name shapes than describe the properties of shapes. This seems to be consistent with the hierarchical property of the van Hiele levels.

- There was not one learner in the sample group for the GIST who used more than one attribute of a shape in naming the shape (Task 1). For example, right-angled isosceles triangles were either named as "isosceles triangle" or "right-angled triangle", with the majority of the learners showing preference for the former name.

- Of the five tasks that constituted the GIST, the task of sorting the shapes into two distinct groups of triangles and quadrilaterals (Task 2) was the easiest for the majority of the learners, since about 52.8% of the Nigerian learners and 71.3% of the South African learners were successful in this task. The task of sorting shapes into smaller subgroups of triangles and quadrilaterals by class inclusions of shapes (Task 3) was the most difficult for the learners, since only 17.8% of the Nigerian learners and 19.2% of the South African learners succeeded in this task.

Taken as a whole, the results of the GIST were found to be consistent with those of many earlier studies. In the chapter that follows, participants' responses to the CPGT are analysed and the results interpreted.

CHAPTER SIX

DATA ANALYSIS, RESULTS AND DISCUSSION 3: THE CPGT

6.1 Introduction

This chapter presents the results of students' responses to the Conjecturing in Plane Geometry Test (CPGT). The first part of the chapter focuses on students' performance in the CPGT based on a quantitative analysis of their mean scores. The second part provides information on learners' performance in the CPGT through an item-by-item analysis of their responses. The CPGT was grade-specific (see Chapter 3, section 3.3.4.1.3) and accordingly the analysis presented here is structured by grade level for each of the participating schools. The reason for the grade level specificity of the CPGT was explained in the third paragraph of section 3.3.4.1.3 of Chapter 3.

6.2 Overall participants' performance in the CPGT per grade

The percentage mean scores in the CPGT were calculated separately for all the grade 10, 11 and 12 learners who wrote this test. The overall percentage mean scores obtained by the grade 10, 11 and 12 learners in the CPGT were, respectively, 17.39%, 22.48% and 36.47%. Note that because the CPGT was grade-specific, it is not possible to compare participants' mean scores across the grade levels. These mean scores should rather be interpreted with respect to the central concepts that the CPGT measures at each grade level (see worksheets 1–3, in Chapter 3, section 3.3.4.1.3).

The CPGT was designed mainly to explore students' abilities to formulate conjectures, draw simple inferences and state definitions of simple geometric shapes (see the rationale for the CPGT in Chapter 3). Thus the low mean scores obtained by learners in their respective grades for the CPGT could be interpreted as evidence that the learners' knowledge was poor in these learning areas. Given that defining, drawing inferences and conjecturing are cognitive activities commonly associated with van Hiele levels 3 and 4 reasoning (see Chapter 2, section 2.8), the low mean scores also indicate that the majority of the learners were not yet at these van Hiele levels of geometric understanding. The results generally identify with those of Pegg (1995), when he stated that only about 25% of high school learners in his study felt comfortable with problems associated with level 4 in the van Hiele hierarchy of geometric thinking levels.

6.2.1 Mean scores in the CPGT of NS and SAS grade 10 learners

The percentage mean scores obtained by the grade 10 learners in the CPGT were calculated separately for learners from NS and SAS. Learners from the NS subsample obtained a percentage mean score of 9.59% on the CPGT and learners from the SAS subsample obtained a percentage mean score of 25.18% (see Table 6.1). By any standard, these are very low means. As was stated in section 3.4.1.3 of Chapter 3 under the subheading 'construction', the Nigerian and the South African geometry curricula for grade 10 learners largely dictated the choice of tasks that comprised the CPGT. As mentioned above, in section 3.3.4.1.3, Worksheet 1 of the CPGT was developed to explore grade 10 learners' knowledge of the side-angle properties of triangles, rectangles, squares and rhombuses. These low mean scores therefore imply that these learners were yet to master that aspect of their geometry curriculum that requires them to be able to make conjectures, state definitions and draw inferences concerning these shapes. It can thus be asserted that these learners were not yet ready for the deductive study of geometry, contrary to their curriculum expectations.

Table 6. 1 Percentage mean scores of grade 10 learners in the CPGT

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	22	9.59	7.42	- 3.81	424	0.0004
SAS	22	25.18	17.68			

As indicated by the mean scores, grade 10 learners from SAS outperformed their Nigerian counterparts from NS on the CPGT. As can be seen in Table 6.1, the difference between the mean scores of the NS learners and that of the SAS learners is statistically significant (t = - 3.81, 42*df*, p < 0.001) in favour of the latter. As with the

TPGT (see chapter 4, sections 4.2.3 and 4.2.4), these results indicate that, in relative terms, the grade 10 learners from the South African subsample had a better knowledge of school geometry as tested by the CPGT than their counterparts from the Nigerian subsample.

6.2.2 Mean scores in the CPGT of NS and SAS grade 11 learners

As with the grade 10 learners, the mean scores obtained for the CPGT by the grade 11 learners from the NS and the SAS subsamples were calculated separately for each of the participating schools. The results are summarized in Table 6.2.

Table 6. 2 Percentage mean scores of grade 11 learners in the CPGT

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	20	24.65	27.17	0.48	41	0.4799
SAS	23	20.30	10.15			

As evident in Table 6.2, the mean score (24.65%) of the NS learners on the CPGT was marginally higher than the mean score (20.30%) obtained by the grade 11 learners from SAS. As further indicated in the table, the difference between the mean scores of these two groups, in favour of the NS learners, was not statistically significant (t = 0.48, 41*df*, $p > 0.05$). That is, in this study, grade 11 learners from the NS subsample did not perform significantly better than their comparative SAS learners on the CPGT. Nevertheless, these results provide support for the point made earlier on that, for the SAS subsample, grade 11 learners' knowledge of school geometry appears suspect (see Chapter 4, sections 4.2.4; 4.2.5.6, para.4; 4.4.2 and 4.4.3). This is the only grade at which learners from the NS subsample ***consistently*** obtained (marginally) higher mean scores than their SAS counterparts in this study.

It was stated in section 3.3.4.1.3 of Chapter 3 that, consistent with the Nigerian and South African geometry curricula, the central concept investigated in Worksheet 2 of the CPGT designed for the grade 11 participants was the similarity properties of triangles. The low mean scores obtained in the CPGT therefore indicate that the grade

11 learners had not yet grasped the concept of similarity as prescribed by their curriculum.

6.2.3 Mean scores in the CPGT of NS and SAS grade 12 learners

Worksheet 3 of the CPGT was designed for the grade 12 learners and it explored learners' knowledge of circle geometry (see Chapter 3, section 3.3.4.1.3). As with the preceding sections, the analysis of the grade 12 learners' performance in the CPGT was based on the percentage mean scores calculated separately for the NS and the SAS learners. Table 6.3 summarizes the results.

Table 6. 3 Percentage mean scores of grade 12 learners in the CPGT

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	22	20.68	22.05	-3.72	40	0.0006
SAS	20	52.25	32.38			

As evident in Table 6.3, the mean score obtained by the grade 12 learners from NS on the CPGT was 20.68% and that of their counterparts from the SAS subsample, 52.25%. Table 6.3 further indicates that the difference between the means of the NS and the SAS learners in the CPGT, in favour of the latter, is statistically significant at the 0.001 confidence level (t = - 3.72, 40df, p < 0.001). This means that the grade 12 learners from SAS performed significantly better than their peers from NS.

Given the fairly impressive mean score obtained by the grade 12 learners from SAS on the CPGT, it follows that the overall low mean (36.47%) calculated for all the grade 12 learners (section 6.2) can be attributed mainly to the very low mean score obtained by the grade 12 learners from NS. Considering the mean score (53%) obtained by the grade 12 learners from SAS on the TPGT for the concept of the circle (Chapter 4, section 4.4.2) and the mean score (52.25%) they obtained in the CPGT, it would seem that these learners demonstrated a consistent knowledge of the geometric concept of the circle even though the two tests cannot be said to be comparable in terms of their overall structure.

6.3 Grade level item-by-item analysis of learners' performance in the CPGT

AS stated earlier, the CPGT was designed to assess students' ability to make conjectures and state definitions of plane geometric shapes. The test made use of a constructivist investigative approach using geometrical construction (see Chapter 3). Since the constructivist investigative approach used for the CPGT involved several stages of activities which were intended ultimately to lead the learners to a certain conclusion (or conjecture) about a given geometric shape and its properties, an item-by-item analysis of learners' responses was considered necessary. This analysis helped to disclose what activity the learners were able to perform in a given geometry investigative task.

A separate worksheet was developed and used at each of the three grade levels involved in this study. Worksheet 1 was for the grade 10 learners, Worksheet 2 was for the grade 11 learners and Worksheet 3 for the grade 12 learners (see Chapter 3, section 3.3.4.1.3). The matter of the type and number of investigative activities that comprised each worksheet was explained in that section. Details of the contents of each of the worksheets are contained in Appendix 5.A.1–3, pp.51, 59 and 66. For a better understanding of the analysis that follows in the next three subsections, it is suggested therefore that the reader consult also section 3.3.4.1.3 (for a description of the respective worksheets) and Appendices 5.A.1–5.A.3 (for the contents of the worksheets).

6.3.1 Item analysis of the CPGT for the grade 10 learners

In section 6.2.1, grade 10 learners' performance in the CPGT was reported by computing their percentage mean scores based on a grading system that was explained in section 3.4.1.3 of Chapter 3. In the present section, the analysis provided is based on the number of grade 10 learners in each of the NS and the SAS subsamples who were able to perform specific activities for each of the 6 investigations that made up Worksheet 1. The results are as summarised in Table 6.4.

Table 6. 4 Analysis of Worksheet 1 of the CPGT

Investigation No.	**Expected activity**	**No. successful**	
		NS (n = 22)	SAS (n = 22)
1	• To obtain, by addition, the sum of the angles of a triangle to be 180°	21	19
	• To conjecture that the sum of the angles of a triangle is 180°	5	9
2	• To recognise, through own construction, and name an isosceles triangle	4	8
	• To state, through own construction, that the base angles of an isosceles triangle are equal	4	11
	• To conjecture that if two sides of a triangle are equal, then two of its angles are also equal	1	5
3	• To recognise, through own construction, and name an equilateral triangle	5	16
	• To conjecture that if in a triangle all the sides are equal, then all the angles are also equal (with each = 60°)	4	8
4	• To recognise, through own construction, and name a rectangle	0	13
	• To list, at least, three properties of a rectangle	0	5
	• To conjecture that if the diagonals of a parallelogram are equal, then the parallelogram is a rectangle	0	0
	• To define a rectangle	0	1
5	• To recognise, through own construction, and name a square	0	9
	• To list, at least, two special properties of a square	0	6
	• To list unique properties of a square that a rectangle does not have	0	4
	• To conjecture that a parallelogram having equal diagonals that bisect each other at right angles is a square	0	0
	• To define a square	0	1
6	• To recognise, through own construction, and name a rhombus	3	4
	• To list, at least, two special properties of a rhombus	0	0
	• To list, at least, two specific properties common to a square and a rhombus	0	0
	• To list one unique property of a square that a rhombus does not have	0	2
	• To recognise, with justification, that a square is a special rhombus	0	2
	• To define a rhombus	0	1

As stated in section 3.3.4.1.3, investigation 1 of Worksheet 1 was to lead the learners to formulate a conjecture that *the sum of the (interior) angles of a triangle is 180°*. This investigation involved two separate activities. The first activity was for the learners to draw (or construct) any triangle and obtain the sum of the angles through measurement and addition of the angles. The second activity required the learners to compare their individual result for the first activity with those of others near them and state their observation as a conjecture. The assumption here was that if the learners noticed that the sum of the angles of each of the different triangles they had drawn was 180°, then they would be able to formulate the conjecture (or draw the conclusion) that the sum of the angles of any triangle is (always) 180°.

For investigation 1, Table 6.4 indicates that although 21 (95%) out of the 22 grade 10 learners from the NS subsample were successful in calculating the angle sum of a triangle to be 180°, only 5 (22%) of them managed to generalize their observation that the sum of the angles of a triangle will always be 180°. Because of a possible language difficulty, it was not expected that these learners should formulate their conjectures in formal terminologically correct statements. For example, Suberu and Abayomi, two of the five learners who conjectured that the angle sum of a triangle is 180°, put it this way:

Suberu: Sometimes the angles may be the same with corresponding answer, and sometimes the angles will be different while the answer will be the same.

Abayomi: What I can conclude about the sum of the angles of a triangle is that no matter the sides [meant sizes] of angles you may have, the addition of the three angles must give you 180°.

Suberu most likely saw both triangles drawn by some students which had the same angle measures as hers (many drew equilateral triangles), and triangles drawn by other students with different angle measures from her own, and noticed that in either case, the sum of the angles (what she called "the answer") is 180° (what she referred to as "the same"). Abayomi, on the other hand, probably compared his work only with those of other learners who drew triangles that had different angle measures from his own, and observed that each of them obtained 180° as the sum of the angles of their separate triangles. The point being made here is that even with this level of flexibility in accepting as correct such responses from the learners as these, many could still not provide an acceptable response to the second activity of investigation 1. It looks probable that these learners had only had limited experience of the kind that could enable them to successfully make conjectures.

Stating a definition of a shape (investigations 4, 5 and 6) proved the most difficult for the grade 10 learners from NS, as none of them was able to do this. Many simply avoided responding to that section of the question or task. However, Abayomi, who named the square that he drew a kite (investigation 5) defined his drawn shape as follows: "Kite is a parallelogram in which all the sides and angles are equal". Where it not for the incorrect name associated with his drawn shape, what Abayomi gave is surely an acceptable definition of a square.

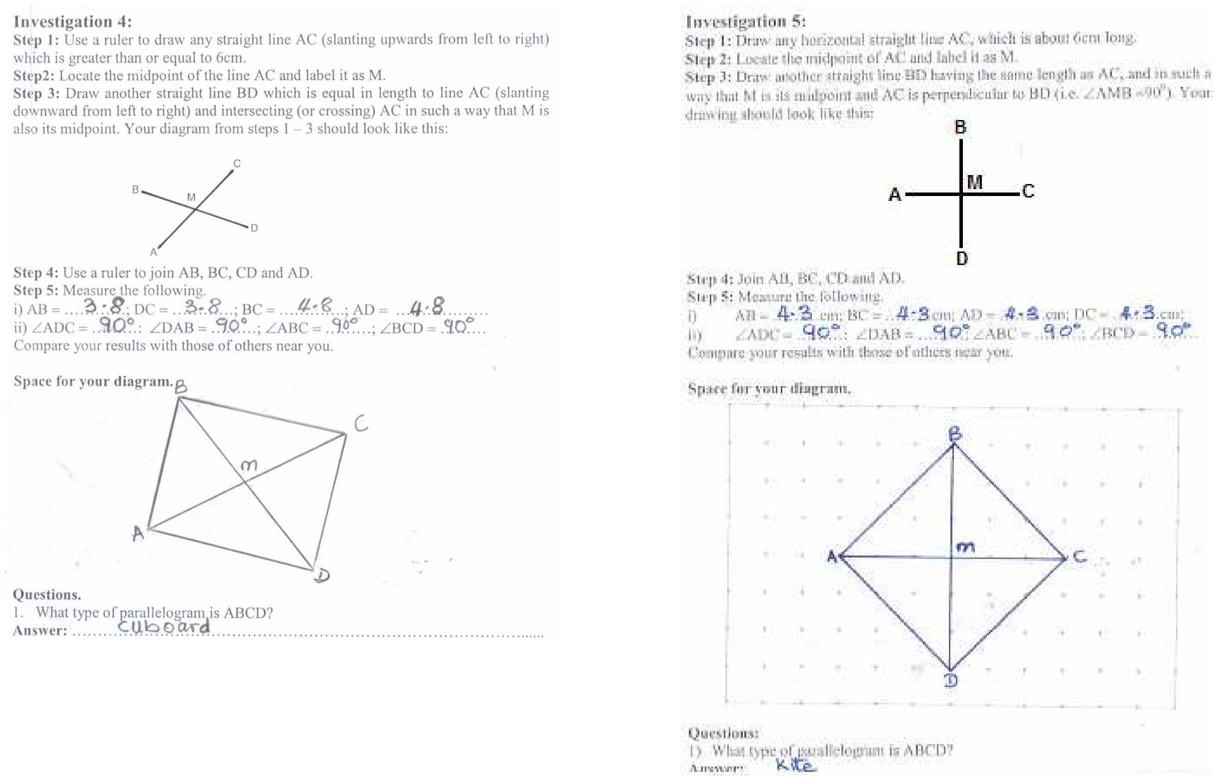
For learners from the SAS subsample, the performance was not much different from that of the NS learners. As could be seen in Table 6.4, 19 (86%) out of the 22 grade 10 learners from SAS who wrote the CPGT succeeded in computing the sum of the angles of a triangle to be 180° in investigation 1. Only 9 (41%) of them, however, were able to generalize their observation as a conjecture. Like their NS counterparts, many of the grade 10 learners from the SAS subsample had difficulty formulating conjectures in formal technical language. Having compared his work with those of his peers, Kondile, for example, generalized his observation as follows: "I conclude that when I'm drawing a triangle and add angle A, B and C and I'm going to get 180° all the time". The language may not be very formal, but the idea is clear: for every triangle that is drawn, the sum of the angles is always 180°.

As with the NS learners, stating a definition of a shape was very difficult for nearly all the learners from SAS as only 1 of them was able to define a rectangle (investigation 4) and a square (investigation 5), while 1 other student was able to define a rhombus (investigation 6). Interestingly though, the two learners stated a hierarchical definition (see De Villiers, 1994; 1998) of these shapes, thereby exhibiting traces of level 3 reasoning according to the van Hiele theory. For example, the learner who defined the rectangle and the square stated that "a rectangle is a parallelogram with one angle equal to 90°" and that "a square is a rectangle with two adjacent sides equal". This was one of the strongest grade 10 students (cognitively speaking) in the study sample as well as in SAS for the study year.

The results for investigation 1 further indicate that some of the learners had difficulty determining the measure of an angle using a protractor despite our (the mathematics teachers' and my) efforts to guide them. Since assessing what the learners were able to do (as opposed to developing and implementing an intervention teaching program) was the general aim of the CPGT, an effort was made only to explain procedures to the learners rather than to ensure that each and every one of them made accurate measurements. As evident in Table 6.4, 1 learner from the NS subsample and 3 learners from the SAS subsample were unable to compute (by measuring and adding) the angle sum of a triangle as 180°. For one of the three learners from the SAS subsample, the sum of the angles of a triangle was 170° (with angles 90°, 50° and 30°), for another it was 184° (with angles 91°, 56° and 37°), and for the third learner it was 140° (with angles 60°, 50° and 30°). The only learner from NS who could not obtain the angle sum of a triangle to be 180° represented the angles of his triangle in centimetres (5cm + 5cm + 5cm = 15cm) – revealing yet another form of learning difficulty among the participants. This learner was actually adding the lengths of the sides of her triangle instead of the angles. There were indeed many learners for whom the unit of measurement of angles was centimetres instead of degrees or radians. This situation would require that teachers explicitly direct learners' attention to the units of measurement for angles, even though they ought to have done work on this at lower levels of their schooling.

Similar interpretations to that of investigation 1 would hold for investigations 2 through 6 of Worksheet 1 (Table 6.4). As evident in Table 6.4, formulating conjectures and stating definitions were more difficult for the majority of the grade 10 learners than the other activities featured in Worksheet 1, such as identifying and listing the properties of shapes. This of course links up with the hierarchical property of the van Hiele levels (see Chapter 2, section 2.8.1).

A point that perhaps deserves separate mention is that none of the learners from the NS subsample was able to identify and name a rectangle (investigation 4) and a square (investigation 5) through their own constructions (Table 6.4). The difficulty encountered by these learners cannot be excused entirely by the nature of the tasks, i.e. the supposition that they were not used to the constructivist investigative approach to learning. In fact, many of these learners had no problem following detailed instructions on the worksheet concerning how to construct (or draw) the required shape in each of the investigations. The problem they had was rather that of identifying and naming the shapes in a nonstandard orientation – the very problem that was reported in paragraph 2 of section 5.3.1 in Chapter 5. Abayomi, for example, correctly constructed a rectangle and a square (Figure 6.1), but without attending to the properties of the shapes or possibly distracted by the orientation of the shapes, he named the rectangle a cuboard (he meant cuboid) and the square a kite (compare with interview episode 3, section 5.3.1 of Chapter 5). There were many other learners with this learning problem.


Investigation 4:
Step 1: Use a ruler to draw any straight line AC (slanting upwards from left to right) which is greater than or equal to 6cm.
Step2: Locate the midpoint of the line AC and label it as M.
Step 3: Draw another straight line BD which is equal in length to line AC (slanting downward from left to right) and intersecting (or crossing) AC in such a way that M is also its midpoint. Your diagram from steps 1 – 3 should look like this:

Step 4: Use a ruler to join AB, BC, CD and AD.
Step 5: Measure the following.
i) AB = ….3.8.; DC = ..3.8…; BC = …4.8…; AD = ..4.8……..
ii) ∠ADC = ..90°.: ∠DAB = .90°..; ∠ABC = .90°…; ∠BCD = .90°…
Compare your results with those of others near you.

Space for your diagram.

Questions.
1. What type of parallelogram is ABCD?
Answer: ……cuboard……

Investigation 5:
Step 1: Draw any horizontal straight line AC, which is about 6cm long.
Step 2: Locate the midpoint of AC and label it as M.
Step 3: Draw another straight line BD having the same length as AC, and in such a way that M is its midpoint and AC is perpendicular to BD (i.e. ∠AMB = 90°). Your drawing should look like this:

Step 4: Join AB, BC, CD and AD.
Step 5: Measure the following.
i) AB = 4.3 cm; BC = 4.3cm; AD = 4.3 cm; DC = 4.3 cm;
ii) ∠ADC = 90°; ∠DAB = 90° ∠ABC = 90°; ∠BCD = 90°
Compare your results with those of others near you.

Space for your diagram.

Questions:
1) What type of parallelogram is ABCD?
Answer: Kite

Figure 6. 1 Illustrating learners' difficulty with identifying and naming shape.

Note in Figure 6.1 that Abayomi had determined (through his own constructions) all that was needed (two pairs of opposite sides equal, 90° angles and equal diagonals etc. for the rectangle; and all sides equal, 90° angles, diagonals bisect each other at right angles etc. for the square) to correctly identify and name the two shapes, yet he named them incorrectly. This was the situation with the majority of the grade 10 learners.

6.3.2 Item analysis of the CPGT for the grade 11 learners

As stated in section 3.3.4.1.3 of Chapter 3, Worksheet 2 was developed for learners in grade 11, and the central concept investigated was the similarity properties of triangles as prescribed by the Nigerian and South African mathematics curricula. In section 6.2.2, grade 11 learners' performance in the CPGT was analysed according to their percentage mean scores. The analysis that is provided in this section, however, focuses on the number of grade 11 learners in each of the participating schools who were able to perform a specific activity for each of the 6 investigations that made up worksheet 2 of the CPGT. The activities that were expected to be performed as well as the corresponding number of learners who successfully performed these activities are shown in Table 6.5.

Table 6. 5 Analysis of Worksheet 2 of the CPGT

Investigation No.	Expected activity	No. successful	
		NS (n = 20)	SAS (n = 23)
1	• To obtain, through own construction, equal ratio for the corresponding sides of two similar triangles • To obtain, through own construction, equal measures for the corresponding angles of two similar triangles • To conjecture that if two triangles are similar, then their corresponding sides are in a constant ratio • To conjecture that if two triangles are similar, then their corresponding angles are equal • To state the NASCO for two triangles to be similar	12 10 5 7 2	12 15 1 14 0
2	• To obtain, through own construction, equal ratio for the corresponding intercepts between three parallel lines cut off by a pair of transversals • To conjecture that if three parallel lines are cut by a pair of transversals, then the corresponding intercepts cut off on each one are in the same ratio	13 5	17 1
3	• To obtain, through own construction, equal ratio for the corresponding sides of two similar triangles • To tell, through investigation, how the line drawn parallel to one side of a triangle divides the other two sides • To conjecture that the line drawn parallel to one side of a triangle divides the other two sides proportionally	13 6 2	16 0 0
4	• To state, through own construction, that the line joining the midpoints of two sides of a triangle is parallel to the third side • To state, through own construction, that the line joining the midpoints of two sides of a triangle is equal to half of the third side • To conjecture that the line joining the midpoints of two sides of a triangle is parallel to the third side • To conjecture that the line joining the midpoints of two sides of a triangle is equal to half of the third side • To conjecture that the line joining the midpoints of two sides of a triangle is parallel and equal to half of the third side	7 6 3 3 3	16 2 0 0 0
5	• To obtain, through own construction, a constant ratio for the corresponding sides of two given similar triangles • To conjecture that if the corresponding angles of two (similar) triangles are equal, then their corresponding sides are proportional	6 0	8 0
6	• To obtain, through own construction, equal measure for the corresponding angles of two similar triangles • To conjecture that if the corresponding sides of two triangles are proportional, then their corresponding angles are equal	4 1	1 0

It was stated in section 3.3.4.1.3 that investigation 1 of Worksheet 2 was to guide the learners to formulate two conjectures: a) *if two triangles are similar, then their* *corresponding sides are proportional*; and b) *if two triangles are similar, then their corresponding angles are equal*. From these conjectures, it was required that the learners should be able to deduce the necessary and sufficient conditions (NASCO) for two triangles to be similar. This investigation required the learners to perform five activities, as indicated in Table 6.5 (see also Appendix 5.A.2, p.59).

As indicated in Table 6.5, for investigation 1, of the 20 grade 11 learners from the NS subsample who wrote the CPGT, 12 (60%) of them successfully obtained, through their own construction, an equal (or a constant) ratio for the corresponding sides of two similar triangles. Only 5 (25%) of them, however, were able to generalize their observation that if two triangles are similar, then their corresponding sides are in a constant ratio, even though the learners were here required only to fill in either 'proportional' or 'in constant ratio' (see investigation 1 in Appendix 5.A.2, p.59). Similarly, although 10 (50%) of the learners from NS were successful in obtaining equal measure for the corresponding angles of two similar triangles, only 7 (35%) of them managed to state their observation as a conjecture. Two learners (10%) from the NS subsample were able to state the necessary and sufficient conditions (NASCO) for two triangles to be similar.

As with the grade 10 learners, Table 6.5 shows that formulating conjectures was generally more difficult than the other activities in Worksheet 2 for the grade 11 learners from NS. Language problems possibly played a role in learners' difficulty with conjecturing, as many stated their conjectures in specific rather than in general and technical terms. In investigation 3, for example, Adeleke correctly observed that the line DE drawn parallel to the side BC of triangle ABC divides the two opposite sides AB and AC of triangle ABC in the same ratio. However, when asked to state his observation as a conjecture, he wrote as follows:

> From observation, it is clear that Δ ADE and Δ ABC are similar. It is also obvious that the parallel line DE to BC divides AB and AC equally in the same ratio.

Clearly, like many other learners from the NS subsample, Adeleke was not generalizing his observation beyond the particular triangle that he had drawn.

Responses like these indicate that the majority of the learners were not able to form conjectures in the way that the curriculum expected them to do.

The response patterns of grade 11 learners from the SAS subsample to the CPGT were similar to those of their NS counterparts. As evident in Table 6.5, although 12 (52%) learners from the SAS sample group successfully obtained equal ratio for the corresponding sides of two similar triangles, only 1 (4%) of them was able to generalize her observation as a conjecture that if two triangles are similar, then their corresponding sides are in a constant ratio. There were 15 (65%) learners from the SAS subsample who successfully obtained equal measure for the corresponding angles of two similar triangles, and 14 (61%) of them were able to state their observation as a conjecture. No grade 11 learner in the SAS sample group was able to state the necessary and sufficient conditions for two triangles to be similar. Perhaps the problem was that they were not familiar with these terms, for some of them explicitly stated that they were encountering the terms for the first time.

It can be seen from Table 6.5 that more learners from the SAS subsample had problems with formulating conjectures than learners from the NS subsample. Considering investigation 1 through 6, Table 6.5 indicates that grade 11 learners from NS were generally more successful in the CPGT compared with their counterparts from SAS. Similar results were earlier reported in section 6.2.2.

6.3.3 Item analysis of the CPGT for the grade 12 learners

Worksheet 3 was designed for the grade 12 learners and it explored learners' mathematical knowledge of circle geometry (see section 3.3.4.1.3). As stated in section 3.3.4.1.3 of Chapter 3, Worksheet 3 of the CPGT consisted of 10 investigations, with each investigation requiring the learners to perform two sets of activities (as is evident in Table 6.6). Investigations 1 through 4 were on the chord properties of a circle, investigations 5 through 8 focused on the arc-angle properties of a circle, and investigations 9 and 10 explored the tangent properties of a circle (see 3.3.4.1.3). Table 6.6 summarizes participants' performance in each of the investigations that made up worksheet 3 of the CPGT.

Table 6. 6 Analysis of worksheet 3 of the CPGT

Investigation No.	Expected activity	No. successful	
		NS (n = 22)	SAS (n = 20)
1	• To obtain, through own construction, angle 90° between the line drawn from the centre of a circle to the midpoint of a chord • To conjecture that the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord	14 7	15 14
2	• To obtain, through own construction, equal measure for the sides of a chord from the point of intersection of the perpendicular from the centre of a circle • To conjecture that the line drawn from the centre of a circle perpendicular to a chord bisects the chord	13 3	14 10
3	• To obtain, through own construction, equal measure for the lines drawn from the centre of a circle perpendicular to chords of equal length • To conjecture that equal chords are equidistant from the centre of a circle	8 1	15 4
4	• To obtain, through own construction, equal central angles for equal chords of a circle • To conjecture that equal chords subtend equal angles at the centre of a circle	8 0	14 6
5	• To obtain, through own construction, angle at centre = 2 x angle at circumference • To conjecture that the angle which an arc of a circle subtends at the centre is twice the angle which the same arc subtends at the circumference	8 2	10 10
6	• To obtain, through own construction, equal angle subtended by the same arc of a circle at two different points on the circumference • To conjecture that angles in the same segment of a circle are equal	7 2	13 11
7	• To obtain, through own construction, angle 90° for the angle subtended by the diameter of a circle • To conjecture that the angle in a semicircle is a right angle	8 1	12 8
8	• To obtain, through own construction, 180° as the sum of the opposite angles of a cyclic quadrilateral • To conjecture that opposite angles of a cyclic quadrilateral are supplementary	3 0	10 7
9	• To obtain, through own construction, angle 90° as the angle between a radius and a tangent at the point of contact • To conjecture that a tangent to a circle is perpendicular to the radius at the point of contact	5 1	15 7
10	• To obtain, through own construction, equal measure for two tangents to a circle drawn from the same external point • To conjecture that tangents to a circle from the same external point are equal in length	0 0	12 2

Investigation 1 of Worksheet 3 was to guide the learners to form a conjecture that *the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.* Table 6.6 indicates that 14 (64%) of the learners from the NS subsample were successful in obtaining, through their own construction, angle 90° between a chord and the line drawn from the centre of a circle to the midpoint of the chord. Only 7 (32%) of them, however, were able to state their observation as a conjecture.

Although Table 6.6 clearly indicates that formulating conjectures was generally difficult for the grade 12 learners from NS, it was particularly so for them with concepts that dealt with the tangent properties of a circle (investigations 9 and 10). Although the inability of some of these learners to construct (or draw) and take accurate measurements may have partly influenced their response patterns in these investigations, that alone cannot justify their poor performance in this learning area, given the fact that they must (or ought to) have had experience in these skills (constructing, drawing and measuring) in their lower grades (see Siyepu, 2005). Most of these learners could simply not perceive the interrelationships between the properties of a circle in the various investigations. The majority of them were not yet at van Hiele level 3. This partly explains why even the many learners who successfully constructed/drew the required shape, noting all its essential properties, were still not able to state their observations as a conjecture.

The results in Table 6.6 appear to support those reported in section 6.2.3, which indicated that the grade 12 learners from SAS had a fairly impressive knowledge of circle geometry compared to their counterparts from NS. In investigation 1, 15 (75%) of the grade 12 learners from the SAS subsample successfully obtained, through their own construction, angle 90° between a chord and the line that is drawn from the centre of a circle to the midpoint of the chord. As indicated in Table 6.6, an impressive number of these learners, 14 (70%), were able to generalize their observation as a conjecture that the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord. It is equally impressive to note that for investigation 5, all the 10 (50%) grade 12 learners from SAS who successfully established, through their own construction, that the angle which an arc of a circle subtends at the centre is twice the angle it subtends at the circumference, were also able to state their observation as a conjecture.

Like the grade 10 and 11 learners, the grade 12 learners from SAS did not necessarily formulate their conjectures in formal technical statements even though many of them presented their conjectures in general terms. Siyabulela, for example, having established, through construction, that the perpendicular lines from the centre of a circle to two nonparallel congruent chords are equal (investigation 3), stated his conjecture in this way: “Two lines from circle centre perpendicular to equal chords are equal”. This may not be as technical as *equal chords are equidistant from the centre of a circle*, yet his idea is clearly understandable. There were a few other learners in this group (SAS) who stated their conjectures in general terms as Siyabulela did, demonstrating evidence of level 3 or 4 reasoning in the van Hiele hierarchy of the levels of geometric thought.

Although conjecturing was generally difficult for the grade 12 learners from the SAS subsample (as it was for their NS counterparts), Table 6.6 indicates that these learners were more successful with investigations that dealt with the arc-angle properties of a circle (investigations 5 through 8) than those concerning the chord and tangent properties of a circle. Like their NS counterparts, the grade 12 learners from SAS were least successful with investigations that dealt with the tangent properties of a circle (investigations 9 and 10). It is possible that only a few teachers get to teach this aspect of circle geometry, probably because of time constraints, as these theorems are among the last required to be taught in high school circle geometry in Nigeria and South Africa.

The results reported in sections 6.3.1 through 6.3.3 partly support and partly refute those of Siyepu (2005). That many learners in this study were initially unable to construct and measure angles, but were soon able to do so when sufficiently guided, generally links up with Siyepu’s (2005) findings in which in South Africa, “these grade 11 learners were unable to construct and measure angles…” (p.64), but “as we (the researcher and the sample groups) proceeded with the investigations, learners became familiar with the constructions, although they displayed small problems of inaccuracy” (p.67). However, the finding that many learners in this study were unable to form conjectures seems to run counter to that of Siyepu (2005), even though initially he had stated that “all the participants in the sample group could not understand the concept of ‘conjecture’ and were thus unable to generalize” (p.65). Siyepu (2005) later reports that “as we proceeded with the investigation, learners in the sample group became familiar with the formation of a conjecture” (p.70) and that “**all** (emphasis mine) the learners in the groups managed to make a conjecture that a tangent to a circle is perpendicular to the radius at the point of contact" (p.71) (compare with result for investigation 9 in Table 6.6).

It looks probable that in the course of his research, Siyepu gradually unfolded a program of instruction, thereby enhancing the learners' performance in given tasks in relation to their entry knowledge. Since the aim of the CPGT was to assess what the learners were able to do rather than to develop a teaching program (the worksheets are useful for this purpose, though), participants were only provided guidance on how to construct/draw and measure and were left to form conjectures on their own. This hypothesized difference in approach possibly accounts for the difference between the findings of Siyepu's research and those of the present study.

Importantly, according to the van Hiele theory, formulating (and testing/proving) conjectures is a cognitive activity of which only learners who are functioning at at least van Hiele level 3 or 4 are capable (see Burger & Shaughnessy, 1986). In a separate test designed to assess his learners' attainment of van Hiele level 1, Siyepu (2005) reports that "the results indicate that 50% of the sample is not even at the van Hiele level 1 with regard to the circle geometry" (p.76). Yet these were the same students whom he had earlier reported were able to form conjectures about circle concepts! This inconsistency is not easily explained, except in terms of the hypothesis outlined above.

6.4 Chapter conclusion

This chapter presented a grade-level analysis of learners' performance in activities included in the worksheets of the CPGT. The analysis was presented in two forms. The first form examined learners' performance in the CPGT according to their percentage mean scores based on a grading system developed for that purpose (see Appendix 5.B.1–3, pp.77, 80 and 82). The second form was based on an item-by-item analysis of the learners' responses, with a focus on the number of learners able to perform specific activities that made up the worksheets for each grade level. The CPGT generally explored students' abilities to form conjectures, draw simple inferences and state definitions of simple geometric shapes (see section 3.3.4.1.3 for details about the worksheets). The major findings include the following:

- The overall percentage mean scores obtained by the grade 10, 11 and 12 learners for the respective worksheet of the CPGT (Worksheet 1 for grade 10, Worksheet 2 for grade 11 and Worksheet 3 for grade 12) were, respectively, 17.39%, 22.48% and 36.47%. The low mean scores obtained by learners in their respective grades was interpreted as evidence that these learners had difficulty in formulating conjectures and stating definitions regarding simple geometric shapes. This implies that they were ill-prepared for the formal deductive study of high school geometry as prescribed by their respective curricula.

- The grade 10 learners from the NS subsample obtained a percentage mean score of 9.59% on the CPGT and their counterparts from the SAS subsample obtained a mean score of 25.18%. Although the difference between the mean scores of NS and SAS learners, in favour of the latter, was found to be statistically significant ($t = -3.81$, $42df$, $p < 0.001$), the performance by learners from both sample groups as indicated by the low mean scores was considered unsatisfactory. The conclusion was reached that these learners were yet to master that aspect of their geometry curriculum that requires them to be able to make conjectures, state definitions and draw inferences with regard to the geometric concepts of triangles, squares, rectangles and rhombuses. That is, these learners were not yet ready for the deductive study of these shapes.

- The mean scores obtained by the grade 11 learners from NS and SAS in the CPGT were 24.65% and 20.30%, respectively. The difference between the means was not statistically significant at the 0.05 level. That is, learners from the NS subsample did not perform significantly better than their peers from SAS. Given the low mean scores obtained by these learners, the conclusion was that these learners were yet to acquire an adequate grasp of the concept of similarity as prescribed by their curriculum.

- The grade 12 learners from SAS performed significantly better than their counterparts from NS. The difference between the mean score (52.25%) obtained by the grade 12 learners from the SAS subsample and the mean score (20.68%) obtained by the NS learners for the CPGT was found to be statistically significant (t = - 3.72, 40*df*, $p < 0.001$). Considering the mean score (53%) obtained by the grade 12 learners from SAS on the TPGT for the concept of circles (Chapter 4, section 4.4.2) and the mean score (52.25%) they obtained on the CPGT, it can be concluded that these learners demonstrated a consistent knowledge of the geometric concept of the circle.

An item-by-item analysis of participants' responses yielded further results, which include the following.

- On the whole, at each grade level, forming a conjecture was much more difficult for the majority of the learners than the other activities that constituted their respective worksheets (defining, constructing, drawing, measuring, comparing). Among the few learners who managed to formulate conjectures, most could not do so in formal technical language.

- For some learners across all three grades, difficulties with measurement were evident in their responses. In grade 10, for example, there were 3 learners from the SAS subsample who obtained angle sums of 170° (with angles 90°, 50° and 30°), 184° (with angles 91°, 56° and 37°) and 140° (with angles 60°, 50° and 30°) for the triangles that they constructed (or drew) by themselves. There were many other learners for whom the unit of measurement of angles was the centimetre. For example, one of the learners from the NS subsample gave the sum of the angles of his triangle as 15cm (5cm + 5cm + 5cm = 15cm). Many of the learners appeared to have had inadequate preparation at lower school levels for the successful study of high school geometry.

- Some of the learners had difficulty in constructing simple geometric shapes using a ruler and a pair of compasses even when provided with detailed instructions for the procedure. When given adequate guidance, however, many were then able to construct the required shapes. Some of the learners in grade 10 who constructed the required shapes could not name them correctly because they were easily misled by the orientations of the shapes.

- Stating a definition for a rectangle, a square and a rhombus was generally difficult for the grade 10 learners. In fact, no learner from the NS subsample was able to define any of these shapes. Only 1 learner from the SAS subsample was able to define a rectangle and a square, and 1 other learner was able to define a rhombus.

- The results reported here were found to be partly consistent with and partly counter to those of Siyepu (2005). That many learners in this study were initially unable to construct shapes and measure angles, but were soon able to do so when guided appropriately, generally links up with Siyepu's work in South Africa. However, the finding that many learners in this study were unable to form conjectures seems to run counter with that of Siyepu (2005). Possible reasons for the difference were suggested.

In the chapter that follows, an analysis of participants' performance in the VHGT is used to identify their van Hiele levels of geometric thinking.

CHAPTER SEVEN

DATA ANALYSIS, RESULTS AND DISCUSSION 4: THE VHGT

7.1 Introduction

In this chapter, an analysis of students' performance in the van Hiele Geometry Test (VHGT) is presented. Phase 1 of this study concerns the determination of the van Hiele levels of geometric conceptualization among the participating learners (see Chapter 3, section 3.3.4.1). Although various instruments (the TPGT, GIST, CPGT and VHGT) contributed to this determination, the major instrument used to assign the learners to various van Hiele levels was the VHGT (see "rationale for the VHGT" in section 3.3.4.1.4). This chapter begins with an analysis of learners' performance in Part A of the VHGT by first examining their mean scores for the test and then allocating them to van Hiele levels. The second part of the analysis in this chapter provides information about learners' performance in Part B of the VHGT.

7.2 Analysis of Part A of the VHGT

7.2.1 Learners' performance in the VHGT according to percentage means

As was stated in section 3.3.4.1.4 of Chapter 3, Part A of the VHGT consisted of 4 subtests with each subtest testing learners' attainment of a specific van Hiele level (see Appendix 6.A.1–3, pp.84, 94 and 104). In the analysis that follows in sections 7.2.1.1 through 7.2.1.8, participants' performance in Part A of the VHGT is provided (regardless of the levels) by examining their percentage mean score in the test, consistent with the first grading method stated in section 3.4.1.4 of Chapter 3. However, in sections 7.2.2 through 7.2.2.3, an analysis of learners' performance is provided based on their percentage mean score at each of the van Hiele levels. The last part of the analysis of learners' performance in Part A of the VHGT focuses on their distribution into the van Hiele levels in accordance with the second grading method explicated in section 3.4.1.4 of Chapter 3.

7.2.1.1 Overall participants' performance in the VHGT

Learners' performance in the VHGT was described in terms of their percentage mean score. Using the first grading method explained in section 3.4.1.4 of Chapter 3, the mean score obtained by the learners in Part A of the VHGT was 7.14 points out of a possible 20 points. This figure represents an overall percentage mean score of 35.68%.

The relatively low percentage mean score obtained by the learners for Part A of the VHGT was found to be consistent with the findings of Usiskin (1982). The highest mean score obtained by any one of the 13 schools involved in Usiskin's (1982) study in a comparative[10] VHGT was 3.69 points (out of 31 points), corresponding to a percentage mean score of 11.90%. It should, however, be pointed out that the percentage mean scores (that of this study and those of Usiskin's reported here) are not necessarily comparable, in that one might be led to think that learners in this study performed better in the VHGT than their American peers in Usiskin's study because of the latter's lower mean scores. On the contrary, Usiskin's sample actually performed better than the sample in the present study. If the VHGT is graded according to the classical van Hiele levels as Usiskin did (see Usiskin, 1982), the overall percentage mean score obtained by the participating learners in this study actually becomes 6.09%, which is less than those of 12 (92%) of the 13 schools in Usiskin's study in which the percentage mean scores were between 7.42% – 11.90% inclusive (see Usiskin, 1982).

Regardless of the grading method, the fact remains that in both the current study and that of Usiskin (1982), learners obtained very low mean scores in the VHGT. This indicates that the majority of the learners in this study (as in Usiskin's) were at low van Hiele levels, possibly levels 0, 1 or 2.

[10] Recall that the VHGT was adapted from Usiskin's CDASSG van Hiele geometry test, hence comparative, which was originally designed to determine the van Hiele levels of the American school children (see Chapter 3, section 3.3.4.1.4).

7.2.1.2 Mean scores on the VHGT of NS and SAS learners

The mean scores obtained by learners in Part A of the VHGT were calculated separately for the NS and SAS subsamples. Table 7.1 summarizes the results.

Table 7. 1 School percentage mean scores for learners in the VHGT

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	68	31.84	12.98	-3.36	137	0.0010
SAS	71	39.37	13.44			

The mean score calculated for learners from the NS subsample was 31.84% and that of their peers from the SAS subsample was 39.37% (see Table 7.1). The difference in these mean scores in favour of the SAS learners was found to be statistically significant at the 0.005 level as evident in Table 7.1. What this means is that there were more NS learners at the lower van Hiele levels than there were SAS learners at those levels. Or conversely, there were fewer NS learners at the higher van Hiele levels compared to the number of SAS learners at those levels. These mean scores could further be interpreted to mean that more learners from the SAS subsample had attained higher van Hiele levels than learners from the NS subsample, which partly explains why the SAS learners consistently outperformed their NS counterparts in the other three tests (the TPGT, GIST and CPGT, see Chapters 4, 5 and 6, respectively).

As with the entire study sample, the low mean scores obtained by learners from the participating schools were found to be consistent with those reported by Usiskin (1982), in whose study the students from all 13 schools surveyed obtained very low mean scores.

7.2.1.3 Grade level comparison of mean scores in the VHGT

As with the TPGT (sections 4.2.3 and 4.2.4), grade level analysis of learners' performance in the VHGT focused on the relative performance of grade 10, 11 and 12 learners in NS and SAS. The aim was to compare performance at each grade level.. For this comparison, the percentage mean scores, correct to the nearest whole number, were computed for learners in their respective grades, as illustrated in Chart 7.1.

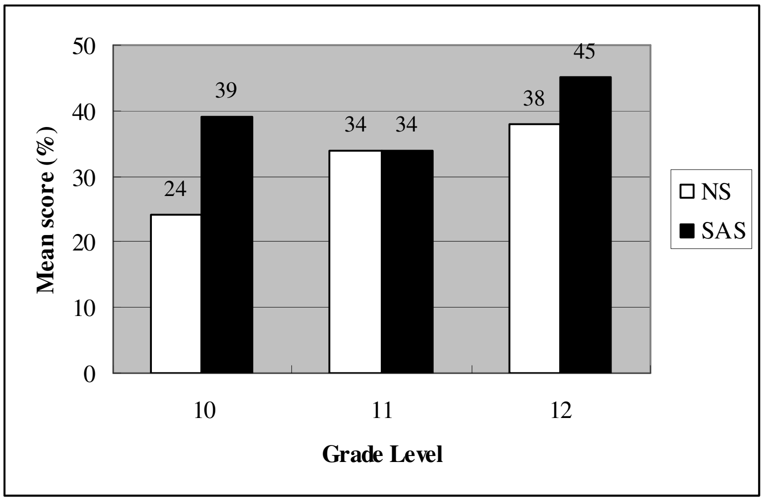


Chart 7. 1 Grade level performance of learners in the VHGT

As indicated in Chart 7.1, from the NS subsample, the mean score (38%) obtained by the grade 12 learners was marginally higher than that of the grade 11 learners (34%), which was in turn marginally higher than the mean score (24%) of the grade 10 learners. This implies that at each successive grade level in NS, there were more learners at higher van Hiele levels than there were at the adjacent lower grade level. This explains the marginal progressive increase in performance in the TPGT along the grade levels for the NS learners reported in section 4.2.3 of Chapter 4.

For the SAS subsample, although the mean score (45%) obtained by the grade 12 learners for the VHGT was higher than that of the grade 11 and 10 learners, the mean score (34%) of the grade 11 learners was lower than that of the grade 10 learners, who obtained a mean score of 39% in Part A of the VHGT. What this means is that there were more grade 12 learners than there were grade 10 learners at the higher van Hiele levels, and more grade 10 than grade 11 learners at those levels. These results are consistent with and provide support for those reported in section 4.2.3 of Chapter 4, in which the grade 12 learners from SAS obtained a higher mean score on the TPGT than the grade 10 learners, whose mean score was in turn higher than that of the grade 11 learners (see Chart 4.1).

By comparing performance at each grade level in both NS and SAS, Chart 7.1 indicates that with the exception of grade 11, in which the learners from both subsamples obtained an equal mean score of 34% in the VHGT, learners from the SAS subsample generally outperformed their NS counterparts. As evident in Chart 7.1, the grade 10 learners from SAS obtained a higher mean score (39%) than their peers from NS whose mean score on the VHGT was 24%. Similarly, the mean score (45%) obtained by the grade 12 learners from SAS was higher than the mean score (38%) of their counterparts from the NS subsample.

These results imply that in grades 10 and 12, either there were more SAS learners at higher van Hiele levels than NS learners, or there were more learners from NS at lower van Hiele levels than SAS learners. Given the rather low means obtained by these learners for the VHGT, the latter case appears more probable. But whichever is the case, the results indicate that grade 10 and 12 learners from SAS demonstrated a better understanding of geometric ideas/concepts than their peers from NS, consistent with the results presented in section 4.2.3.

Further analysis was done to determine whether or not the differences in the mean scores of NS and SAS learners in Part A of the VHGT at each grade level just reported are significant. Table 7.2 represents the results of this analysis.

The results presented in Table 7.2 indicate that there was a statistically significant difference in the mean score of NS grade 10 learners and SAS grade 10 learners in favour of the latter at the 0.001 level (t = - 4.13, 46*df*, $p < 0.001$). This means that grade 10 learners from the SAS subsample performed significantly better than their grade 10 peers from NS. Another way of putting this is to say that there were far more NS grade 10 learners than SAS grade 10 learners at the lower van Hiele levels.

Table 7. 2 Grade level percentage mean scores in the VHGT

Grade	NS			SAS			*t-value*	*df*	*p-value*
	N	Mean	Std Dev.	N	Mean	Std Dev.			
10	24	24.17	11.86	24	38.75	12.62	- 4.13	46	0.0002
11	21	34.29	12.17	23	34.35	11.21	- 0.02	42	0.9860
12	23	37.61	11.27	24	44.79	14.63	- 1.88	45	0.0666

As evident in Table 7.2, although the mean score of SAS grade 11 learners was marginally higher than that of their grade 11 peers from NS, the difference in the mean scores of these two groups of learners was not significant (t = - 0.02, 42df, p > 0.05). This means that an approximately equal number of NS and SAS grade 11 learners were at similar van Hiele levels.

The *t-test* further revealed that there was no significant difference between the mean score of NS grade 12 learners and that of their SAS peers, even though the latter had a higher mean score than the former (t = - 1.88, 45df, p > 0.05). This indicates that grade 12 learners from SAS did not perform significantly better than their NS counterparts in the VHGT. Given that the mean score of the grade 12 learners from SAS was marginally higher than that of their NS peers, the results could also be interpreted to mean that at lower van Hiele levels, there was an approximately equal number of grade 12 learners from both subsamples, but that at higher van Hiele levels, there were more SAS grade 12 learners than NS grade 12 learners.

7.2.1.4 Mean scores in the VHGT of all learners by gender

Learners' performance in Part A of the VHGT was further analysed for a possible gender difference in the entire study sample. As Chart 7.2 indicates, there was a differential gender performance in the VHGT in favour of the male learners. As was the case with the TPGT (see Chapter 4, section 4.2.5.1), on average the male learners obtained higher scores in the VHGT, with a mean score of approximately 38% compared to the female learners' 34%.

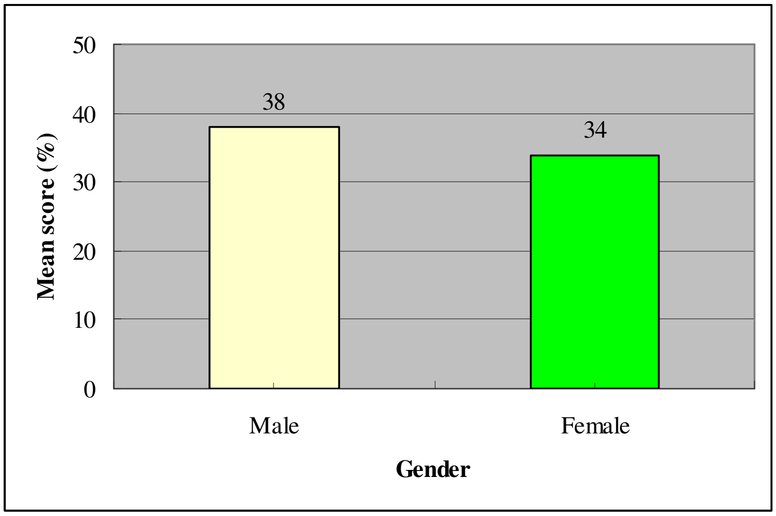


Chart 7. 2 Gender difference in mean score in the VHGT

A test of significance conducted indicated that the difference between the male and female mean scores on the VHGT was not statistically significant at the 0.05 level, as shown in Table 7.3.

Table 7. 3 Mean scores of learners in the VHGT by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	66	37.50	15.55	-1.49	137	0.1378
Female	73	34.04	11.66			

These results were found to be strongly aligned with those of Usiskin (1982) on two counts. First, the finding that the difference between the male and female mean scores was not significant matches Usiskin's (1982, p.75), that "no sex differences in [results of] 'fall' van Hiele levels" were evident among the American school children who wrote a comparable similar van Hiele geometry test. Secondly, the finding that the difference between the male and female mean scores of the learners favours the males is consistent with Usiskin's study (1982, pp.75–76), where in terms of results of 'spring' van Hiele levels (VHS), sex differences tended to "… favour the males".

7.2.1.5 Mean scores in the VHGT of NS subsample by gender

Analysis of the scores of learners from the NS subsample in Part A of the VHGT revealed that there was a differential gender performance in favour of the male learners. A *t-test*, however, indicated that the difference between the male mean score (33.47%) and the female mean score (30.00%) was not significant ($p > 0.05$). The results are as shown in Table 7.4.

Table 7. 4 Mean scores in the VHGT of NS learners by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	36	33.47	15.44	-1.10	66	0.2743
Female	32	30.00	9.42			

As with the entire study sample (section 7.2.1.4), Table 7.4 indicates that the male learners from the NS subsample did not perform significantly better than their female counterparts in the VHGT. Granted the hierarchical property of the van Hiele levels, the low mean scores obtained by NS male and female learners imply that the majority of these learners were at the lower van Hiele levels of geometric understanding. Furthermore, the male and female mean scores of the NS learners (Table 7.4) were lower than the male and female mean scores for the study sample (Table 7.3). This is an indication that NS male and female learners performed more poorly in the VHGT than their SAS peers, as is evident in sections 7.2.1.7 and 7.2.1.8.

7.2.1.6 Mean scores in the VHGT of SAS subsample by gender

As with the NS subsample, the male learners from the SAS subsample obtained a marginally higher mean score (42.33%) than their female peers whose mean score in Part A of the VHGT was 37.20%. The difference between these means was not however statistically significant ($p > 0.05$), as indicated in Table 7.5. This means that the SAS male learners, like their NS counterparts, did not perform significantly better than their female peers in the VHGT.

Table 7. 5 Mean scores in the VHGT of SAS learners by gender

Gender	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
Male	30	42.33	14.49	-1.61	69	0.1122
Female	41	37.20	12.35			

When one compares the male and female mean scores in Table 7.5 with the respective male and female mean scores in Table 7.3, one observes that the male and female mean scores of the SAS learners were higher than those of the study sample. In fact, the male and female mean scores of the SAS learners in the VHGT were both higher than the mean score for the entire study sample (see section 7.2.1.1). This further provides support for the point made earlier in section 7.2.1.2, that in general learners from the SAS subsample performed better than their counterparts from NS. These results also indicate that the SAS male and female learners performed better than their peers from NS in the VHGT.

In terms of the levels of geometric understanding, these mean scores (42.33% and 37.20%) obtained by the male and female learners from SAS are rather low. This means that the majority of the male and female learners, like their NS counterparts, were at the lower levels on the van Hiele scale of geometric conceptualization.

7.2.1.7 Mean scores in the VHGT by male gender

The mean scores obtained by the NS and SAS male learners for Part A of the VHGT were computed separately for learners in each of these subsamples. The mean score calculated for the male learners from NS was 33.47%, while the mean score calculated for the SAS male learners was 42.33%. A test of significance (Table 7.6) indicated that the difference in these means was statistically significant at the 0.05 level (t = - 2.39, 64df, p < 0.05). The meaning here is that the male learners from the NS subsample performed significantly more poorly in the VHGT than their peers from the SAS subsample. That is, there were more male learners from NS than SAS at some lower van Hiele levels (possibly levels 0 or 1).

It is worth remarking here that the SAS male learners obtained a higher mean score than the NS male learners not only in the VHGT, but also in the TPGT (section 4.2.5.4) and the GIST (section 5.2.2.4), which tends to suggest that on the whole, the male learners from the NS subsample had a poorer knowledge of school geometry than their peers from SAS.

Table 7. 6 Mean scores in the VHGT by male gender

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	36	33.47	15.44	-2.39	64	0.0199
SAS	30	42.33	14.49			

7.2.1.8 Mean scores in the VHGT by female gender

As with the male learners, SAS female learners obtained a higher mean score (37.20%) in Part A of the VHGT than NS female learners, who obtained a mean score of 30.00% in this test. A *t-test* analysis indicated that the difference between these means in favour of the SAS female learners was significant ($p < 0.01$), as shown in Table 7.7. This means that the female learners from SAS, like their male peers, performed significantly better than their counterparts from the NS subsample. Granted the hierarchical property of the van Hiele levels in relation to the low mean scores of these learners, these results indicate that the majority of the female learners from both the NS and SAS subsamples were at lower van Hiele levels, though it is indicated that more NS learners were at these (lower) levels than SAS learners.

Table 7. 7 Mean scores in the VHGT by female gender

School	N	Mean score	Std Dev.	*t-value*	*df*	*p-value*
NS	32	30.00	9.42	-2.73	71	0.0079
SAS	41	37.20	12.35			

7.2.2 Analysis of the VHGT according to the van Hiele levels

Part A of the VHGT consisted of 4 subtests, with each subtest being made up of 5 items testing learners' attainment of a specific van Hiele level (see Chapter 3, section 3.3.4.1.4, para.4). In the section under reference, it was stated that items 1–5, 6–10, 11–15 and 16–20 of the VHGT (subtests 1, 2, 3 and 4) tested learners' attainment of van Hiele levels 1, 2, 3 and 4, respectively. The analysis of learners' performance in the VHGT that is presented in the next three sections (sections 7.2.2.1 – 7.2.2.3) is based on the percentage mean scores of the participating learners on each of the four van Hiele levels examined in this study. The assumption here is that if the fixed sequential (or hierarchical) property of the van Hiele levels is valid, as argued in section 2.8.1 of Chapter 2, then a relationship of inverse proportion would be expected to exist between the van Hiele levels and learners' mean scores at these levels.

7.2.2.1 Mean scores of learners at each van Hiele level in the VHGT

The percentage mean score, rounded off to the nearest whole number, at each van Hiele level was calculated for the entire study sample. The results are as represented in Chart 7.3.

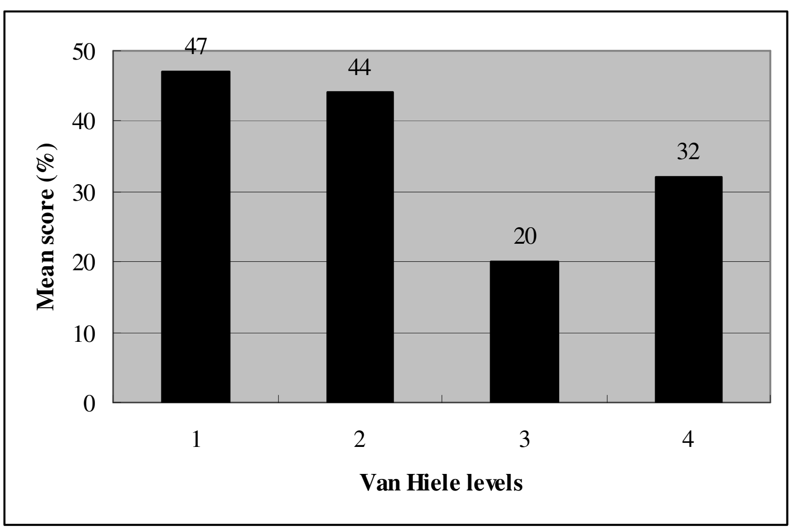


Chart 7. 3 Mean score of learners in the VHGT at each van Hiele level

As indicated in Chart 7.3, the mean score obtained by all the learners (n = 139) for the van Hiele level 1 subtest of the VHGT was 47%. For the van Hiele levels 2, 3 and 4 subtests, the mean scores obtained by these learners were, respectively, 44%, 20% and 32%. As evident in these mean scores, learners' performance in the VHGT decreases progressively at each successively higher van Hiele level between levels 1–3, which provides support for the hierarchical property of the van Hiele levels as stated in the preceding section. It turned out, however, that the level 4 subtest of the VHGT was easier for many learners in this study than the level 3 subtest. As a result, these learners obtained a higher mean score (32%) for the former than for the latter in which the mean score was 20%. This seems to be the general pattern for students' responses to questions typifying the van Hiele levels. Usiskin (1982, p.32), for example, reports a similar situation in which "some level 5 items turned out to be easier for students than items at lower levels . . . so, some students would satisfy the [classification] criterion at levels 1, 2 and 5…", but not at level 3 or 4.

The results from this study (and of course those of earlier studies, e.g. Usiskin, 1982; Atebe & Schafer, 2008) indicate that many learners experience difficulty with geometry problems typifying van Hiele level 3 reasoning. Evidence for this claim is that of all the five tasks that constituted the GIST (section 3.4.1.2), the majority of the learners in this study found Task 3 the most difficult. It was in this task, which explored learners' knowledge of class inclusions and relationships among shapes and their properties – typical of van Hiele level 3 tasks – that the learners in this study obtained their lowest mean score, as reported in section 5.3.6 of Chapter 5. The implication of these results (learners' lack of van Hiele level 3 thought processes) is that many of the learners have only a slight chance of succeeding at high school geometry. This is because empirical study has shown that level 3 thought is needed to begin formal geometry study at high school, and "if prior to the beginning of deductive geometry students have not had experiences leading to the development of level 3 thought processes, they may not benefit from a course in formal geometry" (Mayberry 1983, p.68).

7.2.2.2 Mean scores of NS and SAS learners at each van Hiele level in the VHGT

The percentage mean score, correct to the nearest whole number, at each van Hiele level in the VHGT was calculated separately for the NS and the SAS subsamples. Chart 7.4 provides a summary of the results.

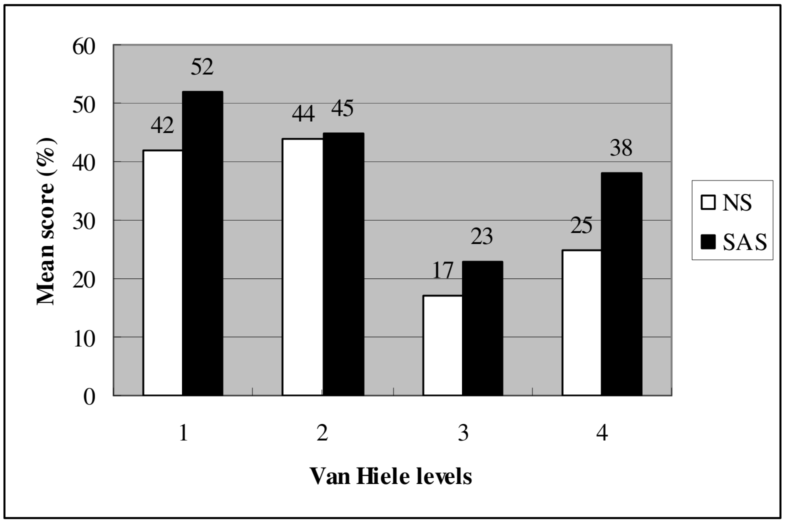


Chart 7. 4 School means at each van Hiele level in the VHGT

The mean scores calculated for learners from the NS subsample for the van Hiele levels 1, 2, 3 and 4 subtests of the VHGT were, respectively, 42%, 44%, 17% and 25% (Chart 7.4). The performance by the NS subsample in the VHGT as indicated by these mean scores was only partly consistent with the hierarchical attribute of the van Hiele levels, since the mean score of these learners at level 2 was marginally higher than their mean score for the van Hiele subtest at level 1.

According to Dina (as cited in Usiskin, 1982, p.6), it takes about “20 lessons” to raise students’ thought from level 1 to level 2, and “50 lessons” to get them from level 2 to level 3. Dina seems to claim that there exists a wider cognitive gap between van Hiele levels 2 and 3 than there is between levels 1 and 2. The data in this study (like that in earlier studies, e.g. Siyepu, 2005) tends to support this claim, since the difference in the mean scores of these learners between the levels 2 and 3 subtests was much wider than the difference in their mean scores between the levels 1 and 2 subtests, as is evident in Charts 7.3 and 7.4. This offers a possible explanation why some groups of learners may obtain a higher mean score for a level 2 subtest than a level 1 subtest, as was the case with the NS learners in the current study.

For learners from the SAS subsample, the mean scores obtained for the van Hiele levels 1, 2, 3 and 4 subtests of the VHGT were 52%, 45%, 23% and 38%, respectively. Consistent with those of the entire study sample (section 7.2.2.1), these mean scores provide support for the hierarchical property of the van Hiele levels, since they decrease progressively at each higher van Hiele level between levels 1–3. As with the NS subsample, the majority of the SAS learners found the level 4 subtest easier than the level 3 subtest and thus obtained a higher mean score (38%) for the former than for the latter.

On a comparative level, Chart 7.4 shows that at each van Hiele level, learners from the SAS subsample obtained higher mean scores in Part A of the VHGT than their counterparts from NS. What this means is that in relative terms, NS learners in this study had a poorer knowledge of geometric ideas than their peers from SAS.

Given that for the van Hiele 1 subtest learners needed only to recognise and name shapes, and that the level 2 subtest required the learners only to identify the properties of shapes, it would be fair to say that on the whole, the low mean scores of the learners at these and other levels can be interpreted as evidence of their weak knowledge of school geometry.

7.2.2.3 Grade level means of NS and SAS learners at each van Hiele level in the VHGT

Mean scores were calculated at each van Hiele level in Part A of the VHGT for each grade category of learners from NS and SAS. The results are represented in Table 7.8. The grade level analysis of learners' performance at each van Hiele level in the VHGT (Table 7.8) indicates that, with the exception of grade 10 learners from NS, there was a progressive decrease in the mean scores of these learners at each successively higher van Hiele level from 1 to 3 (though NS grade 11 learners had equal means at levels 1 and 2). This provides evidence of the hierarchical property of the van Hiele levels. For grade 12 learners from the NS subsample, for example, the mean score (56.52%) at level 1 was higher than their mean score (50.43%) at level 2, which was in turn higher than their mean score (19.13) at level 3.

Table 7. 8 Grade level means at each van Hiele level in the VHGT per school

	NS			SAS		
Van Hiele level	**Grade**			**Grade**		
	10 (N = 24)	11 (N = 21)	12 (N = 23)	10 (N = 24)	11 (N = 23)	12 (N = 24)
1	25.83	44.76	56.52	53.33	45.22	55.83
2	35.83	44.76	50.43	37.50	43.48	54.17
3	13.33	18.10	19.13	19.17	18.26	30.83
4	21.67	29.52	24.35	45.00	30.43	38.33

As evident in Table 7.8, the van Hiele level 3 subtest remains problematic for learners across all three grades in each of the participating schools. The learners in their respective grade categories obtained their lowest mean score for the van Hiele level 3 subtest. This implies that the majority of the learners in this study had difficulty in dealing with problems concerning class inclusions and the relationships between the properties of various simple geometric shapes, and between different shapes. This is consistent with Mayberry's (1983, p.65) research in which within her sample of 19 American pre-service elementary teachers, "class inclusions, relationships, and implications were not perceived by many of the students". These results also corroborate the findings reported in section 5.3.3 of Chapter 5, which indicate that knowledge of class inclusion was simply absent among the GIST sample.

Table 7.8 further indicates that, with the exception of grade 11 in which NS learners obtained a marginally higher mean score than SAS learners, at each grade level and for each van Hiele level subtest, learners from the SAS subsample outperformed their NS counterparts in the VHGT. This provides support for the point made earlier in section 7.2.1.2, that SAS learners in this study had a better understanding of geometric concepts as measured by the VHGT than their peers from NS.

7.2.3 Assignment of levels

Two classification methods were used to assign the learners to various van Hiele levels, according to the '3 of 5 correct' success criterion as explicated in section 3.4.1.4 of Chapter 3 (see second grading method in the section under reference). The two classification methods were adopted from Usiskin (1982) and they are as follows:

1. **Modified/classical van Hiele levels**. A learner's van Hiele level was defined to be the highest consecutive level (beginning from level 0) he or she has mastered. If, for example, a learner satisfies the criterion at levels 1, 2 and 4, he/she would be assigned to van Hiele level 2. Note that under the classical theory, a student's skipping of level 3 would not be condoned (see Usiskin, 1982).
2. **Forced van Hiele levels**. Having assumed the fixed sequence of the levels to be valid, Usiskin (1982) believes that a learner whose responses do not fit the sequence is demonstrating a random fit. He has therefore developed a method for assigning levels to such learners as follows: A student is assigned to level n if "(a) the student meets the criterion at levels n and n-1 but perhaps not at one of n-2 or n-3, or (b) the student meets the criterion at level n, all levels below n, but not at n+1 yet also meets the criterion at one higher level" (Usiskin, 1982, p.34). The forced van Hiele method of level assignment, as will be seen later on in this chapter, allows for many more students to be assigned to van Hiele levels than the modified or classical van Hiele method of classification.

7.2.3.1 Distribution of NS learners into van Hiele levels

Usiskin's (1982) schematic description of the 32 possible profiles of meeting or not meeting the criteria at the 5 van Hiele levels and the corresponding weighted sum and assignment of forced van Hiele levels were adapted to provide 16 profiles for this study. In Table 7.9, the number and percentage of NS learners at each forced van Hiele level (Forced VHL) and the modified (M)/classical (C) van Hiele level based on the '3 of 5' success criterion are given. An x in the table means that the learner has satisfied the criterion at that level

Table 7. 9 Schematic description and number of NS learners at each level of forced van Hiele assignment

Forced VHL	Weighted Sum	Level 1	2	3	4	3 of 5 Criterion	Total (%) at level
Forced VHL 0 =	0 C0, M0					23	
	2		x			11	
	4			x		2	
	8				x	0	36 (53)
Forced VHL 1 =	1 C1, M1	x				12	
	5	x		x		0	
	9	x			x	3	15 (22)
Forced VHL 2 =	3 C2, M2	x	x			14	
	11	x	x		x	2	16 (24)
Forced VHL 3 =	6		x	x		1	
	7 C3, M3	x	x	x		0	1 (1)
Forced VHL 4 =	13	x		x	x	0	
	14		x	x	x	0	
	15 C4, M4	x	x	x		0	0 (0)
Forced No fit =	10		x		x	0	
	12			x	x	0	0 (0)
Total							68 (100)

Table 7.9 indicates that of the 68 learners from the NS subsample who wrote the VHGT, 36 (53%) were at the pre-recognition level (i.e. level 0) of geometric reasoning as measured by the van Hiele geometric scale, while 15 (22%), 16 (24%) and 1 (1%) were, respectively, at levels 1, 2 and 3. None of the learners from NS was at level 4 in the van Hiele hierarchies of geometric conceptualization. As can be seen in the table, of the 36 learners who were at level 0, 23 did not meet the '3 of 5 correct' success criterion at any one van Hiele level, while 11 learners met the criterion at level 2, 2 others at level 3 and none at level 4. These were learners whose weighted sum scores were 0, 2, 4 and 8, respectively, as is evident in Table 7.9. Similar interpretations hold true for learners who were at levels 1 through 4 in Table 7.9. There were no learners from the NS subsample whose response pattern did not 'fit' the forced van Hiele level classification using the '3 of 5 correct' success criterion. That is to say, as Table 7.9 indicates, all the NS learners were classifiable in terms of van Hiele levels.

7.2.3.2 Distribution of SAS learners into van Hiele levels

As with the NS subsample, Usiskin's (1982) schematic description of the 32 possible profiles of meeting or not meeting the classification criteria at the 5 van Hiele levels and the corresponding weighted sum and assignment of forced van Hiele levels were adapted to yield 16 profiles for the SAS subsample in this study. Table 7.10 shows the number and percentage of SAS learners at each forced van Hiele level (Forced VHL) and modified (M)/classical (C) van Hiele level according to the '3 of 5 correct' success criterion. An x in the table indicates that the learner has met the criterion at that level.

Table 7. 10 Schematic description and number of SAS learners at each level of forced van Hiele assignment

Forced VHL	**Weighted Sum**	**Level**				**3 of 5 Criterion**	**Total (%) at level**
		1	**2**	**3**	**4**		
Forced VHL 0 =	0 C0, M0					20	
	2		x			5	
	4			x		0	
	8				x	4	29 (41)
Forced VHL 1 =	1 C1, M1	x				13	
	5	x		x		1	
	9	x			x	2	16 (22)
Forced VHL 2 =	3 C2, M2	x	x			10	
	11	x	x		x	7	17 (24)
Forced VHL 3 =	6		x	x		2	
	7 C3, M3	x	x	x		0	2 (3)
Forced VHL 4 =	13	x		x	x	1	
	14		x	x	x	1	
	15 C4, M4	x	x	x		2	4 (6)
Forced No fit =	10		x		x	2	
	12			x	x	1	3 (4)
Total							71 (100)

As evident in Table 7.10, of the 71 learners from the SAS subsample who wrote the VHGT, 29 (41%) were at the pre-recognition level (i.e. level 0) of geometric reasoning, while 16 (22%), 17 (24%), 2 (3%) and 4 (6%) were, respectively, at van

Hiele levels 1, 2, 3, and 4. Of the 29 learners at level 0, 20 did not meet the '3 of 5' success criterion at any one van Hiele level with a weighted sum score of 0; 5 met the criterion at level 2 with a weighted sum score of 2; none met the criterion at level 3 and; 4 met it at level 4 with a weighted sum score of 8. The number of learners at levels 1 through 4 could similarly be interpreted. There were 3 learners from the SAS subsample whose responses did not 'fit' the forced van Hiele level classification. These learners had weighted sum scores of 10 and 12, as can be seen in Table 7.10.

7.2.3.3 Number of learners at each modified and forced van Hiele level

Part A of the VHGT was further analysed in order to determine the number and percentage of NS and SAS learners at each van Hiele level according to the modified van Hiele level classification and the forced van Hiele level assignment methods. The '3 of 5 correct' success criterion was used in both classification methods. The forced van Hiele figures are the ones in parenthesis in Table 7.11.

Table 7. 11 Number and percentage of learners at each modified and forced van Hiele level

NS			**SAS**		
Level	N	%	Level	N	%
0	23 (36)	34 (53)	0	20 (29)	28 (41)
1	12 (15)	18 (22)	1	13 (16)	18 (22)
2	14 (16)	20 (24)	2	10 (17)	14 (24)
3	0 (1)	0 (1)	3	0 (2)	0 (3)
4	0 (0)	0 (0)	4	2 (4)	3 (6)
Total fitting	49 (68)	72 (100)	Total fitting	45 (68)	63 (96)
No fit	19 (0)	28 (0)	No fit	26 (3)	37 (4)
Totals	**68 (68)**	**100 (100)**	**Totals**	**71 (71)**	**100 (100)**

As stated at the beginning of this section and as shown in Table 7.11 the two classification methods – modified van Hiele and forced van Hiele level assignments – were used to assign learners from both the NS and the SAS subsamples to different van Hiele levels using the '3 of 5 correct' success criterion in each case. The Table indicates that using the modified van Hiele level assignment scheme, a total of 49 learners (72%) from the NS subsample were assignable to various van Hiele levels, while 19 (28%) of them did not 'fit' this classification scheme. Of the 49 learners that 'fitted' the modified level assignment scheme, 23 (34%), 12 (18%) and 14 (20%)

were at van Hiele levels 0, 1 and 2, respectively. There were no learners from NS at van Hiele levels 3 and 4 under this classification scheme.

When the forced van Hiele level method was used, all 68 learners (100%) from the NS subsample who wrote the VHGT were assignable to van Hiele levels. The number and percentage of NS learners at each van Hiele level also increased under the forced van Hiele level assignment scheme, with 36 (53%), 15 (22%), 16 (24%) and 1 (1%) of the learners at van Hiele levels 0, 1, 2 and 3, respectively. Thus Table 7.11 indicates that forced van Hiele level assignment can be a very useful (though less strict) determination scheme because it enables almost every learner to be assigned to a van Hiele level. This is consistent with Usiskin's study (1982), in which 99.7% of American school children were assigned a level. Consequently, unless otherwise stated, learners' assigned van Hiele levels in this study were established according to the forced van Hiele level determination scheme.

As far as the SAS subsample is concerned, Table 7.11 indicates that 45 learners (63%) were assignable to van Hiele levels under the modified van Hiele level assignment scheme, while responses from 26 (37%) of them did not 'fit' this classification scheme. Of the 45 learners that 'fitted' the scheme, 20 (28%), 13 (18%), 10 (14%) and 2 (3%) were, respectively, at levels 0, 1, 2 and 4 of the van Hiele scale of geometric conceptualization. As was the case with the NS subsample, there was no learner in the SAS subsample that met the '3 of 5 correct' success criterion at level 3 in the VHGT. This revelation reinforces the claim made earlier in section 7.2.2.1 that the majority of the learners in this study tend to have difficulty in dealing with geometry problems that typify van Hiele level 3 reasoning.

As with the NS subsample, when the forced van Hiele level assignment scheme was introduced, a much greater number of SAS learners (68 or 96% of them) proved assignable to van Hiele levels than was possible under the modified theory (Table 7.11). The number of SAS learners at each van Hiele level assignment also increased under the forced van Hiele level assignment scheme. Using this scheme, as Table 7.11 indicates, there were 29 (41%), 16 (22%), 17 (24%), 2 (3%) and 4 (6%) SAS learners at van Hiele levels 0, 1, 2, 3 and 4, respectively.

On a comparative level, it is evident in Table 7.11 (figures in parenthesis) that the percentage of NS learners (53%) who were at van Hiele level 0 was higher than that of SAS learners (41%). What this implies in relative terms is that there were more learners from the SAS subsample who were at higher van Hiele levels and hence had a better knowledge of school geometry than their peers from the NS subsample. Indeed, Table 7.11 clearly shows that the percentage of SAS learners at van Hiele levels 3 (3%) and 4 (6%) was higher than that of NS learners at van Hiele level 3 (1%) and level 4 (0%). There were, however, equal but low percentages of learners from both the NS and the SAS subsamples at van Hiele level 1 (22%) and level 2 (24%).

To summarize, Table 7.11 indicates that a large majority of the learners from both the NS and the SAS subsamples were at van Hiele level 0. The number of learners at levels 1 and 2 was very small, and there were almost none at levels 3 and 4.

The overall interpretation of these results is that the majority of the learners who wrote the VHGT were at level 0 on the van Hiele geometric scale, which means that their knowledge of school geometry was poor. The near absence of learners at levels 3 and 4 implies that most of them did not possess the experience necessary for the formal study of high school geometry. These results offer possible explanations for the participants' poor performance in the TPGT, GIST and CPGT, as discussed in chapters 4, 5 and 6, respectively.

7.2.3.4 Grade level distribution of learners into van Hiele levels

Part A of the VHGT was further analysed separately for learners in each of the three grades of the participating schools in order to determine the distribution of these learners into the van Hiele levels. The forced van Hiele level determination scheme was used to assign the learners into van Hiele levels according to the '3 of 5 correct' success criterion. The results are summarized in Table 7.12.

Table 7. 12 Number and percentage of NS and SAS learners at each van Hiele level per grade

Level	Grade 10				Grade 11				Grade 12			
	NS		SAS		NS		SAS		NS		SAS	
	N	%	N	%	N	%	N	%	N	%	N	%
0	18	75	9	38	10	47	12	52	8	35	8	33
1	4	17	8	33	5	24	4	17	6	26	4	17
2	2	8	4	17	5	24	5	22	9	39	8	33
3	0	0	0	0	1	5	0	0	0	0	2	8
4	0	0	2	8	0	0	1	4	0	0	1	4
Total fit	24	100	23	96	21	100	22	96	23	100	23	96
No fit	0	0	1	4	0	0	1	4	0	0	1	4
Totals	**24**	**100**	**24**	**100**	**21**	**100**	**23**	**100**	**23**	**100**	**24**	**100**

The point made earlier in section 7.2.3.3, that the forced van Hiele level determination scheme allows nearly all learners to be assigned to levels, is again demonstrated in Table 7.12. As the table clearly indicates, in each of grades 10, 11 and 12, 100% of the NS learners and 96% of the SAS learners were assignable to van Hiele levels. Only 1 (4%) of the learners did not 'fit' the forced van Hiele level assignment scheme in each grade in the SAS subsample. The consistency with which learners were assigned van Hiele levels across the three grades in each of the participating schools makes the forced van Hiele level assignment scheme a very useful one indeed.

Although the majority of the learners in each grade from both schools were at van Hiele level 0, the proportion of NS grade 10 learners at level 0 — 75% — perhaps deserves special mention. This percentage represents three-quarters of the grade 10 learners from the NS subsample who wrote the VHGT. The very large number of NS grade 10 learners at van Hiele level 0 explains the very low mean score obtained by these learners in the VHGT, as reported earlier in section 7.2.2.3.

Another important point about Table 7.12 is that with the exception of grade 11, there were higher percentages of NS learners at level 0 than SAS learners. This indicates that there were more learners in the NS subsample who had a weak knowledge of geometric concepts than there were in the SAS subsample. This offers a plausible explanation of why SAS learners outperformed their NS counterparts in all the tests (TPGT, GIST and CPGT) used in this study. It is important to point out, however, that the percentage of NS grade 12 learners at van Hiele levels 1 (26%) and 2 (39%) was higher than that of the SAS learners at these levels, even though no NS grade 12 learner was at van Hiele levels 3 and 4 (see Table 7.12).

As is evident in Table 7.12, learners across the grade levels in each subsample had difficulty with van Hiele level 3 geometry problems, with no grade 10 learners meeting the '3 of 5 correct' success criterion at this level. From the NS subsample, only 1 grade 11 learner was at van Hiele level 3, while from the SAS subsample, only 2 grade 12 learners were at this level. Hence geometry problems typifying van Hiele level 3 reasoning tended to be problematic for nearly all the learners in this study.

7.2.3.5 Analysis of items 8, 11, 12 and 17 of the VHGT

It was stated in Chapter 3 (section 3.3.4.1.4, para.3) that items 8, 11, 12 and 17 of Part A of the VHGT were adopted from Usiskin's (1982) van Hiele geometry test in which they occurred as items 10, 15, 14 and 20, respectively. The purpose of adopting these items was explained in the section referred to. For ease of reference in this section, Usiskin's (1982, p.155) sample for the "van Hiele Geometry Test", which was made up of a total of 2699 U.S. high school learners, was designated VHS sample, while the sample in the present study was referred to as the VHGT sample. It is worth pointing out that the analysis that follows does not assume that these samples are necessarily comparable, beyond the simple facts that the majority of the learners in both samples were roughly within the same age bracket (14–17 years for the VHS sample and 15–19 years for the VHGT sample), and that many of them had been taught high school geometry at the senior phase of secondary education for at least one year (see Usiskin, 1982).

Usiskin (1982) administered his van Hiele geometry test twice to his sample of 2361 American high school learners. At the beginning of the school year, within the "first week of school", he administered what he called 'Fall van Hiele geometry test' (VHF), and towards the end of the school year (about 3 to 5 weeks before the end of school), he administered an identical 'Spring van Hiele geometry test' (VHS) to his sampled learners. Since the VHGT, like the three other tests used in this study, was administered to the learners just a few weeks before the end of the school year, only the scores of Usiskin's (1982) VHS (and not VHF) sample were compared with those of the samples tested in the present study. Since the options for each of the four items were juggled in the present study, the item analysis, in terms of the percentage of learners who chose a particular option (A, B, C, D or E) as presented in Table 7.13, matches the questions set for this study alone. The VHS figures represent the percentage of learners who chose an option with the same wording but not necessarily with the same letter option (A, B, C, D or E) as the VHGT. For the contents of these four items, reference should be made to Appendix 6.A.1, p.84. The percentage of learners who made the correct choice appears underlined and in bold in Table 7.13.

Table 7. 13 Item analysis for items 8, 11, 12 and 17 in the VHGT

Item	**Choice**	**Percentage with Choice**		
		VHS sample	VHGT sample NS	VHGT sample SAS
8	A	9	9	8
	B	6	20	10
	C	**58**	**34**	**38**
	D	9	11	23
	E	16	25	21
11	A	7	11	11
	B	**50**	**13**	**21**
	C	7	13	14
	D	10	25	13
	E	26	37	41
12	A	15	22	35
	B	16	9	10
	C	8	13	11
	D	**34**	**7**	**13**
	E	39	49	31
17	**A**	**44**	**22**	**34**
	B	11	24	18
	C	5	12	20
	D	3	16	10
	E	35	20	18

It is immediately visible in Table 7.13 that for each of the four items, the percentage of learners who made the correct choice in the VHS sample was higher than that of the VHGT (SAS) sample, which was in turn higher than that of the VHGT (NS) sample. This means that in respect of all the four items, learners in this study performed worse than their American peers.

The percentage of correct answers as presented in Table 7.13 for both the VHS and the VHGT samples tends to be consistent with the hierarchical property of the van Hiele levels. The percentage of learners who correctly answered the van Hiele level 2 question (item 8), the level 3 questions (items 11 and 12) and the level 4 question (item 17) decreases between levels 2 and 3 in each of the samples. In particular, fewer learners from each of the samples managed to answer item 12 – a van Hiele level 3 question – correctly. This is an indication that van Hiele level 3 geometry problems are difficult not only for African (Nigerian and South African) school children, but also for American high school learners. For this particular item, reproduced hereunder, it is important to note that a high percentage of the learners from each of the samples chose option E as the correct answer.

Item 12 Which is **true**?

A. All properties of rectangles are properties of all parallelograms.
B. All properties of squares are properties of all rectangles.
C. All properties of squares are properties of all parallelograms.
D. All properties of rectangles are properties of all squares.
E. None of (A) – (D) is true.

The students' responses to these four items, and indeed to all the items in the VHGT that exemplify van Hiele level 3 questions, tend to indicate that learners generally have difficulty with the ordering of the properties of simple geometric shapes, consistent with the results reported earlier for Task 3 of the GIST (see section 5.3.3).

7.2.3.6 Item analysis of the VHGT

A comprehensive item analysis of the VHGT for each of the participating schools and at each grade level is contained in Appendix 6.D.1–8, pp.121–128. Also, individual learners' performance in the VHGT is presented in Appendix 6.C.1–6, pp.115–120. In this section, learners' performance in the VHGT is analysed only at school level and with reference to a few selected items.

In both the NS and the SAS subsamples, the highest percentage of learners correctly answering any item in the VHGT occurred in the van Hiele level 1 subtest. In the NS subsample this was 65%, while in the SAS subsample it was 73%. This was the response to item 1 of subtest 1. There were 26% of NS learners and 23% of SAS learners who thought that a long thin triangle was not a triangle (item 1). The lowest percentage of learners from both subsamples to answer correctly any item in the VHGT was recorded in respect of item 12 of the van Hiele level 3 subtest. For the NS subsample, this percentage was 7%, for the SAS subsample, 13%.

For the NS and the SAS subsamples, 29% and 14% of the learners, respectively, thought that a long narrow rectangle was not a rectangle (item 2). There were respectively 56% and 41% of the NS and the SAS learners who thought none of a square, a rhombus and a rectangle could be called a parallelogram (item 5). This reinforces the claim that nearly all the learners in this study could not perceive class inclusions of shapes, as reported earlier in section 5.3.3 of Chapter 5.

No item in the VHGT that required reasoning to a conclusion (items 14, 15, 17 and 20) was correctly answered by more than 40% of the learners in either of the two subsamples. This implies that the majority of the learners in this study were not fully ready for a deductive study of school geometry.

The results reported in this section are generally consistent with those of Usiskin (1982, p.70). For example, Usiskin observes that although many American students were able to identify rectangles, "over two-thirds think a square is not a rectangle;" he also notes that "no item dealing with reasoning to a conclusion…was correctly answered by more than half the students in the fall or two-thirds of the students in the spring". In the next section, results of students' performance in Part B of the VHGT are presented.

7.3 Learners' performance in Part B of the VHGT

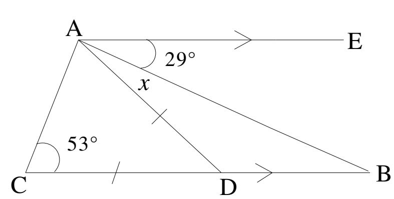
It was stated in section 3.3.3.1.4 of Chapter 3 that Part B of the VHGT consisted of 3 items (see the section under reference for details of each of these items) and that it was grade specific. The general criteria for grading this section of the VHGT were articulated in section 3.4.1.4. Mean scores were not used to describe learners' performance in Part B of the VHGT owing to the very poor performance of many of the learners. Instead, it was considered more useful and informative to describe learners' performance in terms of the percentage of them obtaining certain percentage scores for each item in this test. Due to the grade-level specificity of the questions, the analysis that is given here is according to the grade levels in each of the participating schools. Performance with regard to each of the 3 items was analysed separately in each grade, as all examined different though interrelated abilities on the part of learners (ability to solve geometric riders, ability to supply reasons for steps in proofs in geometry, and ability to write proofs in geometry).

7.3.1 Analysis of grade 10 learners' performance in Part B of the VHGT

Performance on item 1 and SVHGT: Of the 4 points maximum obtainable for this item, no learner in either subsample obtained even a single point. It was disappointing that despite the multipath approaches to this problem as illustrated in section 3.3.4.1.4 (see Figures 3.6 and 3.7), none of the learners offered any meaningful solution that could allow for the assignment of even 1 point. The responses of the learners generally indicated that the majority of them could not deal with geometry problems that require two or more lines of reasoning to get to the answer. There were some common patterns in the learners' responses to this item (reproduced hereunder for ease of reference) that deserve comment.

Item 1

In the diagram, AE // CB and $|AD| = |CD|$. $\angle$ BAE = 29° and $\angle$ACD = 53°. Find the value of $\boldsymbol{x}$. You are to show your workings, giving reasons for each step.



Of the 24 learners from NS who answered this question, 29% of them simply added the given values in the diagram to obtain the value of x. Five others, representing 21% of the learners, simply subtracted the smaller from the bigger of the two given values in the diagram to obtain the value of x. Thus these learners carried out mathematical operations that involved only one line of reasoning. The rest gave answers that were patently meaningless.

Although SAS learners, like their NS counterparts, provided incorrect answers to this question, the majority of them were able to perform mathematical operations requiring two lines of reasoning. Of the 24 grade 10 learners from the SAS subsample that answered this question, 25% of them added x to the given values in the diagram, equated their answer to 180° and then solved for x. These learners appear to be wrongly applying the notion that the angle sum of a triangle is 180°. There were 10 other learners representing 42% of the group who, like their NS counterparts, simply added the given values in the diagram to obtain the value of x. For another 8% of the learners, the smaller of the two given values in the diagram was subtracted from the bigger to get the value of x. There were yet another 13% of the learners who added the given values in the diagram and deducted the sum from 180° to arrive at the value of x.

Performance on the SVHGT: The reason for administering the SVHGT (Supplementary van Hiele Geometry test) to the grade 10 learners and the composition of this test were explained in section 3.3.4.1.4 of Chapter 3. As was stated in the section under reference, question 1 of the SVHGT required only one line of reasoning to get to the answer, while question 2 required two lines of reasoning. Learners' responses to this test (reproduced in Figure 7.1 for ease of reference), indicated that many learners from the SAS subsample but just a few from the NS subsample could handle triangle problems requiring only one line of reasoning. A triangle problem requiring two lines of reasoning to reach the answer tended to be difficult for the majority of the learners in both subsamples.

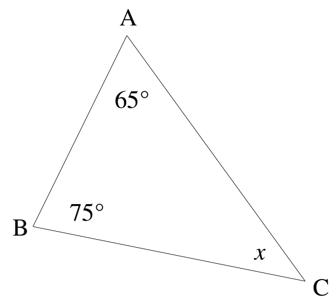
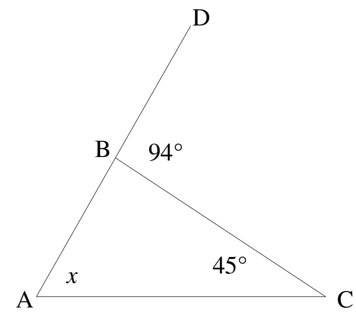
Question 1. Find the value of x in ΔABC drawn below. Give a reason for each step in your answer. 	**Question 2.** Find the value of x in the diagram below. Give a reason for each step in your answer. 

Figure 7. 1 Items in the SVHGT for the grade 10 learners

Only 2 learners representing 8% of the grade 10 learners from NS achieved the maximum score of 4 points for the SVHGT (2 points for each question). Two other learners obtained 1 point (or 25%) each for question 2 of the SVHGT. The rest of the learners in this group scored zero points for the SVHGT. The results here were found to be consistent with those of Clements and Battista (1992), as reported in section 2.7.3.8 of Chapter 2.

It was evident from the responses of these learners that many of them not only had a weak conceptual knowledge of geometry, but also lacked problem-solving ability in this learning area. Just as they did with item 1 of Part B of the VHGT, the majority of the learners from NS simply added the given values in each of the two diagrams as the respective values of x.

As with the NS subsample, only 2 (or 8% of the) learners from the SAS subsample obtained 100% on the SVHGT, i.e. answered both questions correctly. There were, however, many more learners ─ 16 of them, or 67% ─ from the SAS subsample who correctly answered question 1 of the SVHGT. Two other learners scored 1 point each for question 1 of the SVHGT. These were learners who made subtraction errors while trying to solve for x. The responses of 4 other learners were simply meaningless. These results indicate that the majority of the SAS grade 10 learners were comfortable with a triangle problem that required only one line of reasoning to get to the answer, but like their NS counterparts, had difficulty in dealing with a triangle problem that required two lines of reasoning.

Performance on item 2: As stated in section 3.3.4.1.4 of Chapter 3, item 2 of Part B of the VHGT required the learners to fill in statements or reasons in an almost completed geometry proof (see Appendix 6.A.1, p.84). As with item 1, learners' performance on this item was very poor. Of the maximum of 4 points for this item, only 1 learner from the NS subsample managed to obtain 1 point, the rest scoring zero points. For the SAS subsample, 1 learner obtained 2 points, and 2 other learners scored 1 point each on this item. The rest, like their NS counterparts, obtained zero points.

Given that the ability to supply reasons for steps in a proof belongs to level 4 in the van Hiele hierarchy of geometric thinking levels (see section 2.8), and in view of the findings in section 7.2.3.4 that the majority of the learners in this study were at levels 0 and 1, then the result here should come as no surprise. These learners were simply not ready for the study of geometry that uses deductive approaches.

Performance on item 3: This item required the learners to write a complete proof of the theorem that states that the sum of the angles of a triangle is 180° (see Appendix 6.A.1, p.84). All the learners failed to do this, with only 1 NS learner out of both the NS and the SAS subsamples managing to obtain 1 point out of the 4 points maximum.

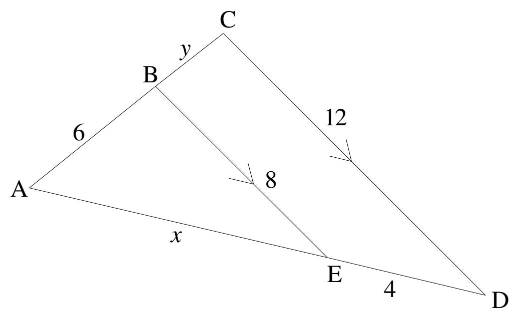
When learners' performance on this item is compared with their performance on Investigation 1 of the CPGT as reported in section 6.3.1 of Chapter 6, it is evident that although the majority of the grade 10 learners could determine (through investigation) that the angle sum of a triangle is 180°, only a few of them could generalize their observation as a conjecture, and fewer still (if any) could prove their conjecture. What is disturbing in the poor performance of these learners is that teachers' questionnaires (Appendix 2, p.11) indicated that these learners had been taught not only the proof of this theorem, but also its application to solving geometric riders.

7.3.2 Analysis of grade 11 learners' performance in Part B of the VHGT

Performance on item 1: As stated in section 3.3.4.1.4 of Chapter 3, item 1 (reproduced hereunder for ease of reference) of Part B of the VHGT for grade 11 learners only required the learners to determine missing values (*x* and *y*) in a triangle using their knowledge of proportion/similarity.

Item 1 for the grade 11 learners

Find the values of *x* and *y* in the diagram



Although teachers' questionnaires indicated that these learners had been taught the concepts of proportion and similarity, learners' responses to this item did not reflect this claim. Of the 4 points maximum obtainable on this item, all 23 learners from the SAS subsample who wrote this test scored zero points. Of the 21 learners from NS who wrote the test, only 1 (or 5%) scored 2 points while another 3 (14%) scored 1 point each. The rest, like their SAS counterparts, achieved zero points.

Learners' written responses to this item did not show any clear pattern of logical thinking. It was indeed difficult to understand the reasoning that underlay the responses of almost all of them. The majority, probably impulsively, added together the given numerical values in the diagram as the value of the unknown, as is illustrated in Lamani's response to this item. Lamani did not attempt to find the value of *y*.

Lamani's response to item 1:

$BE \parallel CD$ [parallel]

$AB = BE$ [$\angle$ of Δ]

$6 + 8 = x$ [$\angle$ of the Δ]

$x = 8 + 6$

$x = 14$

Clearly, responses such as Lamani's are indicative of the fact that these learners had a problem with solving geometric riders that require a knowledge of proportion. These learners did not seem to understand the concepts of proportion and similarity in geometry, which runs counter to curricular expectations of them.

Performance on item 2: This item, as with the grade 10 learners, required the grade 11 learners to fill in statements or reasons in an almost completed geometrical proof (see Appendix 6.A.2, p.94). In relative terms, learners from both the NS and the SAS subsamples did better on this item than they did on item 1. Out of a maximum of 4 points, 2 learners from the NS subsample obtained 3 points apiece for this item. Three other learners from the NS sample scored 2 points, and another two learners scored one point each. The rest of this group of learners obtained zero points.

Among the grade 11 learners from SAS, one managed to obtain 2 points for this item, while 3 others each scored 1 point. The rest obtained zero points. Supplying reasons for steps in a geometric proof appears to be marginally easier for these learners than solving geometric riders entirely on their own.

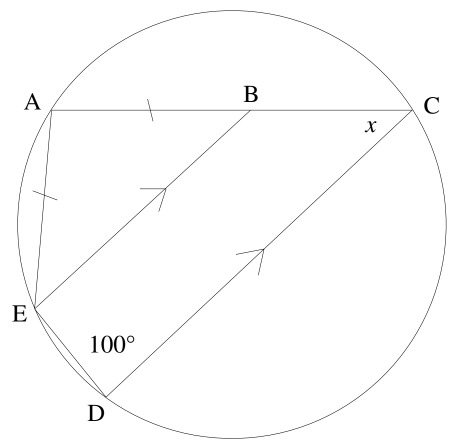
Performance on item 3: This item required the learners to write a complete proof of the theorem that states that a line (drawn) parallel to one side of a triangle divides the other two sides proportionally. This item was generally difficult for virtually all the learners from both subsamples. Only 1 learner from the NS subsample managed 2 points out of the maximum of 4 points. This learner actually showed some level of consistency in performance across all three items of Part B of the VHGT, scoring 2 points for item 1, 3 points for item 2 and 2 points for item 3. The rest of the learners in this group achieved zero points for item 3. Every grade 11 learner in the SAS subsample scored zero points on item 3 of Part B of the VHGT. It was evident from their responses that the majority of the grade 11 learners in this study had problems with the writing of proofs in geometry.

7.3.3 Analysis of grade 12 learners' performance in Part B of the VHGT

Performance on item 1: This item (reproduced hereunder) required the grade 12 learners to determine the value of an unknown angle (x) in a circle geometry problem.

Item 1 for the grade 12 learners

In the diagram, ACDE is a circle with EB // DC and $|AB| = |AE|$. Find the value of x. You are required to show your workings, giving reasons for your steps.



As elementary as this item seems, no learner from the NS subsample managed to score any points at all. This shows that these learners had difficulty dealing with riders in circle geometry.

The performance by the SAS learners on this item was particularly impressive, not in terms of the number of learners doing it correctly, but in terms of the number of points obtained by those who solved it correctly. Five learners from this group obtained the maximum of 4 points each on this item. The ability to solve riders in circle geometry was evident in the responses of these 5 learners. Two of their responses (those of Asanda and Xola) are as shown in Figure 7.2. The rest of the group, like their NS counterparts, scored zero points.

Asanda	**Xola**
In cyclic quad ACDE: $\hat{A}+\hat{D}$ = 180° (opp.∠'s of cycl. quad suppl) ∴ $\hat{A}$ = 180° - 100° = 80° In Δ ABE: $\hat{A}$ + $\hat{B}+\hat{E}$ = 180° (int. ∠'s of Δ) ∴ $\hat{B}+\hat{E}$ = 180° - 80° = 100° ∴ $\hat{B}$ = $\hat{E}$ = 50° (base ∠'s of isosceles Δ) ∴ x = $\hat{B}$ = 50° (corr ∠'s are equal, BE // CD)	$E\hat{A}B = 180^{\circ}$ (opp.angles a cycl quad are suppl) In Δ AEB; $\hat{E}$ = $\hat{B}$ (base ∠'s of an isosceles Δ) ∴ 80° + $\hat{B}+\hat{C}$ = 180° (∠'s of Δ) ∴ 2 $\hat{B}$ = 180° - 80° ($\hat{B}=\hat{E}$) ∴ $\hat{B}$ = 50° ∴ x = 50° (corr ∠'s are equal)

Figure 7. 2 Exemplifying grade 12 learners' solution to item 1 of Part B of the VHGT

Performance on item 2: As with grades 10 and 11, this item required the grade 12 learners to fill in statements or reasons in an almost completed geometrical proof (see Appendix 6.A.3, p.104). Many of the grade 12 learners from both subsamples attained partial success on this item. Out of the 23 learners from NS that wrote the test, 2 (7%) obtained 3 points out of the maximum of 4 points for this item. There were 10 learners (43%) who scored 2 points, and 3 (13%) who scored 1 point. The remainder scored zero points.

All 24 grade 12 learners from the SAS subsample wrote this test and 2 (8%) of them scored the maximum of 4 points on this item. Two (8%) learners obtained 3 points and 2 (8%) obtained 2 points, while 9 (38%) managed to score 1 point. The rest, like their NS counterparts, got zero points. As was the case with the grade 11 learners, supplying reasons for steps in a geometric proof seemed to be easier for the majority of the grade 12 learners than having to solve riders in circle geometry.

Performance on item 3: This item required the grade 12 learners to write a complete proof of the theorem that states that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. The grade 12 learners from the NS subsample performed poorly on this item. Only 1 of the 23 managed to score 1 point out of a maximum of 4, the rest getting zero points. The results here are in line with those reported in section 6.3.3 of Chapter 6, where formulating conjectures was found to be generally difficult for the majority of the NS learners. That is, while many of these learners could not formulate conjectures, a still greater number of them could not prove their conjectures.

The performance on this item was slightly different for the SAS learners. Five of the SAS learners were able to obtain the 4 points maximum for this item. With one exception, these were the same five learners who scored the maximum of 4 points for item 1 of Part B of the VHGT, as reported earlier in this section. These learners therefore demonstrated a fairly consistent knowledge of circle geometry. One other learner from the SAS subsample obtained 1 point for this item. The rest of the learners got zero points.

That learners from the SAS subsample did better on this item than their NS counterparts seems to be consistent with the results reported in section 6.3.3, in which for the CPGT, far more SAS learners were able to formulate conjectures than learners from the NS subsample. These results seem to suggest that learners who are successful with conjectures in geometry are likely to do better in proof writing than their peers who are unsuccessful with conjectures.

7.4 Chapter conclusion

This chapter has focused on the analysis of learners' performance in the VHGT. Their percentage mean scores in Part A of the VHGT was examined before they were assigned to van Hiele levels. The chapter concluded with analyses of learners' performance in Part B of the VHGT, which examined their problem-solving and proof-writing abilities in school geometry. Among the major findings are the following:

- An overall low percentage mean score of 35.68% obtained by the learners in Part A of the VHGT was considered to be evidence that the majority of the learners in this study were at a low van Hiele geometric thinking level, possibly level 0 or 1. The result reported here was found to be consistent with that of Usiskin (1982), in which the American high school learners from each of his sample of 13 schools achieved very low mean scores for a similar van Hiele geometry test.

- At the participating schools level, the mean score of 39.37% obtained by learners from the SAS subsample for Part A of the VHGT was significantly higher than that of their peers from NS, which was 31.84% at the 0.005 level. The conclusion was reached that in this study, there were more NS learners than SAS learners at the lower van Hiele levels (level 0 and 1); or conversely, that there were fewer NS learners at the higher van Hiele levels (levels 3 and 4) than SAS learners.

- At grade level, the NS grade 12 learners' mean score (38%) for Part A of the VHGT was marginally higher than that of their grade 11 peers (34%) which was in turn higher than the mean score (24%) of their NS grade 10 peers. The conclusion reached was that at each successive grade level in the NS subsample, there were more students at a higher van Hiele level than there were at an adjacent lower grade level.

- At grade level, the mean score (45%) of SAS grade 12 learners for Part A of the VHGT was marginally higher than the mean score (39%) of their grade 10 peers, which was in turn marginally higher than the mean score (34%) obtained by the SAS grade 11 learners. These results were interpreted to be consistent with and provide support for those reported in section 4.2.3 of Chapter 4, where grade 12 learners from SAS obtained a higher mean score on the TPGT than the grade 10 learners, whose mean score was in turn higher than that of the grade 11 learners. The conclusion reached was that there were more grade 12 learners than grade 10 learners at the higher van Hiele levels, but that there were fewer grade 11 learners at those levels.

- At each grade level, SAS learners obtained a higher mean score in Part A of the VHGT than their peers from NS. It was, however, only in grade 10 that the difference between the respective means was found to be statistically significant ($p < 0.01$). The conclusions reached were as follows: 1) There were far more NS grade 10 learners at a low van Hiele level than their counterparts from SAS. 2) An approximately equal preponderance of NS and SAS grade 11 learners were at a low van Hiele level. 3) There were an approximately equal number of grade 12 learners at the lower van Hiele levels, but at the higher levels there were more learners from SAS than NS.

- There was no significant difference between the mean scores of the male learners (38%) and the female learners (34%) ($p > 0.05$) for Part A of the VHGT. The results were found to be consistent with those of Usiskin (1982, pp.75–76), where sex differences in spring van Hiele levels (VHS) "favour[ed] the males," albeit to no significant degree of difference.

- Male learners from the NS subsample did not perform significantly better than their female counterparts in Part A of the VHGT. The difference between the male mean score (33.47%) and the female mean score (30.00%) was not significant ($p = 0.2743$). Given these low mean scores, it was concluded that both the male and female learners from the NS subsample were at a low van Hiele level of geometric understanding.

- Male learners from SAS, like their NS counterparts, obtained a marginally higher mean score (42.33%) in Part A of the VHGT than their female peers whose mean score was 37.20%. The difference between these means was, however, not significant ($p = 0.1122$). As with the NS subsample, these low means were interpreted as evidence that the majority of the male and female learners from SAS were at a low van Hiele level of geometric conceptualization.

- In this study, SAS male learners performed significantly better than their NS counterparts in Part A of the VHGT, as the former obtained a significantly higher mean score of 42.33% than the latter whose mean score was 33.47% ($p < 0.05$). Similarly, the female learners from the SAS subsample obtained a significantly higher mean score (37.20%) than their peers from the NS subsample whose mean score was 30.00% ($p < 0.01$). The conclusion reached was that there were more male and female learners from the NS subsample at low van Hiele levels (evident in the low mean scores) than there were learners from the SAS subsample at those levels.

Further analysis of learners' performance in Part A of the VHGT was done according to the van Hiele levels. Among the findings are the following:

- For the entire study sample, learners' mean scores for Part A of the VHGT decreased progressively at each successively higher van Hiele level between levels 1–3. The mean score of these learners for the van Hiele level 4 subtest of the VHGT was, however, higher than their mean score for the van Hiele level 3 subtest. This pattern of achievement in Part A of the VHGT was interpreted as providing supporting evidence of the hierarchical property of the van Hiele levels. The conclusion was also reached that learners in this study experienced more difficulty with geometry problems typifying van Hiele level 3 reasoning than they did with problems at the other levels. These results were consistent with those of Usiskin (1982).

- At each van Hiele level, learners from the SAS subsample obtained higher mean scores than their counterparts from NS. The conclusion reached was that in relative terms, NS learners in this study had a poorer knowledge of geometric ideas than their peers from SAS as measured by Part A of the VHGT.

- At each grade level in each of the participating schools, learners obtained their lowest mean score on the van Hiele level 3 subtest. The conclusion drawn was that geometry problems typifying level 3 reasoning were generally difficult for learners across all three grades in this study.

- All 68 learners (100%) from the NS subsample who wrote the VHGT were assignable to van Hiele levels. Of this figure, 36 (53%) were at the pre-recognition level, i.e. at van Hiele level 0, while 15 (22%), 16 (24%) and 1 (1%) were, respectively, at van Hiele levels 1, 2 and 3. No learner from this group was at level 4. With the large number of these learners at level 0, and with only 1 at level 3 and none at level 4, it was concluded that they had a poor understanding of school geometry.

- Of the 71 learners from SAS that wrote the VHGT, 68 (or 96%) of them were assignable to van Hiele levels. Of the total that wrote the test, 29 (41%) were at the pre-recognition level, while 16 (22%), 17 (24%), 2 (3%) and 4 (6%) were at van Hiele levels 1, 2, 3 and 4, respectively. There were 3 learners from this group whose responses did not 'fit' the forced van Hiele level determination scheme. Given the large number of these learners at level 0, and with nearly none at levels 3 and 4, the conclusion reached was that the majority of them had a weak knowledge of school geometry.

- For selected items in Part A of the VHGT, Usiskin's (1982) sample of American high school students performed better than SAS learners in the present study, who in turn performed better than NS learners in the present study.

- There was no item in Part A of the VHGT that required reasoning to a conclusion that was correctly answered by more than 40% of the learners in each of the subsamples, which is consistent with Usiskin's (1982) findings.

- The performance in Part B of the VHGT was generally poor for learners from both the NS and the SAS subsamples. For the grade 10 learners, many from the SAS subsample could find the third angle of a triangle in which two of the angles were given. The majority of them, however, could not successfully deal with a triangle problem in geometry requiring two or more lines of reasoning to get to the answer. Among the NS grade 10 learners, many could not find the third angle of a triangle given two of its angles, and none of them could successfully handle a triangle problem in geometry requiring two or more lines of reasoning.

- Writing a complete proof of even some familiar theorems in high school geometry was particularly difficult for all the grade 10 and grade 11 learners in both subsamples, and for the grade 12 learners in the NS subsample. Nevertheless, some grade 12 learners from the SAS subsample were successful in this task.

In the next chapter, correlation analyses are performed among learners' performance in the TPGT, GIST and CPGT as reported in Chapters 4 through 6, as well as their end-of-the-year examination scores in mathematics, and their performance in the VHGT, in order to determine what relationship exists between performance in those learning areas and students' van Hiele levels.

CHAPTER EIGHT

DATA ANALYSIS, RESULTS AND DISCUSSION 5: THE CORRELATIONS

8.1 Introduction

It was stated in section 3.3.4.2 of Chapter 3 that Phase 2 of this study concerned the determination of the possible relationship that might exist between the van Hiele levels and the general mathematics achievement of the participating learners. The technique used for this determination was explained in the section under reference. This chapter reports on the results of Phase 2 of this study. The first part of the chapter (sections 8.2 through 8.2.6) presents an analysis of the various correlations calculated between learners' scores in the van Hiele Geometry Test[11] (VHGT) and their scores in the School Examination in Mathematics (SEM), Terminology in Plane Geometry Test (TPGT), Conjecturing in Plane Geometry Test (CPGT) and Geometric Items Sorting Test (GIST). As indicated in section 3.3.4.2, participants' end-of-year examination scores in mathematics for the study year were obtained from the archival records of the participating schools. In the second part of this chapter (sections 8.3 through to 8.3.3), various comparisons are made among learners who are at different van Hiele levels in relation to their performance in the SEM, TPGT, CPGT and the GIST.

It would be recalled that although the analyses of the results of each of these tests as reported in Chapters 4 through 6 complement those of the VHGT (Chapter 7) to furnish us with and extend our insight into participants' knowledge of school geometry, their main purpose is to provide information on how learners' knowledge in these aspects of geometry relate to their exhibited van Hiele levels. This chapter, therefore, interconnects learners' performance in each of these tests with their performance in the VHGT.

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[11] Since learners' assigned van Hiele levels were based on their performance in Part A of the VHGT alone, the VHGT as used in this chapter refers only to that part of this test.

8.2 Correlation between learners' VHGT scores and their SEM, TPGT, CPGT and GIST scores

Consistent with Phase 2 of this study, sections 8.2.1 through 8.2.6 of this chapter present the results of the analysis of the correlations between learners' scores for the VHGT and their school examination scores in mathematics, as well as their scores for the other three tests used in this study (i.e. the TPGT, CPGT and GIST). These correlations were determined for the entire study sample and separately for each of the participating schools. The correlations were also calculated separately for learners in each of the three grade categories at the two schools involved in this study.

8.2.1 Correlation between learners' scores in the VHGT and the SEM, TPGT, CPGT and GIST

The scores obtained by all the participating learners in the VHGT were correlated separately with their sets of scores for the SEM, TPGT, CPGT and the GIST. The results of these correlations are as illustrated in Table 8.1.

Table 8. 1 Correlation for all learners between the VHGT and the SEM, TPGT, CPGT and GIST scores

	SEM	**TPGT**	**CPGT**	**GIST**
VHGT	$r = 0.01$ $p = 0.912$ N = 139	$r = 0.52$ $p = 0.000$ N = 136	$r = 0.52$ $p = 0.000$ N = 127	$r = 0.25$ $p = 0.134$ N = 36

Table 8.1 indicates that there were no significant correlations, for the entire study sample, between learners' VHGT scores and their SEM and GIST scores ($p > 0.05$). That is, success in the VHGT did not necessarily imply success in the SEM and GIST for the majority of the learners in this study. In other words, learners who achieved high (or low) scores for the VHGT did not necessarily obtain equally high (or low) scores for the SEM and the GIST.

However, Table 8.1 indicates significant positive correlations between learners' VHGT scores and their TPGT and CPGT scores ($p < 0.001$). This implies that, for the majority of the learners in this study, success (or failure) in the VHGT also meant success (or failure) in the TPGT and the CPGT. That is, learners who had high (or low) scores in the VHGT had equally high (or low) scores in the TPGT and the CPGT. This would seem to suggest that a learner's performance in the VHGT would be a good indicator as to the learner's performance in the TPGT and the CPGT.

Since the correlations reported in this section made use of the combined scores of both the NS and the SAS learners, it is possible that the coefficients of correlation obtained were affected by this combination. It was therefore deemed necessary to determine these correlations separately for each of the participating schools. This is the focus of the next two sections.

8.2.2 Correlation between NS learners' scores in the VHGT and the SEM, TPGT, CPGT and GIST

Correlation coefficients were calculated for the NS learners between their scores in the VHGT and each of their SEM, TPGT, CPGT and GIST scores. Table 8.2 summarizes the results.

Table 8. 2 Correlation of NS learners' VHGT and the SEM, TPGT, CPGT and GIST scores

	SEM	**TPGT**	**CPGT**	**GIST**
VHGT	$r = 0.38$ $p = 0.001$ N = 68	$r = 0.46$ $p = 0.000$ N = 65	$r = 0.42$ $p = 0.001$ N = 62	$r = 0.23$ $p = 0.361$ N = 18

The results in Table 8.2 clearly show that the correlations between NS learners' VHGT scores and their scores for the other four tests all had positive coefficients (*r-values*). With the exception of the correlation between the VHGT and the GIST scores, which is not statistically significant ($p > 0.05$), all the other correlations are statistically significant even though they are weak. The correlations between the

VHGT and the SEM and CPGT were both significant at the 0.005 level, while the correlation between the VHGT and the TPGT was significant at the 0.001 level (Table 8.2).

The correlations presented in Table 8.2 need to be interpreted with caution. The correlation coefficients between the VHGT scores and the SEM, TPGT and CPGT tend to suggest that to perform well in the school mathematics examination (SEM) and geometry content tests (TPGT and CPGT), one needs to be at a high van Hiele level. But the converse is just as tenable, that to be at a high van Hiele level, one needs to have a good knowledge of school mathematics generally and standard geometry topics in particular (see Usiskin, 1982). Whichever is the case, one conclusion that clearly stands out from these results is that for the NS subsample, there is a relationship (even though weak) between the van Hiele levels (as measured by the VHGT) and performance in school mathematics examination (SEM) and geometry content tests (TPGT and CPGT).

The correlation between the VHGT and the TPGT for the NS subsample was found to be consistent with that of Usiskin's (1982, p.44) study in which in a comparative and similar Entering Geometry test (EG), the "entering geometry knowledge [of American high school children] correlates between 0.58 and 0.61 with fall van Hiele level". In fact, Usiskin (1982, p.46) generally reports "a strong correlation between performance on geometry tests and van Hiele level" for his American high school sample. Indeed, as would be seen later on in this chapter, correlation coefficients as high as Usiskin's, $r = 0.61$, were calculated between the VHGT and each of the TPGT and CPGT for some categories of learners in this study.

8.2.3 Correlation between SAS learners' scores in the VHGT and the SEM, TPGT, CPGT and GIST

As with the NS subsample (section 8.2.2), significant positive correlation coefficients were calculated for the SAS subsample between the VHGT scores and the SEM, TPGT and the CPGT. It is notable that, with the exception of the SEM, the correlation coefficients between the VHGT and these three tests were higher than their corresponding NS figures. That is, the VHGT correlates more strongly with the TPGT and the CPGT for the SAS subsample than for the NS subsample. As indicated in Table 8.3, the VHGT did not correlate significantly with the GIST for the SAS subsample.

Table 8. 3 Correlation of SAS learners' VHGT and the SEM, TPGT, CPGT and GIST scores

	SEM	**TPGT**	**CPGT**	**GIST**
VHGT	$r = 0.35$ $p = 0.003$ N = 71	$r = 0.54$ $p = 0.000$ N = 71	$r = 0.53$ $p = 0.000$ N = 65	$r = 0.07$ $p = 0.772$ N = 18

As is evident in Table 8.3, the correlation between SAS learners' VHGT and SEM scores was statistically significant ($r = 0.35$, n = 71, $p < 0.005$). This implies that there is a positive relationship between performance in school mathematics examinations and the van Hiele levels. That is, for the SAS subsample, learners who performed well in the VHGT equally did well in their school mathematics examination, and vice versa.

Table 8.3 reveals that there exists a strong significant correlation between the VHGT and the TPGT scores for the SAS subsample ($r = 0.54$, n = 71, $p < 0.001$). This suggests that performance in the VHGT signals performance in a geometry terminology test for the SAS learners, consistent with Usiskin's (1982) study in which similar correlation coefficients were obtained between his entering geometry test sample and the van Hiele levels. Similarly, Table 8.3 shows that the VHGT strongly correlates significantly with the CPGT ($p < 0.001$). This suggests that for the SAS subsample, learners who did well in the VHGT did just as well in the CPGT.

As with the entire study sample (section 8.2.1) and the NS subsample (section 8.2.2), there was no significant correlation between the VHGT and the GIST scores for the SAS subsample (Table 8.3). It is not clear why this is so, but the small sample size for the GIST is a possible reason. Furthermore, despite the use of a well-designed marking scheme to assign scores to the learners for the GIST, it is possible that the actual allocation of scores to individual learners for this test nevertheless deviated slightly from this scheme, which is common in the scoring of free-response tests like the GIST. This latter possibility may have also contributed to the lack of correlation between the VHGT and the GIST scores in this study.

8.2.4 Grade level correlation between learners' scores in the VHGT and the SEM, TPGT and CPGT

In this section, results of the correlations between the VHGT and each of the SEM, TPGT and the CPGT scores are presented separately for learners in each of the three grades (10, 11 and 12) involved in this study. Table 8.4 summarizes the results of the grade level analyses of these correlations. It must be pointed out that because of the small sample of learners at each grade level involved in the GIST (6 learners per grade per school), no correlation analysis was done at grade level for this test. (See the second last paragraph in section 3.3.4.1.2 of Chapter 3 for the reason for involving so small a sample in the GIST.)

Table 8. 4 Grade level correlation between the VHGT and the SEM, TPGT and CPGT scores

Grades	Correlations			
		SEM	**TPGT**	**CPGT**
10	**VHGT**	$r = -0.10$ $p = 0.508$ n = 48	$r = 0.61$ $p = 0.000$ n = 45	$r = 0.61$ $p = 0.000$ n = 44
11	**VHGT**	$r = -0.02$ $p = 0.892$ n = 44	$r = 0.32$ $p = 0.035$ n = 44	$r = 0.30$ $p = 0.055$ n = 42
12	**VHGT**	$r = 0.02$ $p = 0.896$ n = 47	$r = 0.53$ $p = 0.000$ n = 47	$r = 0.54$ $p = 0.000$ n = 41

As can be seen in Table 8.4, there are no significant correlations between the VHGT and the SEM across all three grades ($p > 0.05$). In grades 10 and 11, the correlation coefficients are actually negative, suggesting an inverse relationship between the VHGT and the SEM for these sample groups. There could be several reasons for the lack of a significant correlation between the VHGT and the SEM scores across the grades in this study. The SEM scores obtained from the participating schools may have been awarded arbitrarily in one or both of the schools, thus awarding learners with low VHGT scores high SEM scores. In effect, this was the situation with the NS learners. Despite their very poor performance in the VHGT, and indeed all the tests used in this study, these learners were assigned very high SEM scores (see Appendix 7.A, p.129). The SAS learners' SEM scores (see Appendix 7.A, p.129) did not contrast with their performance in the VHGT as sharply as did those of the NS subsample. Combining the NS and SAS learners' SEM scores, as was done for the results displayed in Table 8.4, could therefore have skewed the correlations.

The VHGT correlates significantly with the TPGT across all the three grade categories (Table 8.4). The correlation coefficient, $r = 0.61$, calculated between the VHGT and the TPGT scores for the grade 10 learners, was the highest of the three grades and was significant at the 0.001 level. This was followed by that of the grade 12 learners, for whom $r = 0.53$, significant also at the 0.001 level. Table 8.4 further indicates that the VHGT correlates rather weakly with the TPGT for the grade 11 learners ($r = 0.32$, n = 44, $p < 0.05$). These results suggest that there is a relationship between performance on a geometry terminology test and the van Hiele levels, and that this relationship is stronger for the grade 10 and 12 learners than for the grade 11 learners. The rather inconsistent response patterns of the grade 11 learners reported in section 7.3.2 of Chapter 7 may have contributed to the weak correlation between the VHGT and the TPGT for this group of learners in this study.

Table 8.4 reveals a significant positive correlation between the VHGT and the CPGT scores for the grade 10 and 12 learners at the 0.001 level, but no significant correlation for the grade 11 learners ($p > 0.05$). With correlation coefficients of $r = 0.61$ and $r = 0.54$ between the VHGT and the CPGT for the grade 10 and 12 learners, respectively, the results in Table 8.4 indicate that there is a strong relationship between the van Hiele levels and performance in a geometry test that requires the learners to formulate conjectures, draw simple inferences and state definitions. Such a relationship, however, does not seem to exist for the grade 11 learners in this study, possibly for the reason stated in the previous paragraph.

These correlations were further considered at each grade level separately for each of the participating schools, as reported in sections 8.2.5 and 8.2.6.

8.2.5 Grade level correlation between NS learners' scores in the VHGT and the SEM, TPGT and CPGT

Correlation analyses between scores for the VHGT and the SEM, TPGT and CPGT were done at each grade level for the NS subsample, in order to reduce the effect which the combining the scores of the learners from both subsamples may have had on the correlation coefficients reported in the preceding section. The results of these analyses are as represented in Table 8.5.

Table 8. 5 Grade level correlation between NS learners' VHGT and SEM, TPGT and CPGT scores

Grades	**Correlations**			
		SEM	**TPGT**	**CPGT**
10	**VHGT**	$r = 0.39$ $p = 0.063$ n = 24	$r = 0.32$ $p = 0.163$ n = 21	$r = 0.24$ $p = 0.284$ n = 22
11	**VHGT**	$r = 0.31$ $p = 0.178$ n = 21	$r = 0.43$ $p = 0.052$ n = 21	$r = 0.47$ $p = 0.042$ n = 19
12	**VHGT**	$r = 0.26$ $p = 0.227$ n = 23	$r = 0.38$ $p = 0.075$ n = 23	$r = 0.32$ $p = 0.163$ n = 21

As evident in Table 8.5, with the exception of grade 11 in which there was a weak significant correlation between the VHGT and the CPGT ($p < 0.05$), there were no significant correlations between the VHGT score and any of the SEM, TPGT and CPGT scores across all three grade categories ($p > 0.05$). Given that for the entire NS subsample there were significant correlations between the VHGT and each of these tests (see section 8.2.2), the lack of significant correlations between the VHGT and these tests at each grade level for the NS subsample could be attributed to the small sample size at each of these grades.

Furthermore, since the VHGT correlates significantly with the TPGT and the CPGT at each grade level for the entire study sample (section 8.2.4), it could be argued that the VHGT generally correlates significantly better with the TPGT and the CPGT for the SAS subsample at each grade level than for the NS subsample, given the results in Table 8.5. That this is indeed the case is made clear in the next section.

8.2.6 Grade level correlation between SAS learners' scores in the VHGT and the SEM, TPGT and CPGT

As with the NS subsample, correlation analyses between the VHGT and the SEM, TPGT and CPGT scores of the learners were done at each grade level for the SAS subsample. Table 8.6 summarizes the results.

Table 8. 6 Grade level correlation between SAS learners' VHGT and SEM, TPGT and CPGT scores

Grades	Correlations			
		SEM	**TPGT**	**CPGT**
10	**VHGT**	$r = 0.46$ $p = 0.023$ n = 24	$r = 0.61$ $p = 0.002$ n = 24	$r = 0.61$ $p = 0.003$ n = 22
11	**VHGT**	$r = -0.40$ $p = 0.056$ n = 23	$r = 0.13$ $p = 0.551$ n = 23	$r = -0.01$ $p = 0.949$ n = 23
12	**VHGT**	$r = 0.43$ $p = 0.034$ n = 24	$r = 0.58$ $p = 0.003$ n = 24	$r = 0.55$ $p = 0.012$ n = 20

The correlation analyses in Table 8.6 indicate that there were significant positive correlations between the VHGT scores and each of the SEM, TPGT and the CPGT scores for the grade 10 and 12 learners from the SAS subsample. But the table also shows that for the SAS grade 11 learners there were no significant correlations between the VHGT scores and those for the other three tests. This is probably due to inconsistency in the response patterns of this group of learners, alluded to earlier in section 8.2.4.

The correlation coefficients for the SAS grade 10 and 12 learners in Table 8.6 are strikingly similar. The coefficients of correlation between the VHGT and the SEM for the grade 10 learners (r = 0.46) and the grade 12 learners (r = 0.43) were both moderately low and significant at the 0.05 level. As Table 8.6 clearly indicates, the correlation coefficients between the VHGT and the TPGT for the grade 10 learners (r = 0.61) and the grade 12 learners (r = 0.58) were both high and significant at the 0.005 level, though that of the grade 10 learners indicates a stronger correlation. A similar interpretation holds true for the correlation coefficients between the VHGT and the CPGT scores for these two groups of learners.

To summarize, for the grade 10 and 12 learners from SAS, performance in geometry tests (the TPGT and CPGT) correlates more strongly with the van Hiele levels than does performance in a school mathematics examination. For the grade 11 learners from this group, no significant correlations between the van Hiele levels and the other tests were evident in this study.

8.3 Comparison of performance between learners at different van Hiele levels

Having established the existence of correlations (varying from weak to strong) between the VHGT and the SEM, TPGT, CPGT and the GIST, separately for the NS and the SAS learners (sections 8.2.2 and 8.2.3), I thought it would be useful to determine whether learners at adjacent van Hiele levels performed significantly differently from each other in each of these tests. Certainly, if the hierarchical property of the van Hiele levels is valid, then it is to be expected that learners at a given van Hiele level (not level 0) will perform better than their peers at an adjacent lower level on the same geometry task, such as those in the tests used in this study. Recall that the validity of the fixed hierarchy of the van Hiele levels between levels 1–3 reported in Chapter 7 (see, for example, sections 7.2.2.1 and 7.2.2.3) was based solely on the assumed difficulty of the question items that typify each successive van Hiele level, rather than on the differential performance of learners at adjacent levels.

In sections 8.3.1 through 8.3.3, of this chapter, the validity of the postulated fixed hierarchy of the van Hiele levels is further examined by comparing the performance of learners at adjacent van Hiele levels in each of the SEM, TPGT, CPGT and GIST. For this comparison, percentage mean scores of the learners who were at each van Hiele level were computed for each of these tests. It must be stressed, however, that the purpose of these comparisons was not necessarily to confirm or refute the hierarchical property of the van Hiele levels, but rather to compare the performance of learners at different levels in this study.

8.3.1 Comparison of performance between learners at different van Hiele levels in the TPGT, CPGT, GIST and SEM

The results of performance comparison for the TPGT, CPGT, GIST and the SEM between learners at different van Hiele levels are presented in this section. Table 8.7 represents the results of *t-tests* of the equality of means of each of these tests (i.e. the TPGT, CPGT, GIST and the SEM) for learners at adjacent van Hiele levels. For the analyses in this section right through to section 8.3.3, the *t-tests* follow a one-way analysis of variance (ANOVA) with van Hiele level as the independent variable and each of the four tests as the dependent variable. The asterisk (*) in Table 8.7 means that computation was not possible as there was only one learner at that level.

As evident in Table 8.7, for $n \leq 2$, learners at van Hiele level n obtained higher means in each[12] of the four tests than learners at level n–1. It seems likely, however, that the few learners at van Hiele level 3 attained that level by chance through random guessing, which would explain why these learners consistently obtained lower mean scores in nearly all these tests than their peers at the adjacent lower level. For n = 4, learners at level n had higher means on the TPGT, CPGT and the GIST than learners at van Hiele level $n < 4$.

[12] For the GIST, learners at level 1 obtained slightly higher mean than learners at level 2.

Table 8. 7 Mean scores in the TPGT, CPGT, GIST and SEM of learners at each van Hiele level

Test	VH levels	N	Mean (%)	Std. Dev.	*F-value*	*df*	*p-value*
TPGT	0	62	37.77	10.80			
	1	31	45.71	16.58			
	2	33	55.33	16.70	9.17	(4, 128)	0.000
	3	3	42.00	12.77			
	4	4	57.75	21.19			
	Total	133	44.68	15.91			
CPGT	0	59	16.12	15.21			
	1	28	26.75	25.04			
	2	30	40.10	28.85	6.97	(4, 119)	0.000
	3	3	15.33	15.01			
	4	4	47.25	36.81			
	Total	124	25.31	24.33			
GIST	0	11	32.45	8.14			
	1	12	41.92	15.94			
	2	10	41.5	15.86	2.61	(4, 31)	0.054
	3	1	42.00	*			
	4	2	65.00	7.07			
	Total	36	40.19	14.85			
SEM	0	65	44.77	22.40			
	1	31	48.39	21.77			
	2	33	50.15	22.84	0.58	(4, 131)	0.677
	3	3	41.00	26.89			
	4	4	37.25	24.66			
	Total	136	46.6	22.35			

For the SEM, the mean scores in Table 8.7 show almost the reverse of what might be expected for learners' means between levels 2 and 4, but this can easily be explained. Recall that no learner from the NS subsample was at level 4, and only 1 of them was at level 3. That is, the majority of the NS learners were at the lower van Hiele levels (see section 7.2.3.3 of Chapter 7). Therefore, combining their SEM scores (which were very high as stated in section 8.2.4) with the SEM scores of the SAS learners (which were very low), invariably produced combined higher SEM mean scores at the lower van Hiele levels than at the higher levels. This situation suggests that it would be more useful and informative to examine these differential level performances separately for each of the participating schools, which is done in the next two sections.

The results reported in the preceding paragraph tend to provide support for the hierarchical property of the van Hiele levels in general and for levels 0–2 in particular. The results are in general consistent with those of Usiskin (1982, p.47), in which "on all assignment criteria, for $n \leq 3$ students at VHF [van Hiele fall] level n score significantly higher on the EG [Entering Geometry test] than students at level n–1". It must be pointed out here that the problem of random guessing, which may elevate some learners to a level by chance (as is the case with the level 3 learners in this study), could be eliminated or reduced by increasing the number of items at each van Hiele level in the subtest. It is also possible that the relatively small sample in this study (compared to Usiskin's) contributed to the very small number of learners (3 of them) at level 3 and hence the low mean scores of these learners in the various tests. The point is that, with these two constraints taken care of, the results of this study could be said to be in complete agreement with those of Usiskin (1982) concerning the hierarchical property of the van Hiele levels.

Table 8.7 further indicates that the differences in the mean scores of learners at different van Hiele levels for each of the four tests were significant only for the TPGT ($F = 9.17$, $(4, 128)df$, $p < 0.001$) and the CPGT ($F = 6.97$, $(4,119)df$, $p < 0.001$). For the GIST and the SEM, there were no significant differences between the means of learners at different van Hiele levels ($p > 0.05$ in each case). In order to determine at which pair of van Hiele levels these differential mean scores were significant, a *Tukey HSD post-hoc test* was conducted. The result of the *post-hoc* comparisons of means at the different van Hiele levels (Table 8.8) indicates that the differences between the mean scores on the TPGT and the CPGT for learners at van Hiele level 0 and learners at van Hiele level 2, in favour of the latter, are statistically significant ($p < 0.05$). What this means is that learners at van Hiele level 2 obtained significantly higher mean scores on the TPGT and the CPGT than learners at level 0. The differences between the mean scores of learners at any two of the van Hiele levels other than between levels 0 and 2 in each of the three tests[13] were not statistically significant ($p > 0.05$) for the entire study sample, as shown by the *post-hoc* multiple comparisons in Table 8.8.

[13] Inter-level comparisons of means were not done for the GIST since only one learner was at level 3 which made it impossible to compute the standard deviation at that level.

Table 8. 8 *Tukey HSD post-hoc test* for the TPGT, CPGT and the SEM

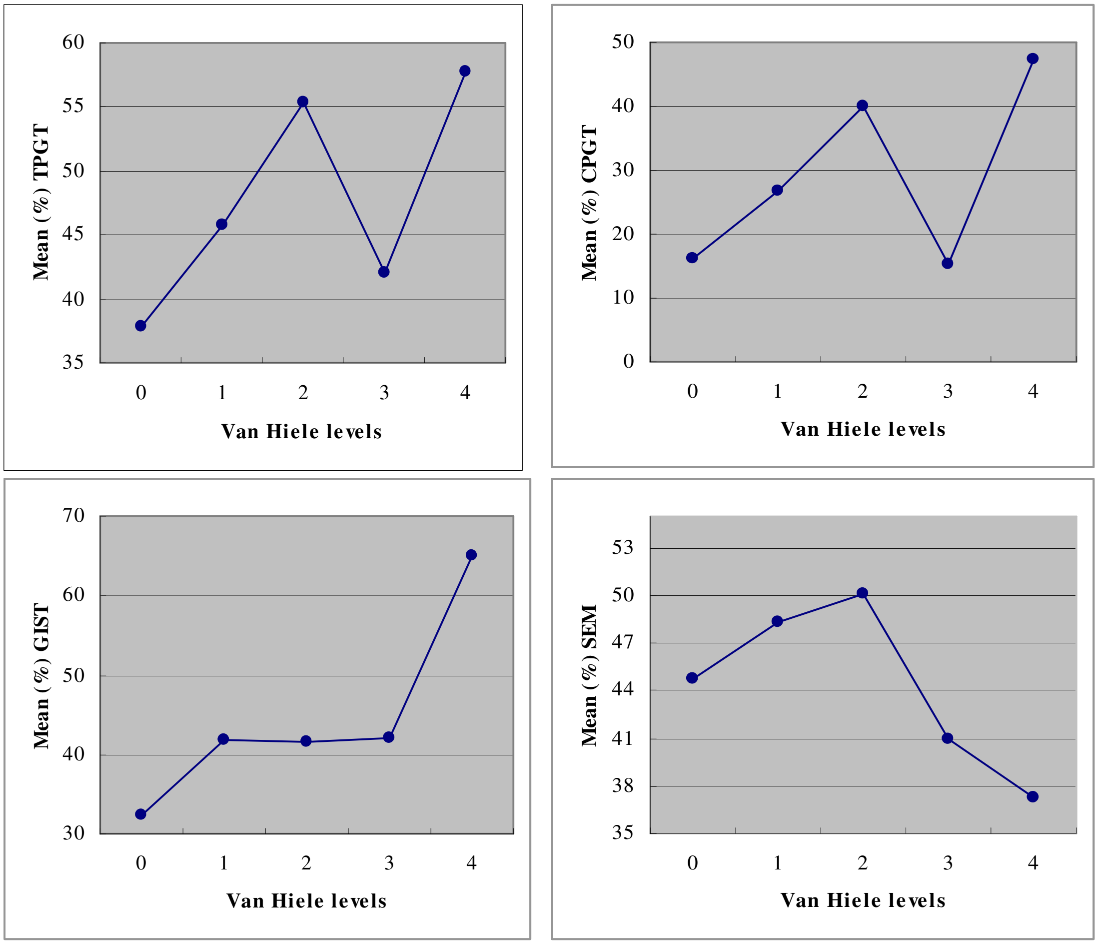
Test	VH levels		p-value
	(I) level	(J) level	
TPGT	0	1	0.090
		2	0.000
		3	0.987
		4	0.057
	1	2	0.059
		3	0.993
		4	0.506
	2	3	0.53
		4	0.998
	3	4	0.598
CPGT	0	1	0.235
		2	0.000
		3	1.000
		4	0.059
	1	2	0.158
		3	0.916
		4	0.424
	2	3	0.357
		4	0.974
	3	4	0.335
SEM	0	1	0.947
		2	0.796
		3	0.999
		4	0.967
	1	2	0.998
		3	0.983
		4	0.884
	2	3	0.962
		4	0.815
	3	4	0.999

NB: Comparing level 0 (I column) with level 1 (J column) will produce the same result as comparing level 1 (I column) with level 0 (J column), hence the latter pair of comparisons (i.e. level 1 versus level 0) is left out in each of the tests in Table 8.8. A similar reason explains why level 4 (I column) does not appear in the table.

In order to illustrate more clearly the differential mean performance in the TPGT, CPGT, GIST and the SEM by learners at adjacent van Hiele levels, means plots of their percentage mean scores were done and are represented in Graph 8.1.

The means plots in Graph 8.1 clearly reiterate the point made earlier, that the results of the comparisons of the mean scores of learners for each of the tests used in this study provide support for the hierarchical property of the van Hiele levels generally and levels 0–2 specifically. It is also evident from Graph 8.1 that learners at van Hiele level 3 attained that level by chance in this study. The implication of this is that more items should be used at each van Hiele level subtest so as to reduce the impact of guessing on students' attainment of the levels.

Graph 8. 1 Means plots for all learners at each van Hiele level



8.3.2 Comparison of performance between NS learners at different van Hiele levels in the TPGT, CPGT, GIST and SEM

The mean scores for the TPGT, CPGT, GIST and the SEM of NS learners at different van Hiele levels were computed and compared. The results of *t-tests* of equality of means of each of these tests for learners at adjacent van Hiele levels are summarized in table 8.9.

As with the entire study sample (section 8.3.1), Table 8.9 indicates that for the NS subsample, learners at van Hiele level n obtained higher means on all four tests than learners at level n–1 for $n \leq 2$. It was only for the TPGT and the CPGT, however, that the differences in the means were statistically significant ($p < 0.05$). These results suggest that the only NS learner at van Hiele level 3 attained that level by chance, which corroborates the claim in the previous section that generally the learners at level 3 in this study got there by random guessing in the VHGT. Nevertheless, the fixed sequence of the van Hiele levels was in general supported by results of the analyses of the responses of the NS learners, particularly between levels 0–2 in this study.

Table 8. 9 Mean scores for the TPGT, CPGT, GIST and SEM of NS learners at each van Hiele level

Test	VH levels	N	Mean (%)	Std. Dev.	*F-value*	*df*	*p-value*
TPGT	0	33	34.33	9.98			
	1	15	41.53	19.58			
	2	16	55.31	18.70	7.23	(3, 61)	0.000
	3	1	28.00	*			
	Total	65	41.06	17.10			
CPGT	0	32	12.16	11.10			
	1	14	18.00	23.05			
	2	15	33.13	29.57	3.79	(3, 58)	0.015
	3	1	16.00	*			
	Total	62	18.61	21.23			
GIST	0	8	31.00	8.69			
	1	4	34.50	12.72			
	2	5	47.40	20.35	1.54	(3, 14)	0.249
	3	1	42.00	*			
	Total	18	36.94	14.45			
SEM	0	36	61.94	7.91			
	1	15	66.87	7.03			
	2	16	66.50	10.01	2.00	(3, 64)	0.123
	3	1	58.00	*			
	Total	68	64.04	8.46			

NB: The asterisk (*) means not computable as there was only one learner at that level.

In order further to determine between which pair of van Hiele levels these differential mean scores were significant, a *Tukey HSD post-hoc test* was conducted only for levels 0–2. It was not possible to do this at level 3 as there was only one learner and it was thus impossible to obtain the standard deviation for learners' scores at that level. The result of the *post-hoc* comparisons of means (Table 8.10) indicates that for the TPGT the difference between the mean scores of learners at van Hiele level 0 and level 2, in favour of the latter, was statistically significant ($p < 0.05$), and the difference between the means of learners at level 1 and level 2 was also significant in favour of the latter at the 0.05 level. Similarly, for the CPGT, the difference between the means of learners at van Hiele level 0 and level 2, favouring the latter, was significant (p < 0.05). For any other pair of levels the differences in the mean scores of learners in each of the four tests were not statistically significant (p > 0.05).

Table 8. 10 *Tukey HSD post-hoc test* for NS learners in the TPGT, CPGT, GIST and the SEM

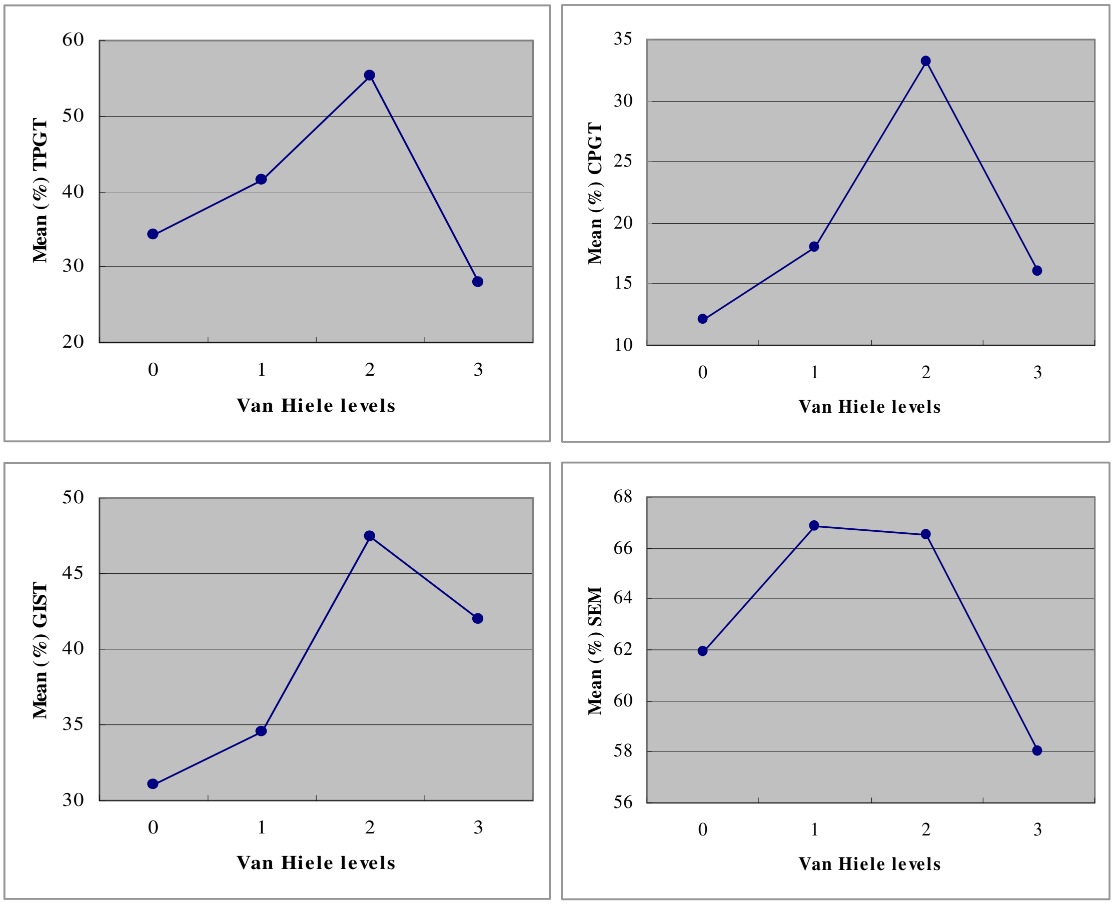
Test	VH levels (I) level	VH levels (J) level	p-value
TPGT	0	1	0.281
		2	0.000
	1	2	0.035
CPGT	0	1	0.632
		2	0.004
	1	2	0.110
GIST	0	1	0.911
		2	0.129
	1	2	0.371
SEM	0	1	0.137
		2	0.167
	1	2	0.992

NB: Comparing level 0 (I column) with level 1 (J column) will produce the same result as comparing level 1 (I column) with level 0 (J column), hence the latter pair of comparisons (i.e. level 1 versus level 0) is left out in each of the tests in Table 8.10. A similar reason explains why level 2 (I column) does not appear in the table.

As with the entire study sample (see Table 8.8), the results of the *post-hoc* comparisons of means in Table 8.10 indicate that NS learners at van Hiele level 1 did not obtain significantly higher means in each of the four tests than learners who were at level 0. This would seem rather surprising. However, van Hiele levels of 0 or 1 are so low that a learner at either level at the end of the school year would not be in possession of the conceptual knowledge of geometry needed to perform well in nearly all these tests, particularly the CPGT and the SEM (see Usiskin (1982). Hence, performance in these tests by learners at levels 0 and 1 would not differ significantly from each other, both being poor. The results as reported here were found to be consistent with those of Usiskin's (1982, p.48) study, in which "in the spring, students at van Hiele levels 0 and 1 … [had] nearly the same geometry knowledge".

The mean scores of NS learners at different van Hiele levels on the TPGT, CPGT, GIST and the SEM were further represented by means plots as illustrated in Graph 8.2.

Graph 8. 2 Means plots for NS learners at each van Hiele level



As with the entire study sample, the means plots for the NS learners clearly provide support for the hierarchical property of the van Hiele levels, especially for levels 0–2, with regard to learners' knowledge of school geometry.

8.3.3 Comparison of performance between SAS learners at different van Hiele levels in the TPGT, CPGT, GIST and SEM

The performances of SAS learners at different van Hiele levels in the TPGT, CPGT, GIST and the SEM were compared by looking at their mean scores in each of these tests. Table 8.11 represents the results of *t-tests* of the equality of means of each of these tests for the SAS learners at adjacent van Hiele levels.

Table 8. 11 Mean scores in the TPGT, CPGT, GIST and SEM of SAS learners at each van Hiele level

Test	VH levels	N	Mean (%)	Std. Dev.	*F-value*	*df*	*p-value*
TPGT	0	29	41.69	10.50			
	1	16	49.63	12.57			
	2	17	55.35	15.17	3.74	(4, 63)	0.009
	3	2	49.00	5.66			
	4	4	57.75	21.19			
	Total	68	48.13	13.94			
CPGT	0	27	20.81	18.07			
	1	14	35.50	24.61			
	2	15	47.07	27.30	3.85	(4, 57)	0.008
	3	2	15.00	21.21			
	4	4	47.25	36.81			
	Total	62	32.00	25.54			
GIST	0	3	36.33	6.03			
	1	8	45.63	16.82			
	2	5	35.60	8.02	2.82	(3, 14)	0.077
	3	0	*	*			
	4	2	65.00	7.07			
	Total	18	43.44	14.92			
SEM	0	29	23.45	14.78			
	1	16	31.06	15.48			
	2	17	34.76	20.75	1.48	(4, 63)	0.220
	3	2	32.50	31.82			
	4	4	37.25	24.66			
	Total	68	29.15	17.81			

NB: No learner at level 3 wrote the GIST in SAS (the asterisks).

As with the entire study sample (section 8.3.1), for $n \leq 2$, SAS learners at van Hiele level n obtained higher means on each[14] of the four tests than learners at level n–1 (Table 8.11). As with the NS subsample (section 8.3.2), the few SAS learners at van Hiele level 3 obtained marginally lower mean scores than their peers at level 2 for each of the tests. This appears to provide further evidence to support the claim that learners at level 3 attained this level by chance in this study. Learners at level 4, however, obtained higher means than learners at lower van Hiele levels in each of the four tests. Thus, as with the entire study sample (section 8.3.1) and the NS subsample (section 8.3.2), the results for the SAS subsample in this section provide support for the hierarchical property of the van Hiele levels in general, and for levels 0–2 in particular.

Consistent with the results for the entire study sample (section 8.3.1) and the NS subsample (section 8.3.2), Table 8.11 also indicates that the differences in the mean

[14] For the GIST, SAS learners at level 1 had a marginally higher mean than learners at level 2.

scores of SAS learners at different van Hiele levels for each of these tests were significant only for the TPGT ($F = 3.74$, $(4, 63)df$, $p < 0.05$) and the CPGT ($F = 3.85$, $(4, 57)df$, $p < 0.05$). For the GIST and the SEM, the differences in the mean scores were not significant for learners at different van Hiele levels. A *Tukey HSD post-hoc test* was conducted in order to determine between which pair of van Hiele levels the differences in the mean scores of SAS learners were significant for the TPGT, CPGT and the SEM. This was not done for the GIST as it turned out that no SAS learner who wrote the GIST was at level 3.

The result of the *post-hoc* comparisons of means at the different van Hiele levels (Table 8.12) indicates that the differences between the mean scores in the TPGT and the CPGT for the SAS learners at level 0 and level 2, in favour of the learners at level 2, are significant ($p < 0.05$). That is, SAS learners at van Hiele level 2 obtained significantly higher mean scores in the TPGT and the CPGT than their peers at level 0. For each of the four tests, the differences between the mean scores of learners at any one pair of the van Hiele levels other than levels 0 and 2 were not statistically significant, as can be seen in Table 8.12.

Means plots of the SAS learners at each van Hiele level for the TPGT, CPGT, GIST and the SEM were done in order graphically to illustrate the differential mean performance of the learners in this study. Graph 8.3 represents the means plots.

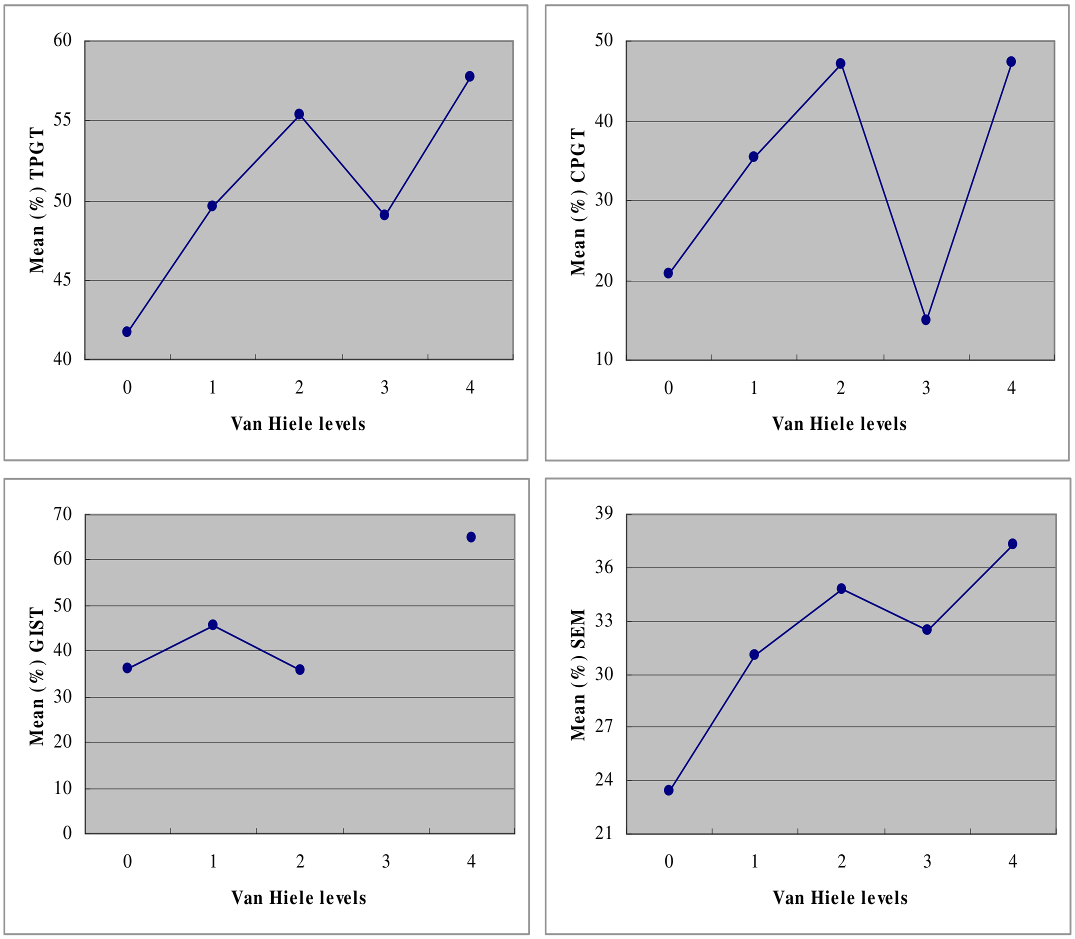
Individual grade level comparisons of performance in these tests (the TPGT, CPGT, GIST and the SEM) for learners at each van Hiele level, separately, for each of the participating schools, did not yield any results markedly different from the combined grades comparisons of performance at each van Hiele level reported in sections 8.3.2 and 8.3.3. Consequently, no such individual grade level comparisons of performance in these tests for learners at adjacent van Hiele levels are reported in this chapter.

Table 8. 12 *Tukey HSD post-hoc test* for the SAS learners on the TPGT, CPGT and the SEM

Test	VH levels		p-value
	(I) level	(J) level	
TPGT	0	1	0.291
		2	0.008
		3	0.937
		4	0.149
	1	2	0.709
		3	1.000
		4	0.793
	2	3	0.965
		4	0.997
	3	4	0.935
CPGT	0	1	0.328
		2	0.008
		3	0.997
		4	0.232
	1	2	0.675
		3	0.775
		4	0.902
	2	3	0.374
		4	1.000
	3	4	0.511
SEM	0	1	0.635
		2	0.229
		3	0.955
		4	0.583
	1	2	0.974
		3	1.000
		4	0.970
	2	3	1.000
		4	0.999
	3	4	0.998

NB: Comparing level 0 (I column) with level 1 (J column) will produce the same result as comparing level 1 (I column) with level 0 (J column), hence the latter pair of comparisons (i.e. level 1 versus level 0) is left out in each of the tests in Table 8.12. A similar reason explains why level 4 (I column) does not appear in the table.

Graph 8. 3 Means plots for SAS learners at each van Hiele level



8.4 Chapter conclusion

This chapter has presented the results of the analysis of the various correlations between learners' scores for the VHGT and their scores for the SEM, TPGT, CPGT and the GIST. The chapter has further reported the results of the various comparisons of the performance in these tests of learners at each van Hiele level. The major findings include the following:

- For the entire study sample, learners' VHGT scores correlate significantly with their TPGT and CPGT scores at the 0.001 level. However, no significant correlations were found between learners' VHGT scores and their SEM or GIST scores. With correlation coefficients of r = 0.52 calculated between learners' VHGT scores and each of their TPGT and CPGT scores, the conclusion reached was that there is a strong relationship between performance in geometry content tests (TPGT and CPGT) and van Hiele levels for the majority of the learners in this study.

- At the participating schools level, NS learners' VHGT scores correlate significantly with their SEM, TPGT and CPGT scores. With correlation coefficients of $r = 0.38$, $r = 0.46$ and $r = 0.42$, respectively for these tests, the conclusion drawn was that there is a relationship between the van Hiele levels and performance in the school mathematics examination (SEM) and geometry content tests (TPGT and CPGT) for learners from the NS subsample.

- At the participating schools level, SAS learners' VHGT scores correlate significantly with each of their SEM, TPGT and CPGT scores. Learners' VHGT scores, however, correlate even more strongly with their TPGT scores ($r = 0.54$, n = 71, $p < 0.001$) than with their CPGT scores ($r = 0.53$, n = 65, $p < 0.005$). The conclusion reached is that for the SAS subsample, learners who did well in the VHGT did just as well in the school mathematics examination (SEM) and geometry content tests (TPGT and CPGT). The converse applies.

- At each grade level for the entire study sample, grade 10 learners' VHGT scores correlate strongly ($p < 0.001$) with their TPGT and CPGT scores, for both of which a correlation coefficient of $r = 0.61$ was calculated. For the grade 11 learners, the VHGT scores correlate only weakly with the TPGT scores ($r = 0.35$, n = 44, $p < 0.05$). In grade 12, strong correlations were found between learners' VHGT scores and their TPGT and CPGT scores at the 0.001 level, with correlation coefficients of $r = 0.53$ and $r = 0.54$, respectively. No significant correlations were found between learners' VHGT scores and their SEM scores across the grade levels.

- For the NS subsample, it was only in grade 11 that there existed a significant positive correlation between learners' VHGT scores and CPGT scores. No significant correlations were found between the VHGT scores and the SEM and TPGT scores for learners across all three grades in NS. There were also no significant correlations between the VHGT scores and the CPGT scores for the grade 10 and 12 learners from NS.

- For the SAS subsample, significant positive correlations were found between grades 10 and 12 learners' VHGT scores and their respective SEM, TPGT and CPGT scores. For the SAS grade 11 learners, no significant correlations were found between learners' VHGT scores and their SEM, TPGT and CPGT scores.

- For the entire study sample, for $n \leq 2$, learners at van Hiele level n obtained higher mean scores for the SEM, TPGT, CPGT and the GIST than learners at level n–1. Learners at level 3 were judged to have attained that level by chance since they consistently obtained lower means on nearly all these tests than their peers at level 2. Learners at level 4, however, had higher means on the TPGT, CPGT and the GIST than learners at the lower van Hiele levels. The results as reported here were interpreted as providing support for the hierarchical property of the van Hiele levels in general, and levels 0–2 in particular.

- For the entire study sample, significant differences occurred between the mean scores of learners at the different van Hiele levels for the TPGT and the CPGT. A *Tukey HSD post-hoc test* indicated that the differences between the mean scores in the TPGT and CPGT for learners at van Hiele level 0 and learners at van Hiele level 2 are statistically significant ($p < 0.05$) in favour of the latter. The difference between the mean scores of learners distributed between any two of the van Hiele levels other than levels 0 and 2, for the SEM, TPGT and the CPGT, was not statistically significant ($p > 0.05$).

- For the NS subsample, for $n \leq 2$, learners at van Hiele level n had higher means on all the four tests than their peers at level n–1. The differences between the means of learners at the different van Hiele levels were, however, significant only for the TPGT and the CPGT. In all four tests, learners at level 3 had lower means than learners at level 2 for the NS subsample. A *Tukey HSD post-hoc test* conducted to determine between which pair of the van Hiele levels these differential mean scores were significant revealed that for the TPGT, the difference between the mean scores of learners at van Hiele level 0 and level 2, in favour of the latter, was statistically significant ($p < 0.05$), as was the difference between the means of learners at level 1 and level 2, in favour of the latter. For the CPGT, the difference between the means of learners at van Hiele levels 0 and 2, favouring the latter, was significant. Besides these, no significant differences were found to exist between the means of NS learners at the different van Hiele levels for the SEM, TPGT, CPGT and GIST.

- For the SAS subsample, for $n \leq 2$, learners at van Hiele level n obtained higher means in the tests (with the exception of the GIST) than learners at level $n-1$. As with the NS subsample, SAS learners at level 3 consistently obtained lower mean scores for each of the four tests (the SEM, TPGT, CPGT and the GIST) than learners at level 2, suggesting that these learners had attained level 3 by chance. Learners at level 4, however, had higher mean scores in each of the SEM, TPGT, CPGT and GIST than learners at the lower van Hiele levels. As with the NS subsample, the differences between the mean scores of the SAS learners at the different van Hiele levels in these tests were statistically significant only for the TPGT and the CPGT. *Tukey HSD post-hoc* comparisons of means further indicated that the differences between the mean scores in the TPGT and the CPGT for the SAS learners at van Hiele level 0 and level 2, in favour of the latter, were statistically significant ($p < 0.05$). For all other combinations of levels for each of the four tests, the differences between the mean scores of the SAS learners were not statistically significant.

In Chapters 4 to 7, participating learners' knowledge of school geometry was explicated and their van Hiele levels determined. The relationship between learners' knowledge of geometry and their exhibited van Hiele levels formed the basis for the analysis and discussion presented in Chapter 8. But, as important as it is to know learners' van Hiele levels, the determination of these levels alone is not enough. As stated in section 3.3.4.3 of Chapter 3, what is also needed is information on the classroom instructional processes that possibly contributed to the production of these levels among the learners. The next chapter presents and discusses the results of analyzing lessons videotaped in the geometry classrooms of the participating teachers and learners.

CHAPTER NINE

DATA ANALYSIS, RESULTS AND DISCUSSION 6: CLASSROOM VIDEO STUDY

9.1 Introduction

Phase 3 of this study concerns instructional methods in geometry classrooms (see section 3.3.4.3). The aim of this phase is to provide information on how geometry is taught in Nigerian and South African high schools, and to elucidate what learning opportunities the teaching methods being used offer to learners in the subject. The procedure followed in studying instruction in geometry classrooms was also explicated in the section under reference, while the analytic processes were explained in section 3.4.2.2. This chapter presents the analysis of geometry lessons that were videotaped separately in three classrooms in NS and three classrooms in SAS. It must be pointed out that although the results of the video study furnish us with useful information on teaching methods in geometry in the participating schools, the videotaped lessons can only be partial portraits or vignettes of the whole process of instruction in these schools. Hence, the findings concerning teaching methods presented in this chapter are tentative.

Given the complexity and dynamics of classroom instructional activities, coupled with the multiplicity of instructional methods, it would seem improbable that any one instructional model would suffice adequately to capture the whole process of instruction in a given classroom. However, it is possible to determine whether a particular classroom instructional practice/strategy or method fits or conforms to a specific model. This would require identifying key features of the instructional model and then checking these against segments of the observed classroom instruction. This is the purpose of the checklist of van Hiele phase descriptors (see section 2.8.4.1). As stated in the last paragraph of section 2.8.4, this checklist is my own careful articulation of key elements of the van Hiele model of geometry instruction, but with important ideas tapped from the instructional modules of Fuys et al. (1988), and from Yager's (1991) constructivist learning model (CLM).

Having transcribed all six videotaped lessons through a painstaking and time-consuming process of repeated playbacks – as a novice researcher, I spent on average about 14 hours transcribing each video – I decided to carry out a pilot application of the checklist of van Hiele phase descriptors to each of the lessons. What I did was to give each of three critical readers the transcribed copies of the videotaped lessons together with a copy of the checklist of van Hiele phase descriptors. I then requested each of them independently to apply the checklist to each of the transcribed lessons. I wanted to test how consistently the application of a given criterion on the checklist to the same segment of a lesson by different people would yield the same objective judgement or result. That is, I wanted to know whether different people could reach the same objective judgement, when applying the same criterion on the checklist to the same lesson, regarding the instructional strategy used by the teacher.

The three critical readers involved in this trial application of the checklist criteria were all from my university. One of them was a PhD student and a university lecturer from Nigeria. The other two were from Namibia, one a master's student and a college of education lecturer, the other a second-year PhD student of education. It turned out that there were differences of judgement among the four participants (I was the fourth) when the same checklist criterion was applied to the same lesson. That is, when one or two of us judged that a teacher had met a criterion on the checklist in his instructional delivery, others would judge that the same teacher had not satisfied that criterion.

These differences in judgement occurred in about 50 percent of the checklist criteria in some lessons, while in others they occurred in just a few of them. As we discussed and debated these differences, it became clear that we had interpreted many of the checklist criteria differently. This suggested to me that the checklist criteria needed refinement, perhaps a definition for each criterion that would clarify what counts as evidence of the criterion in the videotaped lessons.

9.2 Defining the criteria on the checklist of van Hiele phase descriptors

As was stated in the second last paragraph of section 3.4.2.2, the process of turning videos into information involved a consultative panel of four independent observers (including myself). Subsequent to the trial application of the checklist criteria to the transcribed videotaped lessons, and before the final application of the checklist of van Hiele phase descriptors to the lessons, each observer was requested to write a definition of what counted as evidence for each criterion on the checklist. This was done in order to minimize the flexibility in the application of the checklist to the lessons that was evident in the trial application. As rightly noted by Stigler and Hiebert (1999, p.22), "anyone who has engaged in this process knows that it is not easy to write such a definition. But it can be done". For example, what would count as evidence for a teacher's having introduced a topic by recognising and building on learners' prior knowledge, as the checklist requires?

As stated in section 3.4.2.2, the panel of observers met and after careful deliberation adopted a definition for each criterion on the checklist. The aim was to establish common ground upon which anyone viewing the classroom videos or reading the transcriptions could base his/her judgement of the extent to which the checklist criteria were being met.

To define the criteria on the checklist of the van Hiele phase descriptors was a bold move, as there is no doubt considerable scope for interpretation and hence disagreement in this context. But not stating a definition at all was for me even more dangerous, since it would have meant basing judgement of the videotaped lessons on the arbitrary decisions of observers. Each criterion on the checklist of van Hiele phase descriptors was defined by the consultative panel of observers, as indicated in Table 9.1.

Table 9. 1 Definition of the criteria in the checklist of van Hiele phase descriptors

Checklist criterion	Definition
1. Teacher introduces the topic by recognising and building on learners' prior knowledge.	Any instance during the lesson when the teacher explicitly reminds the learners of knowledge covered in previous lessons, by recapping on topics or concepts, or by asking questions which would require learners to recall their knowledge of earlier lessons, topics or concepts.
2. Teacher delays instruction of formal vocabulary, and condones learners' use of common informal terms in the issuing discussion.	Any instance during the lesson when the teacher initially accepts learners' use of informal/layman's terms either when the learners are asking questions, responding to questions or explaining an idea or a concept.
3. Teacher asks questions that seek to clarify students' imprecise terminology and gradually introduces formal mathematical language.	Either any instance during the lesson when the teacher explicitly requests the learners to explain the meaning of any imprecise terminology used by them, or any other such instance when the teacher carefully guides the learners, through constant probing, to come up with the correct terminology themselves.
4. Teacher creates an interactive learning environment and encourages learners to challenge, contest and negotiate meanings and solutions to mathematical problems.	Any instance during the lesson when the teacher explicitly requests the learners to work in groups. This includes any other such instance when the teacher gives room for the learners to 'throw ideas around' and take on each other's ideas, whether as a group or individually, regarding the mathematical problem at hand.
5. Teacher asks questions that steer students' thought toward the central idea being developed.	This includes all instances during the lesson when the teacher poses questions, whether as examples or classwork for the learners, whether such questions are written on the chalkboard or contained in worksheets, which would enable the teacher to assess learners' understanding of the concept developed in the lesson thus far.
6. Teacher uses open-ended questions and encourages learners to seek their own solution strategies.	Any opportunity given by the teacher during the lesson to let the learners, whether individually or as a group, to come up with various types of solutions (strategies) or perspectives on the mathematical problem at hand.
7. Teacher encourages learners to elaborate on their responses.	Any instance during the lesson when the teacher explicitly requests the learners to expatiate or shed more light on their solution (strategy) to a mathematical problem or an idea which they have expressed.
8. Teacher uses questions that encourage the learners to reflect on, refine and summarise their ideas about the concept learned.	This include only such instances during the lesson when the teacher explicitly asks the learners to reflect on what they have just learned by summarising the lesson themselves or the teacher does the summary himself.

With each criterion on the checklist of the van Hiele phase descriptors having been carefully defined, the stage was now set to apply the checklist to the transcribed videotaped lessons. But, before this, it was deemed necessary to establish whether these definitions would serve their intended purpose of enabling different observers to reach the same objective judgement about a lesson by applying the same checklist criterion. Indeed, my critical readers and I were satisfied with the degree of agreement in our independent judgement of the lessons based on the defined checklist criteria in our second application. On average, we reached the same judgement about each of the lessons in no fewer than seven of the checklist criteria. Where disagreements occurred, it was due to an observer citing aberrant or conflicting evidence for the criterion from the transcript. In some such instances, further refinement of the definition of the criterion was necessary before the final application of the checklist criteria to the videotaped lessons.

9.3 Analysis of the videotaped lessons according to the checklist of van Hiele phase descriptors

Defining each of the checklist criteria was inherently advantageous to the analysis of the lesson videos in various ways. For example, in ensuring consistency in the application of each criterion to the lessons, the definitions provided a context that helped to reduce judgements as to whether or not a given lesson met a given checklist criterion to simple 'Yes' or 'No' decisions. This facilitated the onerous process of analyzing the videotaped lessons, with the added advantage that one could practically tell at a glance which lesson conformed to the van Hiele model of geometry classroom instruction. But simple 'Yes' or 'No' decisions tend not to be very informative and leave the reader with little or no clue about the actual activities that took place during the lesson. Thus to be more useful and informative, the 'Yes' or 'No' decisions are supplemented with supporting evidence from segments of the lesson videos, as indicated in Table 9.2. In some instances, the time interval during the lesson within which the evidence was extracted is also given.

In Table 9.2, a 'Yes' indicates that the lesson satisfies the checklist criterion against which it is being measured and a 'No' indicates the reverse. Lessons 1 through 3 represent those delivered by the NS grade 10 teacher (Mr Adeleke), the NS grade 11 teacher (Mr Balogun) and the NS grade 12 teacher (Mr Lawal), respectively. Lessons 4 through 6 were delivered by the SAS grade 10 teacher (Mr John), the SAS grade 11 teacher (Mr Shlaja), and the SAS grade 12 teacher (Mr Andile), respectively. The full transcripts of each of these lessons are contained in Appendix 8.A–F, pp.130, 136, 148, 155, 166 and 167. In these transcripts, all my comments are italicised, comments or responses from the learners are in quotes and written in boldface, while the rest of the transcript represents the teacher's own voice.

Table 9. 2 Analysis of the videotaped lessons in NS (lessons 1–3) and SAS (lessons 4–6)

Checklist criterion	Lesson 1	Lesson 2	Lesson 3	Lesson 4	Lesson 5	Lesson 6
1	Yes, 5 minutes on, Mr Adeleke reminded the learners about their previous week's discussion on types of triangles.	No, 5 minutes on, Mr Balogun introduced the lesson by defining for the learners what a triangle is.	Yes, 5 minutes on, Mr Lawal briefly reminded the learners about their previous week's lesson for which the current lesson is a continuation.	Yes, 5 minutes on, Mr John reminded the learners about the topic they did the previous year – that is triangles.	No, 5 minutes on, Mr Shlaja merely announced to the learners the field of study – properties of parallelograms.	Yes, 13–15 minutes on, just before the proof of the theorem, Mr Andile reminded learners that their knowledge of the concept of congruency was needed.
2	No, being a whole-class instruction, no much opportunity was provided to hear the voice of the learners.	No, being a teacher-dominated instruction, no opportunity was offered the learners to express themselves.	No, as learners' voice was heard only for questions requiring "Yes Sir" responses and those for which Mr Lawal has already given the answers..	Yes, 15 minutes on, Mr John explained in formal terms what a triangle is to his students having condoned their incomplete definition of a triangle.	Yes, 40 minutes on, long after the students had used straight lines to connect opposite angles of the parallelogram did Mr Shlaja use the term 'diagonals' to refer to these lines.	No, as classroom discussion was teacher-led with little chance given for the learners to verbally express themselves.
3	No, since no room was provided for criterion 2.	No, since the learners' voice was little heard.	No, since no room was provided for criterion 2.	No, the learners could not report their findings before the end of the lesson.	Yes, about 40 minutes on, Mr Shlaja guided the students to learn that the diagonals of a parallelogram "bisect each other".	No, since no room was provided for criterion 2.
4	No, as Mr Adeleke dominated the classroom discussion and did not create the opportunity for student-student interactions in the class.	No, since group discussion was not encouraged and Mr Balogun did not pose questions to assess learners' understanding of the concept taught.	No, as Mr Lawal adopted whole-class teaching and dominated classroom discussion and did not create room for student-student classroom interaction.	Yes, throughout the lesson period, Mr John encouraged the learners to work in groups of four. (See his remarks about 43–45 minutes into his lesson).	Yes, throughout his lesson the students worked in interactive groups of two.	No, 13 minutes on, Mr Andile told the students what it means to "bisect" rather than seeking learners' meaning of the word.
5	Yes, about 30 minutes on, Mr Adeleke asked a question for which he quickly demonstrated the application of the theorem he has just proved for the learners.	No, the teacher did not give any form of assessment to the learners other than asking whether the learners understood what he was saying.	Yes, about 25–30 minutes on, Mr Lawal posed questions based on the theorem he has just proved for the learners. However, he solved these questions by himself.	Yes, the worksheets contained questions that tested learners' understanding of the central idea taught.	Yes, about 30–40 minutes on, Mr Shlaja, through constant probing, led the students to establish the properties of parallelograms.	Yes, about 35 minutes on, Mr Andile distributed among the learners worksheets containing geometric riders on the theorem just proved.
6	Yes, about 35 minutes on, Mr Adeleke called one of the learners to write her solution on the chalkboard.	No, all through the lesson, Mr Balogun asked only 'Yes' or 'No' questions, e.g. he would ask: "Are you with me? Am I correct?" And the learners would respond: "Yes Sir".	Yes, about 35 minutes on, Mr Lawal called on a student to come to the front "and demonstrate how you arrived at your answer" on the chalkboard.	Yes, many of the worksheet questions were open-ended and the learners were required to answer them on their own.	Yes, about 40 minutes on, Mr Shlaja asked his students: "what can you say about the opposite angles of a parallelogram?".	No, about 35–37 minutes on, Mr Andile orchestrated the solution strategy by himself, even though the questions were open-ended.
7	No, Mr Adeleke did not invite the learners to challenge the solution offered by their peers, and so, the learners did not elaborate on their solutions.	No, as the learners were not assessed, no room was provided for them to express themselves.	No, Mr Lawal did not encourage the learners to elaborate on their solution.	No, as no feedback was got from the learners before the end of the lesson.	No, as the teacher simply accepted learners' responses as 'right' or 'wrong'.	No, as the learners did not provide any solution to a problem on their own throughout the lesson period.
8	No, by the end of the lesson, Mr Adeleke simply assigned homework and told the learners the topic for the next lesson.	Yes, 45 minutes on, Mr Balogun summarised all that he has taught during the lesson for the learners.	No, Mr Lawal only assigned homework at the end of the lesson.	No, the lesson was brought to an abrupt end as the learners were still working when the change-of-lesson bell was rung.	No, as Mr Shlaja simply assigned homework to the learners at the end of the lesson.	No, 45 minutes on, the students were still doing the classwork when the change-of-lesson bell was rung.

As evident in Table 9.2, two of the lessons delivered by teachers from NS (lessons 1 and 3) and two of the lessons delivered by teachers from SAS (lessons 4 and 6) satisfy criterion 1 on the checklist of van Hiele phase descriptors. These were the lessons in which the teachers, by the very generous definition of this criterion, recognised learners' prior knowledge as a necessary factor in their understanding of the current lesson. However, two of the lessons, Lesson 2 delivered by the NS grade 11 teacher, and Lesson 5 delivered by the SAS grade 11 teacher, did not meet criterion 1 on the checklist. The teachers of these two lessons simply presented the topic for the day's lesson without any attempt at determining what the learners already knew or even reminding them of aspects learned in their previous lessons that could aid their understanding of the present lesson.

Even the teachers who recognised learners' prior knowledge during their lesson did so by merely referring to lessons, topics or concepts which the learners had already been taught. Only about half of these teachers explicitly stated how these earlier concepts related to the topic at hand. And none of them assessed learners' prior knowledge by asking pertinent questions. Mr Adeleke, the grade 10 teacher from NS, for example, started off his lesson by stating that "last week, we discussed about types of triangles" (see the transcript of Lesson 1 in Appendix 8.A, p.130). He then listed the names of the three types of triangles (scalene, isosceles and equilateral) for the learners, stating their properties in each case, rather than asking the learners themselves to state these properties.

Anyone who is familiar with the Nigerian educational system knows that pre-service college teachers are taught, as part of their qualifying programme, to ascertain learners' 'entry behaviour' or prior knowledge at the start of the day's lesson. Most of the teachers who attempted to establish learners' 'entry behaviour' in this study were merely 'paying lip service' to this aspect of the instructional process, even though they all had many years of teaching experience and might have been expected to be more interrogative in this initial stage of the mathematics lesson.

There were only two lessons that met criterion 2 on the checklist. These are Lessons 4 and 5, both of which were delivered by teachers from SAS. In both lessons, the teachers provided an opportunity for the learners to verbally express their understanding of the mathematical idea or concept under discussion. All three lessons from NS (Lessons 1–3) and Lesson 6 from SAS did not satisfy criterion 2 on the checklist. For these four lessons, instruction was whole-class and teacher-dominated with little room provided for the learners to express themselves verbally.

A lesson that satisfies criterion 3 on the checklist of the van Hiele phase descriptors will also have satisfied criterion 2 on the checklist, but not the converse. Asking learners to clarify their imprecise terminology had to follow from giving them the opportunity to express themselves verbally. As can be seen in Table 9.2, only Lesson 5 satisfies criterion 3 on the checklist.

Table 9.2 further indicates that only Lessons 4 and 5 satisfy criterion 4 on the checklist. These are lessons during which the teachers promoted and coordinated healthy classroom interactions. The learners were encouraged to work in small groups and cooperate with one another in their various groups. Instruction during these lessons was activity-based using an investigative approach (see the transcripts of these lessons in Appendices 8.D and 8.E). The remaining four lessons (all three from NS and Lesson 6 from SAS) did not satisfy criterion 4. Instruction during these four lessons was whole-class, with no provision being made by the teachers for learners to work in groups. None of the teachers who delivered these lessons provided any opportunity for the learners to challenge, contest or negotiate the meaning of or solution to a mathematical idea, concept or problem.

Given that criterion 5 only requires the teacher to assess learners' understanding of the central idea being developed in the lesson by assigning questions which are to be solved either by the teacher, as examples, or by the learners, as classwork, it was only Lesson 2 that did not meet this criterion. This suggests that assessment of learners' grasp of the current lesson is a common feature of geometry classroom instructional practices among many teachers in NS and SAS.

Criterion 6 on the checklist of the van Hiele phase descriptors was satisfied by four of the lessons. Two of these lessons (Lessons 1 and 3) were delivered by teachers from NS and the other two (Lessons 4 and 5) were delivered by teachers from SAS. These are lessons during which the teachers asked open-ended questions and encouraged the learners to come up with different solutions to the problems. The learners were left to seek their own solution path to the given mathematical problem during these lessons, with minimal interference from the teacher. In some cases, as was evident in Mr Adeleke's lesson, a learner would be called to come forward and write his/her solution on the chalkboard. Mr Adeleke's apparent intention in doing this was to create lively classroom interaction in which the rest of the learners could challenge, contest or present alternative perspectives on the solution to the given problem. Interestingly though, Mr Adeleke, like his other colleagues from NS, did not provide such an opportunity for the learners, since he failed to invite the learners to engage in such interactive activity.

Criterion 7 on the checklist was the one least satisfied by the lessons delivered by the teachers. In fact, none of the lessons met this criterion (see Table 9.2). In terms of the criterion, teachers are expected to request learners to elaborate on their responses to a mathematical problem or concept. However, the failure of the teachers to invite their learners to challenge, contest, negotiate and/or provide alternative perspectives to each other's solution or idea (see criterion 4) meant that all of the lessons observed fell short of satisfying this criterion.

Despite the very generous definition given to criterion 8 on the checklist (see Table 9.1), only Lesson 2 met this criterion. This was the only lesson during which a summary of what was taught was provided by the teacher towards the end of the lesson. Note, however, that if the criterion were to be strictly defined, then such a summary should have been given by the students and not the teacher (see Chapter 2, section 2.8.2). Even with this level of flexibility in applying the criterion, five out of the six lessons could still not meet it. In all five lessons, the teachers either assigned homework for the learners or asked the learners to complete as homework the classwork which they could not complete before the end of the lesson.

The foregoing analysis centred on each of the checklist criteria across all the six videotaped lessons in this study. The picture that emerges from the analysis is that only three of the checklist criteria were satisfied by four or more of the lessons. This means that the instructional practices in the majority of the lessons failed to conform to the van Hiele model of instruction in the geometry classroom. But what does this lack of conformity tell about geometry instruction in NS and SAS in terms of students' opportunity to learn in the geometry classroom?

The degree of conformity with or deviation from the van Hiele model of geometry instruction as exemplified by the checklist of van Hiele phase descriptors was taken as a measure of the learning opportunities that observed instructional methods offer the learners in geometry classrooms in NS and SAS (section 3.3.4.3). Therefore, given that only a few of the checklist criteria were satisfied by nearly all the observed lessons, it could be argued that in terms of the van Hiele model of instruction, geometry instructional practices in the NS and the SAS offer the learners scant opportunity to learn the subject.

In drawing the above conclusion, no claim that teaching methods in the observed classrooms are 'bad' is made or implied. To be able to make such a claim, one would need to look at the activities that typify instruction separately in the NS and the SAS – the images of teaching in these schools – and I will return to this shortly. In the meantime, an analysis of applying the checklist to the individual lessons is presented. This is done in an effort to highlight the number of lessons that conform to the checklist criteria in each of the participating schools.

As evident in Table 9.2, of the three lessons videotaped in NS, two of them (Lessons 1 and 3) satisfy only three of the checklist criteria (1, 5 and 6), and one of them (Lesson 2) satisfies only criterion 8 on the checklist. This means that in NS, the teaching methods in geometry classroom deviate significantly from the van Hiele model of geometry instruction.

In SAS, Table 9.2 shows that each of Lessons 4 and 5 satisfies five of the eight criteria on the checklist of van Hiele phase descriptors, while Lesson 6 satisfies two of these criteria (1 and 5). This implies that on average, geometry classroom instruction in SAS conforms to the van Hiele model of geometry instruction.

In terms of the learning opportunities that the observed methods offer learners in the geometry classroom, the result in Table 9.2 indicates that, in terms of the van Hiele model, the teaching methods in SAS hold greater opportunities for the learners than the observed instructional methods in NS. It is necessary to stress that this conclusion is based purely on the indicators of the van Hiele model and not on the subjective personal judgement of what constitutes a 'good' and 'bad' instruction. I made the point earlier that no single method of teaching is best for all students and all learning (see section 2.8.4). Nevertheless, learning opportunities are undoubtedly enhanced more by some methods than by others (Stigler & Hiebert, 1999).

Our ascertaining of the extent to which observed instructional methods in geometry classrooms in NS and SAS conformed to the van Hiele model of instruction tells us little about the actual activities that took place during those lessons. To describe the actual teaching processes in these schools, I shall now turn to the concept of 'images of teaching'.

9.4 The images of teaching in geometry classrooms

The phrase 'images of teaching' as used in this study refers to a description that captures the dominant and distinctive activity in classroom instructional processes (see section 3.4.2.2). It was indicated in the section under reference that the purpose of using images of teaching in this study was to extend our knowledge about the nature of instruction in geometry classrooms in NS and SAS beyond an understanding of the extent to which the observed methods in these schools conformed to the van Hiele model of instruction.

If learners are asked to evaluate $\frac{17}{8}$ using a calculator, the possible responses might be 2, 2.1, 2.13 or 2.125. Each of these learners is correct, depending on the number of decimal digits desired. The convention in mathematics, however, is for the teacher to specify the number of decimal digits correct to which each learner is expected to express his/her answer, so that the problem will have only one solution for all the learners. The instance just described is analogous to the images of teaching described in the next few sections. Although I have included the transcripts of the lesson videos in Appendix 8.A–F, pp.130, 136, 148, 155, 161 and 167, different readers of these transcripts may foreground or focus attention on different aspects of these lessons (see, for example, Stigler & Hiebert, 1999).

In order to have a 'common denominator' upon which opinions about the lessons might be based, activities during the lesson have been articulated in themes to facilitate an organised presentation of the images of teaching in evidence. It must quickly be pointed out, however, that since the transcripts of the lessons are included in the thesis, the emergent themes are discussed only briefly in this chapter. The themes around which the images of teaching are foregrounded in the lesson videos of NS and SAS are: exchange of greetings; introducing the day's lesson; the body of the lesson; review of the day's lesson; and assigning homework.

9.4.1 The images of teaching in NS geometry classroom

Before presenting the themes in terms of which the images of teaching in NS are organized, I will briefly report a few of the comments made by two of my three critical readers that were involved in the trial application of the checklist of van Hiele phase descriptors to the transcribed lesson videos (see section 9.1, para.4). The purpose here is not for me to bias the mind of the reader against the teaching processes in these lessons, but rather to demonstrate further that the images of teaching reported for each of the participating schools were not based on my own subjective judgement alone.

I had just asked the Nigerian among my three participating colleagues to comment on the teaching methods in the lesson whose transcripts he had just read. This is what he had to say about teaching in NS:

Ah! I think we should start videotaping our own lessons so that we can see for ourselves how badly we teach. Maybe, I teach equally as bad, but because I don't get to see myself, I do not know. My God! These teachers were just running commentaries throughout the lesson, particularly the grade 11 teacher.

The Namibian master's student and college of education lecturer (see section 9.1, par.4), on a different day and separate occasion, also made the following comments about teaching processes in NS lessons:

I find it difficult to analyze some of these lessons using the criteria on this checklist. The teaching method of teachers from NS does not conform to many of the criteria on the checklist. I think all of them were simply lecturing. In fact, from what I read on these transcripts, the approach they used is highly teacher-centred.

Were these colleagues of mine unjustifiably critical about the teaching processes in NS geometry classrooms? I will now turn to the themes concerning the images of teaching in NS for a possible answer. In the discussion that follows, Appendix 8.A–C, pp.130, 136 and 148 should be consulted for all references made to any of the lessons.

9.4.1.1 Exchange of greetings

In the NS (as in many other public high schools in Nigeria), the learners have permanent classrooms where the teachers go to meet and teach them in turn. The teachers usually stay in the staffroom or in their respective offices when they have no class to teach.

When the teacher enters the class, the learners stand up and greet:[15] "Good morning Sir". The teacher responds: "Good morning". The learners then settle down and the lesson is now officially underway. But is this exchange of pleasantries between the learners and the teacher at the beginning of the lesson of any significance to the classroom instructional processes?

My observation of NS videotaped lessons suggests that this exchange of greetings has a classroom management function. It appears to be a strategy by which the teacher subtly exercises control over the learners. The learners who have been chatting away become quiet soon after the exchange of greetings. They seem to have an implicit understanding that with the exchange of greetings the lesson is about to begin, and that any learner who makes a noise thereafter might interfere with the lesson.

9.4.1.2 Introducing the day's lesson

Teachers in the NS (e.g. Mr Adeleke and Mr Lawal) tend to introduce the day's lesson by recapping on the previous lesson or topic. But the introduction grossly

[15] In some cases, e.g. Mr Balogun's lesson, it is the teacher that greets first and the learners would then respond.

underplays the significance of determining learners' prior knowledge about the current topic (see sections 2.7.4.3 and 2.8.4). Mr Adeleke, for example, introduces the day's lesson by stating for the learners what they had earlier "discussed about types of triangles". Mr Adeleke, like his other colleagues, does not ask questions that would enable him to ascertain what the learner already knows.

The introduction of the lesson, and indeed, the entire lesson delivery in NS, is based on the transmission–absorption model of behaviourism (see Chapter 2, section 2.8.4, para.2). The teacher directly introduces the lesson by 'narrating' to the learners the work that they did in their previous lesson. Some of the references made, during the introduction, to the work done in the previous lesson, only have remote connections with the current topic. For example, how would a learner's knowledge of types of triangle assist him to prove that the angle sum of a triangle is 180°? (See Mr Adeleke's lesson during the first 5 minutes.) Exploring learners' prior knowledge of the angle properties of parallel lines and transversals and straight line angles is clearly more related to the proof of the angle sum of a triangle than exploring their knowledge about types of triangles.

9.4.1.3 The body of the lesson

By the phrase 'the body of the lesson' I mean all the instructional activities during the lesson that exclude the introduction, review of the day's lesson, and the assigning of homework. In other words, 'the body of the lesson' refers to all the instructional activities that lie between those that mark the beginning of the lesson (i.e. the introduction) and those that mark the end of the lesson (i.e. review of the day's lesson and assigning of homework). The body of the lesson in NS is described in terms of the following subthemes: presenting the concept for the day; lesson organisation; and who does the work? Although other subthemes like lesson cohesion, type of tasks given, making connections across topics etc. are important components of the body of the lesson (see Stigler and Hiebert, 1999), the three chosen for reporting in this study appear to be the most pertinent to making decisions about the success of the lesson in terms of student learning.

Presenting the concept for the day: Stigler and Hiebert (1999) distinguish between two forms of concept presentation during a lesson: one in which the teacher just *states* the concept for the learners, and the other in which the teacher *develops* the concept with the learners. If, for example, the teacher tells the learners that the sum of the angles of a triangle is 180° in a lesson whose focus is the proof of the angle sum of a triangle, the teacher has merely *stated* the concept. On the other hand, if for a lesson with a similar focus, the teacher guides the learners to first establish, through measurement, that the angle sum of a triangle is 180° before carefully leading them to write the proof, then the teacher has *developed* the concept. The view expressed by Stigler and Hiebert (1999) is that lessons in which concepts are *developed* hold greater opportunities for the learners to learn mathematics than lessons in which concepts are just *stated*. What form of concept presentation is evident in the lessons delivered by NS teachers?

Just after the brief introduction in NS lessons, the teacher announces the concept or topic for the day orally and then writes it on the chalkboard. Mr Adeleke, for example, announces to his learners that "today we are going to look at how we can prove that the sum of a triangle, ok, is equal to 180°". If it is a theorem to be proved, as was the case with Mr Adeleke and Mr Lawal's lessons, the teacher then goes ahead to write the proof on the chalkboard all by himself, asking only superficial, corroborative 'Yes' or 'No' questions of the learners. The teacher emphasizes the steps for writing the proof to the learners, who are expected to simply copy the proof from the chalkboard. Clearly, this approach 'fits' Stigler and Hiebert's notion of *stating* a concept during lesson presentation, and by implication, it offers little opportunity for the learners to learn mathematics. Had the teacher allowed the learners to find out for themselves that the angle sum of a triangle is 180° before guiding them, through the asking of pertinent, thought-evoking questions, to write the proof, he would have indeed *developed* the concept. In sum, in NS classrooms, concepts are merely *stated* rather than *developed*.

Lesson organization: The way a lesson is organized provides a context within which the teacher engages the learners in learning the concept or subject (Stigler & Hiebert, 199). Generally, class time during the lesson could be divided into periods of *classwork* and *seatwork* (Doyle, 1983). *Classwork* is when the teacher is working with all learners and, usually, orchestrating classroom discussion (Stigler & Hiebert, 1999). Activities during *classwork* include learning a new concept, reviewing a previously learned concept, demonstrating a solution strategy, or solving a problem together. In fact, activities during classwork are usually teacher-led and are based on whole-class discussion or direct instruction. *Seatwork*, according to Stigler and Hiebert (1999, p.67), is "the time when students work individually or in small groups on assigned tasks. Talk is mostly private – teacher-student or student-student". According to Doyle (1983, p.189), "research on effective teaching has generally indicated that … high levels of student [task] engagement are associated with high [academic] achievement". There is evidence from research that tends to indicate that in effective classroom teaching – teaching of a kind that offers learners a real opportunity to learn the subject – approximately 60 to 70 percent of class time is usually spent in seatwork (Doyle, 1983).

Of the three lessons videotaped in NS, about 80 percent of class time was spent in classwork in two of the lessons (those of Mr Adeleke and Mr Lawal), while 100 percent of class time was spent in classwork in Mr Balogun's lesson. The image of teaching created here is that in NS, nearly all the class time is spent in classwork in which the teacher orchestrates classroom discussion in teacher-led whole-class instruction. Less than 10 percent of class time was spent in *seatwork* in NS lessons.

Few who are familiar with general classroom teaching in Nigerian public high schools would express much surprise that the amount of class time spent in classwork surpasses that spent in seatwork in NS. What may be startling, though, is the extent of the difference between the times spent in classwork and seatwork. But even this is not necessarily unique to Nigeria, since according to Stigler and Hiebert (1999, p.67), "in Japan and the United States 60 percent of the [class] time was spent in classwork; in Germany it was 70 percent".

Knowing how much time was spent in classwork and seatwork is of little significance unless we know what happens during these times. We need to know who is doing the mathematical work during classwork and seatwork.

Who does the work? There appears to be consensus among the majority of educators that learning opportunities are enhanced when learners do most of the mathematical work during the lesson (Doyle, 1983; Stigler & Hiebert, 1999). However, merely looking at whether things are being done during classwork or seatwork is not sufficient. For example, during classwork the teacher is often doing the work, but might orchestrate the discussion so that the learners are required to do some of it themselves. On the other hand, during seatwork the learners are often doing the work, but might be assigned tasks for which the teacher has already done the mental work. In order to measure more accurately who is doing the mathematical work, Stigler and Hiebert (1999) suggest that one needs to look at who controls the solution method to a given mathematical problem. In their view, if the teacher suggests a solution method or quickly demonstrates a procedure for solving the problem, then he is doing most of the work. If, however, the teacher guides the learners to invent their own solution strategies then the learners are doing the mathematical work.

As stated earlier, nearly the whole class time is spent in classwork in NS lessons. During this period the teacher *states* (as against *developing*) the concept and quickly demonstrates the application of the concept to solving mathematical problems (see Mr Adeleke and Mr Lawal's lesson transcripts in Appendices 8.A and 8.C). The teacher then assigns a similar problem to the learners to do on their own during seatwork. Hence, another image of teaching evident in the NS classrooms is that during the lesson the teacher does more of the mathematical work than the learners. This implies that teaching methods in the NS, as observed in the lesson videos, do not offer as much opportunity for the learners to learn mathematics as they might.

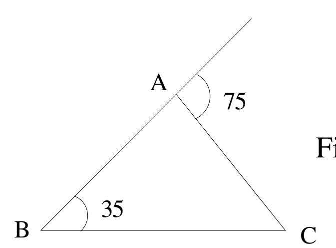
9.4.1.4 Review of the day's lesson

An important instructional activity that typifies the ending point of a lesson, according to the van Hiele phases of learning, is for the learners to summarize the main ideas learned during the lesson (see section 2.8.4.1, and Pegg, 1995). Only one of the three lessons videotaped in the NS provided an overview of the day's lesson at the end of the lesson. Mr Balogun's lesson was the only one in which the main concepts learned in the lesson were summarized at the end. But Mr Balogun himself provided the summary, rather than asking the learners to do so. For example, towards the end of the lesson, 45 minutes on, Mr Balogun says: "so, let's go back to where, where we started from. We talked about em, definition of triangle, which I said that a triangle is … Then from there we moved down to the types of triangles that we have … Then from there we talked about the properties of triangles and I said …" (see Mr Balogun's lesson in Appendix 8.B, p.136).

The significance of allowing the learners to provide an overview of the lesson is that it affords the teacher insight into what aspect of the day's lesson was not properly understood by the learners. It also offers the learners the opportunity to reflect on what they have just learned. So, another image of teaching evident in NS lessons is that the learners are not given the opportunity to reflect on and summarize their ideas about the concept learned.

9.4.1.5 Assigning homework

Assigning homework to the learners forms part of the instructional activities that mark the end of the lesson in the NS. Just after the classwork, Mr Adeleke, for example, says to his students: "Now, I want you to go home with this … I want you to go home with this and find angle A and angle C" (see Mr Adeleke's lesson in Appendix 8.A, p.130). He then writes the following problem on the chalkboard.



Find $\hat{A}$ and $\hat{C}$

It is important to note that although nearly all the teachers in the NS assign homework to the learners, there is no evidence in their lessons that learners' solutions to the assigned homework are checked in the next day's lesson. With the assigning of homework, one would expect that checking of learners' responses would form part of the starting activities or introduction to the day's lesson. As this was not the case in NS lessons, it would seem that another image of teaching in NS classrooms is that the teacher assigns homework to the learners during the lesson, but this assignment is taken as an end in itself. That is, the teacher does nothing with the homework that he assigns the learners.

9.4.2 The images of teaching in SAS geometry classroom

As was done with the NS lessons (see section 9.4.1), I also asked my three colleagues (see section 9.1, para.4) who were involved in the trial application of the checklist of van Hiele phase descriptors to the transcribed lesson videos to comment on the teaching methods in SAS as revealed by the videotaped lessons. The second-year PhD student from Namibia in the Education department of my university has this to say about teaching in SAS:

As I read through the transcripts of the lessons, I found a lot of similarities between some of these lessons and the way teaching is done in my country. Almost all the teachers from SAS used an approach that is learner-centred in that teaching in these lessons is activity-based, and that is how we teach in our country. The learners are allowed to work in small groups while the teacher only mediates between and coordinates interactions within and between the groups.

Does the above comment from my colleague accurately capture the form that teaching takes in SAS? It may be that my colleague has exaggerated the situation a little, but the picture that is portrayed is quite clear: Teaching in SAS is activity-based and learner-centred. To confirm this, one could, of course, read the transcripts of the lesson videos in Appendix 8.D–F, pp.155, 161 and 167, assuming that the reader already knows what the indicators of a learner-centred approach to classroom teaching are (see, for example, Howie, 2004 for indicators of a learner-centred approach). In order not to leave the description of the form that teaching assumes in SAS to the individual reader's subjective judgement, I will now turn to the themes emerging from the images of teaching, as was done for the NS lessons (see sections 9.4.1.1 through to 9.4.1.5).

9.4.2.1 Exchange of greetings

As was stated in the last paragraph of section 3.3.1.2 of Chapter 3, in SAS (as with many State 'township' high schools in the Eastern Cape), the learners do not have permanent classrooms. As a result, during each change-of-lesson time, the learners move to another classroom where the next lesson is scheduled to take place. However, the teachers, in addition to having desks in the staffroom, also have a permanent classroom where the learners go to meet them during each lesson period. This situation whereby the learners move from one class to the other during each change-of-lesson time has important implications for the use of class time and the exchange of greetings between the learners and the teacher.

Just before the lesson period, the teacher is busy arranging the class for the next lesson. Soon, the learners begin to file into the class, chatting away. As they enter into the class, the learners settle on their seats either individually or in small groups of two or four according to the arrangement which the teacher has made of the seats. In some instances the teacher directs the learners to specific seats. Meanwhile, the learners are busy chatting to one another, and valuable class time is ticking away. In some instances, the teacher simply begins his lesson while the learners are still chattering, without exchanging pleasantries with them. In short, of the three lessons videotaped in SAS, it was only in Mr Andile's class that there occurred an exchange of greetings between the teacher and the learners. But what is the significance of this exchange of greetings to the classroom instructional process?

As with NS lessons (see section 9.4.1.1), the exchange of greetings has a classroom management function in SAS. What I observed in the lesson videos of SAS is that in the class where an exchange of greetings occurred between the teacher and the learners (i.e. in Mr Andile's class), the learners settled down for the day's lesson much more quickly than in classes where such an exchange of greetings did not take place. My impression about teaching in the SAS with regard to exchange of greetings between the teacher and the learners is that, because the learners file into the class at different times, often such exchange of greetings does not take place. But when it does occur, it has the same classroom management function of settling down the learners quickly as it has with NS.

9.4.2.2 Introducing the lesson

Teachers in SAS (e.g. Mr John and Mr Andile) tend to introduce the day's lesson by explicitly interrogating learners' prior knowledge. Mr John, for example, having told the learners that the current lesson relates to their previous year's work on triangles, then asks the learners to state what a triangle is as a means of assessing their entry knowledge. Similarly, just before writing the proof of the theorem which formed the main focus of the day's lesson, Mr Andile interrogates his learners' knowledge of congruency necessary for the proof of the theorem.

In SAS the introduction of the lesson and indeed the entire lesson delivery is based on a constructivist approach to teaching (see section 2.8.4). Mr Andile, for example, does not just 'narrate' to his learners (as some teachers did in NS lessons) what knowledge of congruence is needed for the current lesson. Rather, he carefully guides the learners, through prompts, to state the conditions for congruency needed for the proof of the theorem. As noted by my colleague referred to earlier on, teaching in SAS tends to be learner-centred and activity-based. Just after the brief introduction, Mr John, for example, tells the learners that "I've got some worksheets for you to complete".

It is also important to point out here that during the introduction of the lesson in SAS, references made to the work done in their previous lessons were more closely related to the current lesson than was the case with NS lessons (see section 9.4.1.2). The image of teaching here is that during the introduction of the lesson in SAS, the teacher interrogates learners' prior knowledge and relates it to the current topic being taught.

9.4.2.3 The body of the lesson

As with NS lessons, the body of the lesson in SAS is described under these subthemes: presenting the concept for the day; lesson organisation; and who does the work?

Presenting the concept for the day: Following a brief introduction in SAS lessons, the teacher announces the concept or topic for the day orally and circulates around the classroom, distributing worksheets to the learners. Mr Shlaja, for example, says to his learners that "today's lesson is an introductory one. It is about the properties of parallelograms". The teacher then distributes worksheets which he had prepared for the day's lesson to the learners. When measurements and/or constructions are required (as was the case with Mr John and Mr Shlaja's lessons), the teacher distributes measuring and/or construction tools to the learners.

Following the distribution of the worksheets and measuring/construction tools, the teacher tells the learners what is expected of them. For example, having given worksheets on parallelograms to his learners, Mr Shlaja says to them: "Now I want you to measure AB. Measure AB and you write it down. When you finish measuring AB, you measure CD, then after measuring that, measure AC and BD" (see Mr Shlaja's lesson in Appendix 8.E, p.161). As the learners begin to do the activities on the worksheets, either individually or in groups, the teacher moves around the classroom answering their questions, correcting their errors and giving guidance to them. If the teacher identifies a common error in learners' work, he resorts to whole-class instruction by making use of the chalkboard to explain the problem to the learners.

While the learners are working on a problem, the teacher does not quickly demonstrate a procedure for solving the problem. The learners are first allowed to develop their own solution strategy before being assisted where necessary. Mr Shlaja, for example, having made the learners measure and record the angles of the parallelogram on the worksheet, then poses this question: "What can you say about the opposite angles of a parallelogram?" The learners answer correctly that "they are equal" and Mr Shlaja writes this on the chalkboard as one of the properties of a parallelogram. By asking similar questions, Mr Shlaja guides the learners to develop a secure knowledge of the properties of parallelograms, rather than just informing them about them.

The instructional procedures described in the preceding three paragraphs clearly 'fit' into Stigler and Hiebert's notion of *developing* a concept during lesson presentation (referred to earlier in section 9.4.1.3). Hence, another impression or image of teaching evident in SAS geometry classroom is that concepts are *developed* (through investigative activities) and not just *stated*. Given Stigler and Hieber's (1999) assertion that learners' opportunities to learn are enhanced in lessons where concepts are *developed* than in lessons where they are merely *stated*, what this means, in relative terms, is that learners in SAS have greater opportunities to learn geometry than their counterparts from NS.

Lesson organization: As with NS lessons, class time in SAS was divided into periods of classwork and seatwork. However, instruction in these two schools differs significantly in terms of the amount of class time that is spent in classwork and seatwork. Far more of class time was spent in seatwork in SAS lessons than in NS lessons.

In two of the three lessons videotaped in SAS (those of Mr John and Mr Shlaja), between 50 to 60 percent of class time was spent in classwork. In these lessons the learners worked in small groups of two or four on the assigned tasks, using an investigative approach. Whole-class instruction was minimal. However, in Mr Andile's class over 70 percent of class time was spent in classwork, and discussion was mostly whole-class and teacher-led. The image of teaching created here is that in SAS, more class time is spent in classwork than in seatwork, especially for lessons where instruction is whole-class and teacher-led.

Who does the work? In both classwork and seatwork, the SAS learners appear to be doing more of the mathematical work than the teacher. In Mr Shlaja's lesson, for example, during seatwork, the teacher circulates around the classroom, only giving guidance to the learners who are working on their own. And during classwork, the teacher asks probing and thought-provoking questions that lead the learners to provide answers to the mathematical problem at hand.

There was, however, one exception to the claim that the learners did more of the mathematical work during classwork and seatwork in SAS classrooms. In short, in Mr Andile's lesson, it is the teacher who does nearly all the mathematical work during both classwork and seatwork. Although Mr Andile did ask questions that could help the learners to gain ownership of the mathematics being learned, all too often he answers these questions himself without allowing enough time for the learners to think more critically about the problem and come up with their own solution. Was he trying to impress the researcher? Nevertheless, given the kinds of questions that Mr Andile asked during the lesson, one would agree that the whole lesson structure was designed for the learners to do more of the mathematical work. Hence, in general, another image of teaching in SAS classrooms is that during the lesson the learners do more of the mathematical work than the teacher. What this means, in relative terms, is that teaching methods in SAS, as observed in the videotaped lessons, offer greater opportunity for the learners to learn mathematics than the NS methods.

9.4.2.4 Review of the day's lesson

In all three lessons videotaped in SAS, the teachers neither provided an overview of the day's lesson nor did they ask the learners to do so. None of the lessons ended 'smoothly', since the learners were often still working on assigned tasks when the change-of-lesson bell rang. The teachers tended to include more activities in their lessons than available lesson time could accommodate. As a result no provision is made for either the teacher or the learners to summarize the lesson at the end of the lesson period. Therefore, another image of teaching evident in SAS geometry classrooms is that the learners are not provided the opportunity to reflect on and summarize key ideas learned in the day's lesson.

9.4.2.5 Assigning homework

As with NS lessons, assigning homework to the learners forms part of the instructional activities that mark the end of the lesson in SAS. The assigned homework in SAS is often part of the task which the learners were not able to complete during the lesson period. Towards the end of his lesson, Mr Andile, for example, says to the learners who had been working on an assigned task that "you will finish that as homework" (see Mr Andile's lesson in Appendix 8.F, p.167).

It must be mentioned here too that as with NS, although SAS teachers assign homework to the learners, there is no evidence that learners' responses to the assigned task are checked in the next day's lesson. Hence, another image of teaching in SAS classrooms is that the teacher assigns homework to the learners during the lesson, but subsequently does nothing with that homework.

9.5 Chapter conclusion

This chapter presents the results of the analysis of the instructional methods in geometry classrooms in NS and SAS. For this analysis, the van Hiele model of geometry instruction was used as a lens. Each of the six videotaped lessons was analysed first to determine its conformity with criteria on the checklist of van Hiele phase descriptors developed for the purpose. Next, the images of teaching evident in NS and SAS lessons were described. The major findings include the following.

- Overall, only three of the eight criteria on the checklist of van Hiele phase descriptors were satisfied by four or more of the six lessons. This means that instructional processes in the majority of the lessons videotaped in NS and SAS did not conform to the van Hiele model of instruction in the geometry classroom. The conclusion reached was that, in terms of the van Hiele model of geometry instruction, the observed geometry instructional practices in NS and SAS classrooms do not offer learners much of an opportunity to learn the subject.

- Of the three lessons videotaped in NS, two of them (Lessons 1 and 3) satisfy only three of the checklist criteria (criteria 1, 5 and 6), and one of them (Lesson 2) satisfies only criterion 8 on the checklist. The conclusion was reached that in NS, the teaching methods in the geometry classroom deviate significantly from the van Hiele model of geometry instruction, and hence observed methods were interpreted as offering learners scant opportunity to learn the subject.

- In SAS, Lessons 4 and 5 satisfy five of the eight criteria on the checklist of van Hiele phase descriptors, and Lesson 6 satisfies two of the checklist criteria (see Table 9.2). The conclusion reached was that on average, geometry classroom instruction in SAS conforms to the van Hiele model of geometry instruction. Hence, in terms of the van Hiele model, observed teaching methods in SAS hold greater opportunities for the learners to learn the subject than observed instructional methods in NS classrooms.

Beyond describing how observed instructional methods in the participating schools did (or did not) conform with the van Hiele model of geometry instruction, this chapter also gives a description of certain images of teaching evident in the observed classrooms. The findings concerning these images of teaching are as follows:

- **Exchange of greetings**: Just before the lesson starts, there is usually an exchange of greetings between the teacher and the learners in NS classrooms. In SAS, this exchange of greetings is often skipped. Whenever an exchange of greetings occurs, whether in NS or in SAS lessons, it has the classroom management function of settling down the learners quickly for the day's lesson.

- **Introducing the day's lesson**: During the introduction of the lesson in NS classrooms, the teacher merely recaps on the previous lesson or topic. The teacher neither interrogates learners' prior knowledge nor does he relate learners' prior knowledge to the current topic. In SAS, during the introduction of the lesson, the teacher explicitly interrogates learners' prior knowledge and relates it to the current topic or concept being taught.

- **Presenting the concept for the day**: Another image of teaching evident in the observed lessons is that in NS classrooms, concepts are merely *stated* rather than *developed*. In SAS classrooms, concepts are *developed* and not just *stated*. What this means, in relative terms, is that learners in SAS have greater opportunities to learn geometry than their counterparts from NS (see Stigler & Hiebert, 1999).

- **Lesson organization**: In NS, about 80 percent of class time was spent in classwork in two of the three lessons, while 100 percent of class time was spent in classwork in the remaining lesson. In SAS, between 50–60 percent of class time was spent in classwork in two of the three videotaped lessons, while about 70 percent of class time was spent in classwork in the other lesson. The conclusion reached was that although in both NS and SAS classrooms the total class time is usually divided into periods of classwork and seatwork, in NS lessons the teacher spends a much greater percentage of class time in classwork than in seatwork, whereas in SAS lessons class time is shared almost equally between classwork and seatwork.

- **Who does the work**? In NS, the teacher does more of the mathematical work than the learners during the lesson. In SAS, it is the learners who do more of the mathematical work than the teacher. The conclusion reached was that in relative terms, observed teaching methods in SAS offer a greater opportunity for the learners to learn mathematics than NS methods.

- **Review of the day's lesson**: In both NS and SAS lessons, the learners are not given the opportunity to reflect on and summarize their ideas about the concept learned.

- **Assigning homework**: Lastly, another image of teaching evident in the lesson videos is that in both NS and SAS classrooms, the teacher assigns homework to the learners during the lesson, but this assignment is taken as an end in itself as the teacher does nothing with the homework once it has been done.

As a final word in this chapter, the findings from the classroom video study indicate that observed instructional methods in SAS geometry classrooms offer greater learning opportunities for the learners in the subject than observed teaching methods in NS classrooms. These findings could explain why learners from SAS outperformed their counterparts from NS on all the tests used in this study, as reported in Chapters 4 through 8. This was the same conclusion reached by Stigler and Hiebert (1999) when they used classroom video study to explain why Japanese school children usually outperformed many of their counterparts from other countries like Germany and the U.S. in TIMSS. Stigler and Hiebert (1999) concluded that learners from countries like Japan where the teaching methods offer greater opportunities for learning mathematics outperformed their peers from countries where the teaching methods tended to limit learners' mathematics learning opportunities.

Given the findings of this study, it would seem that learners whose instructional experiences align approximately with the van Hiele phases of learning are most likely to demonstrate a better understanding of geometric concepts than their counterparts whose geometry classroom instructional experiences deviate significantly from the van Hiele model. It would be a worthwhile venture for both prospective and practising teachers who seek to enhance their instructional practices and thus promote their learners' geometric understanding to try out the van Hiele model of instruction in their classrooms.

The study is concluded in the chapter that follows.

CHAPTER TEN

CONCLUSION

10.1 Introduction

This study aimed both to explore and explicate the van Hiele levels of geometric thinking of grade 10, 11 and 12 learners in Nigerian and South African schools. Furthermore, it aimed to provide a rich and in-depth description of the geometry classroom instructional practices that might have contributed to the levels of geometric conceptualization exhibited by the sample cohort of high school learners. The latter aim was conceived in order to assess what learning opportunities instructional methods currently being used might offer learners of geometry.

The research reported in this study is a case study. It is oriented in an interpretive paradigm and it utilizes both qualitative and quantitative methods. Employing both purposive and stratified sampling techniques, the study involved a total of 144 high school learners and 6 mathematics teachers: 72 learners and 3 teachers from a State-owned high school in Ojo LED in Lagos State, Nigeria, and 72 learners and 3 teachers from a comparable 'township' high school in the Makana Educational District in the Eastern Cape, South Africa.

The study was structured into three phases, with each phase addressing one of three major goals. Phase 1 concerned determining the van Hiele levels of geometric thinking of the participating learners. Phase 2 explored the possible correlations between the participating learners' van Hiele levels and achievement in 'general' mathematics, and phase 3 examined instructional methods in the geometry classrooms of the participating schools. Many of the findings in this study are discussed in Chapters 4 through 9, in the course of which they are related to the literature review of Chapter 2. In the section designated "chapter conclusion" at the end of each of those chapters, a summary of the relevant research findings is presented. This chapter provides an overall summary of all those findings and, in particular, addresses the following:

- Review of the research goals and the research questions;
- A summary of the findings of greatest significance from the study;
- Significance of the study;
- Limitations of the study;
- Areas for future research;
- Implications and recommendations for teaching and learning;
- A final word of personal reflection.

10.2 Review of the research goals and the research questions

This study sought to achieve three main goals. As stated in the preceding section, each of the three phases into which this study is structured coincides approximately with one of the main goals of the study. In pursuance of these goals, answers were sought to three major research questions. In section 10.2.1, a review of the main research goals is presented, and in section 10.2.2, a review of the research questions.

10.2.1 Review of the research goals

In Chapter 1, the three main goals of this study were articulated as follows:

1. To explore and determine the van Hiele levels of geometric thinking of selected grade 10, 11 and 12 learners in Nigeria and South Africa;
2. To explore and explicate the possible correlations that might exist between the van Hiele levels and general mathematics achievement of the participating learners;
3. To provide information on geometry teaching in selected Nigerian and South African high schools, and hence to elucidate what possible learning opportunities observed instructional methods offer learners in geometry classrooms.

The first goal

This goal was addressed in phase 1 of this study (see section 3.3.4.1) and was achieved by employing both pen-and-paper tests (the TPGT, CPGT and the VHGT – see sections 3.3.4.1.1, 3.3.4.1.3 and 3.3.4.1.4) and a hands-on activity test (the GIST – see section 3.3.4.1.2). In a nutshell, the goal was achieved largely through the administration of the VHGT, a test adapted from Usiskin's (1982) CDASSG van Hiele geometry test. The other three tests (i.e. the TPGT, GIST and the CPGT), though providing useful insight into learners' levels of geometric understanding, were employed mainly in pursuance of the second goal of this study. The VHGT proved to be a useful and an effective van Hiele levels determination test, and the findings relating to learners' van Hiele levels in this study (see Chapter 7, especially sections 7.2.3.1 through 7.2.3.4) were, on the whole, consistent with those of Usiskin (1982) and Siyepu (2005).

The second goal

This goal was addressed in phase 2 of this study (see section 3.3.4.2), and was achieved by computing the correlation coefficients between learners' VHGT scores and each of their SEM, TPGT, CPGT and GIST scores (see Chapter 8, especially sections 8.2.1 through 8.2.6). Generally speaking, the correlation between the van Hiele levels and the general mathematics achievement of the participating learners was determined by computing the correlation coefficient between learners' VHGT scores and their SEM scores. In relating the van Hiele levels specifically to learners' knowledge of school geometry, learners' VHGT scores were correlated with each of their TPGT, CPGT and GIST scores. For each of the participating schools, the results obtained for these correlations (see sections 8.2.2 and 8.2.3) were, in general, consistent with those of Usiskin (1982).

The third goal

This goal furnished the aim of phase 3 of the study (see section 3.3.4.3). It was achieved by engaging in video study of geometry classroom instruction in the participating schools (see Chapter 9). The van Hiele model of geometry instruction provided the overall perspective and analytic lens for the explication of instructional practices in geometry classrooms in the participating schools (see sections 2.8.2, 2.8.4 and 2.8.4.1). The general procedures for studying these classrooms and for reporting the findings were modelled after Stigler and Hiebert's (1999) TIMSS video study of instructional methods in mathematics classrooms in Germany, Japan and the U.S (see section 3.3.4.3). In terms of the learning opportunities that observed instructional methods offer the learners in geometry classrooms, the findings from this study (see section 9.5, second last paragraph) were consistent with those of Stigler and Hiebert (1999).

10.2.2 Review of the research questions

This study addressed three major research questions. As stated in Chapter 1, these questions were not necessarily intended to set limits on what the study aimed to achieve. Instead, they were intended mainly to provide a sharper focus for realizing the broader goals of the study. The three research questions as articulated in Chapter 1 are as follows:

1. What van Hiele level of geometric thinking do selected grade 10, 11 and 12 learners attain by the end of the study year in their respective grades?
2. How does a learner's van Hiele level of geometric thinking correlate with his/her achievement in school mathematics generally and in school geometry specifically?
3. What learning opportunities are evident in selected observed geometry classroom instructions in the participating schools?

The first question

The purpose of this question was to address the first goal of this study, and the VHGT was especially designed to realize this goal. But the findings from the VHGT alone could provide only a part of the general picture of learners' understanding of school geometry. Hence, the other three tests (i.e. the TPGT, CPGT and the GIST) were administered to complement the results from the VHGT and thus provide a more comprehensive picture of learners' levels of geometric understanding. Therefore, answers to this research question were sought not only from the results of learners' performance in the VHGT, but also from the results of their performance in the TPGT, CPGT and the GIST. While the VHGT enabled the determination of the van Hiele levels of the participating learners, the TPGT, CPGT and the GIST, in addition to providing a context for comparing (or correlating) learners' exhibited van Hiele levels with their performance in other aspects of school geometry, also provided a context for extending our insight into the extent of participants' knowledge of school geometry.

The point made in the preceding paragraph is that to provide a specific answer to the first research question would tend to limit and impoverish the very rich and robust findings pertaining to the first research goal of this study. Consequently, the (summary of) findings from this study concerning the first research question, and indeed the other two questions, are presented not only in terms of specific answers to these questions, but also in terms of the broader research goals as described above.

The second question

This research question relates to the second goal of this study. Hence, in conjunction with the second research goal, answers were sought to this question by making various correlations and comparisons between participating learners' van Hiele levels (using their VHGT scores) and their performance in each of the SEM, TPGT, CPGT and the GIST (see Chapter 8). These correlations were calculated at three different levels – at the entire study sample level (see section 8.2.1), at the participating schools level (see sections 8.2.2 and 8.2.3) and at each grade level (see sections 8.2.4 through 8.2.6). In sections 8.3.1– 8.3.3, further comparisons were made in order to determine whether learners at adjacent van Hiele levels performed significantly differently from each other in the SEM, TPGT, CPGT and GIST.

The third question

This research question relates to the third goal of this study, and answers to it are contained in Chapter 9, where the results of the classroom video study are presented and discussed.

10.3 Summary of findings

The findings of this study have been treated as integral to the research process and data narrative. Consequently, together with the literature reviewed in Chapter 2, many of the findings were presented in the chapters, 4 to 9, which dealt with the data analysis, results and discussion. A summary of the findings of greatest significance to this study is presented in this chapter, organized in accordance with the three main goals of the study.

10.3.1 Summary of the findings relating to the first research goal and its associated research question

The summary of findings presented in this section is articulated in terms of the following:

- The TPGT and learners' performance;
- The GIST and learners' performance;
- The CPGT and learners' performance;
- The VHGT and learners' performance.

10.3.1.1 The TPGT and learners' performance

The entire study sample

- Learners in this study had inadequate knowledge of basic geometric terminology, since they were only able to obtain an overall percentage mean score of 44.17% for the TPGT, despite the fact that the items that constituted this test were largely of van Hiele level 1 nature.

- There were positive correlations between participants' ability in verbal geometry terminology tasks (see odd-numbered items of the TPGT) and their ability in visual geometry terminology tasks (see even-numbered items of the TPGT). However, these learners tended to have a better understanding of geometric terminology presented through visual tasks than that presented in verbal form, obtaining higher mean scores in the former than in the latter (see section 4.3.1).

- In this study, learners had a better knowledge of the geometric terminology associated with the concept of lines and angles than of the terminology associated with concepts of circles, triangles and quadrilaterals. Consistent with Clements and Battista (1992), the terminological concept of a right angle was the most familiar to and therefore popular with learners in this study, while perpendicular lines and supplementary angles were among the least popular (see sections 4.4.1–4.4.3).

School differences

- Learners from the SAS subsample performed significantly better than their NS counterparts in the TPGT, since there was a statistically significant difference between the mean score of the former (47.85%) and that of the latter (40.49%) at the 0.05 confidence level (see section 4.2.2).

- Grade level differences in the mean scores of grade 10 and 12 learners on the TPGT favoured learners from the SAS subsample and were statistically significant. The difference in the mean scores of NS grade 11 learners and SAS grade 11 learners in favour of the former was not statistically significant (see sections 4.2.3 and 4.2.4).

Gender differences

- For the entire study sample, the male learners performed significantly better than the female learners in the TPGT (see section 4.2.5.1). At each school level, differential gender performances in the TPGT also favoured the male learners. However, it was only in the NS subsample that these differences were statistically significant (see section 4.2.5.2 and 4.2.5.3).

- In this study, SAS male and female learners obtained higher mean scores for the TPGT than their respective male and female peers from the NS subsample. The differences in the mean scores were, however, statistically significant only for the female learners (see sections 4.2.5.4 and 4.2.5.5).

10.3.1.2 The GIST and learners' performance

The entire study sample

- On the whole, learners' performance in the GIST was poor, since they were only able to obtain an overall mean score of 40.12% for the test, despite the fact that learners who reason even entirely at van Hiele levels 1 and 2 should be able to cope with it (see section 5.2). This suggests that only a few of the learner-participants were at van Hiele levels 1and 2 or higher.

- In the identifying and naming shapes task (Task 1 of the GIST), the majority of the learners made use of imprecise visual qualities, such as shape orientation, to describe the shapes (see section 5.3.1), consistent with Burger and Shaughnessy (1986).

- There was no learner in the sample group for the GIST that used more than one attribute of a shape in naming the shape. For example, right-angled isosceles triangles were either named as "isosceles triangle" or "right-angled triangle", with the majority of the learners showing preference for the former name. The task of naming shapes was, in general, easier than that of stating the properties of shapes for the majority of the learners, consistent with the hierarchical property of the van Hiele levels.

School differences

- Learners from the SAS subsample performed better in the GIST than their peers from the NS subsample, since the former obtained a mean score of 43.44%, while the latter had a mean score of 36.94% for this test. The difference between these means was, however, not statistically significant (see section 5.2.1).

- For each of the five tasks that constituted the GIST, SAS learners obtained marginally higher mean scores than the NS learners (see section 5.3.6).

- Differential gender performance in the GIST favoured the male learners, both for the entire study sample and for each of the participating schools. The differences between the mean scores of the male and female learners were, however, not statistically significant (see sections 5.2.2.1–5.2.2.3).

- The male and female learners from the SAS subsample obtained higher mean scores for the GIST than their male and female peers from the NS subsample.

However, the differences in the mean scores were not statistically significant (see sections 5.2.2.4 and 5.2.2.5).

10.3.1.3 The CPGT and learners' performance

The entire study sample

- The CPGT was grade-specific. Learners in their respective grades had difficulty with formulating conjectures and stating definitions of simple geometric shapes. For the entire study sample, the grade 10, 11 and 12 learners were only able to obtain mean scores of 17.39%, 22.47% and 36.47%, respectively (see section 6.2). These low mean scores indicate that the majority of the learners were at low van Hiele levels, and hence not ready for the deductive study of school geometry (a finding consistent with Pegg, 1995).

- On the whole, at each grade level, formulating a conjecture was much more difficult for the majority of the learners than the other activities (defining, constructing/drawing, measuring, comparing etc.) required of them. Among the few learners who managed to formulate conjectures, most could not do so in formal technical terms.

School differences

- Grade 10 and 12 learners from the SAS subsample obtained higher mean scores (25.18% for grade 10 and 52.25% for grade 12) on the CPGT than their respective grade 10 and 12 counterparts from the NS subsample whose mean scores were, respectively, 9.59% and 20.68%. NS grade 11 learners had a marginally higher mean score (24.65%) than their SAS peers (20.30%). The differences between these means were statistically significant only for the grade 10 and 12 learners (see sections 6.2.1 through 6.2.3).

- Providing a definition of a rectangle, a square and a rhombus was generally difficult for the grade 10 learners, since no learner from the NS subsample was able to define any of these shapes, and only 1 learner from the SAS subsample was able to define a rectangle and a square, and 1 other learner, also from the SAS subsample, was able to define a rhombus (see section 6.3.1).

- Consistent with the findings of Siyepu (2005), for some learners across all the three grades, problems with measurement were evident in their responses to the CPGT. In grade 10, for example, there were 3 learners from the SAS subsample who obtained angle sums of 170° (with angles 90°, 50° and 30°), 184° (with angles 91°, 56° and 37°) and 140° (with angles 60°, 50° and 30°) for the triangles that they constructed (or drew) by themselves. There were many other learners for whom the unit of measurement for angles was the centimetres. For example, one of the learners from the NS subsample gave the sum of the angles of his triangle as 15cm (5cm + 5cm + 5cm = 15cm).

It needs to be reiterated here that the TPGT, GIST and CPGT, in addition to providing the data for learners' performance to be correlated with their van Hiele levels, all further complement the VHGT in providing a more comprehensive picture of participants' knowledge of school geometry (see sections 8.1 and 10.2.1).

10.3.1.4 The VHGT and learners' performance

The entire study sample

- An overall percentage mean score of 35.68% obtained by the participating learners in Part A of the VHGT was regarded as evidence that the majority of the learners in this study were at low van Hiele geometric thinking levels (possibly levels 0, 1 or 2), consistent with Usiskin (1982).

- For the entire study sample, learners' mean scores for Part A of the VHGT decreased progressively at each successive higher van Hiele level subtest between levels 1–3 (see section 7.2.2.1). The mean score of these learners on van Hiele level 4 subtest of the VHGT was, however, higher than their mean score on the van Hiele level 3 subtest. These results, in addition to providing support for the hierarchical property of the van Hiele levels, also indicated that learners in this study experienced more difficulty with geometry problems typifying van Hiele level 3 reasoning than with problems associated with the other levels, which is again consistent with Usiskin (1982) and Burger and Shaughnessy (1986).

- Of the 139 learners who wrote the VHGT, 136 (or 98%) of them were assignable to van Hiele levels, while 3 (or 2%) of them did not 'fit' the forced van Hiele level determination scheme. Of the total that wrote this test, 65 (or 47%), 31 (or 22%), 33 (or 24%), 3 (or 2%) and 4 (or 3%) were at van Hiele levels 0, 1, 2, 3 and 4, respectively. Given the large number of learners at level 0 and the near absence of learners at levels 3 and 4, these results indicate that the majority of the learners in this study were not ready for the formal deductive study of school geometry. This is consistent with Usiskin (1982), Mayberry (1983) and Siyepu (2005).

- There was no item in Part A of the VHGT requiring reasoning and arriving at a conclusion that was correctly answered by over 40% of the learners in each of the subsamples.

- For selected items in Part A of the VHGT, Usiskin's (1982) sample of American high school students performed better than the SAS learners in the present study, who in turn performed better than the NS learners (see section 7.2.3.5).

- Performance in Part B of the VHGT was generally poor, especially by the grade 10 and 11 learners. Writing a complete proof of even some familiar theorems in high school geometry was particularly difficult for a large majority of the learners (see sections 7.3.1 through 7.3.3).

School differences

- The mean score (39.37%) obtained by learners from the SAS subsample for Part A of the VHGT was significantly higher than that of their peers from NS (31.85%) at the 0.005 confidence level (see section 7.2.1.2).

- At each successive higher grade level, NS learners obtained marginally higher mean scores for Part A of the VHGT than learners at an adjacent lower grade level. In SAS, however, the mean score of the grade 12 learners was marginally higher than that of the grade 10 learners which was in turn higher than that of the grade 11 learners. That is, in SAS, the grade 10 learners demonstrated a better knowledge of geometric ideas than their grade 11 peers (see section 7.2.1.3).

- At each grade level, SAS learners obtained a higher mean score for part A of the VHGT than their respective peers from the NS subsample. It was, however, only in grade 10 that the difference between the means was statistically significant (see section 7.2.1.3).

- For each van Hiele level subtest, learners from the SAS subsample obtained higher mean scores than their counterparts from the NS (see section 7.2.2.2).

- For each grade level in each of the participating schools, learners obtained their lowest mean score in the van Hiele level 3 subtest of the VHGT. This indicates that the geometry problems that typify level 3 reasoning were generally difficult for learners across all three grades in this study (see section 7.2.2.3).

- More learners from the NS subsample were at van Hiele level 0 than learners from the SAS subsample. At levels 1 and 2, there were equal percentages of learners from the subsamples, but at levels 3 and 4, there were more learners from the SAS subsample. In short, of the 68 learners from the NS subsample who wrote the VHGT, 36 (or 53%) were at level 0, while 15 (or 22%), 16 (or 24%) and 1 (or 1%) were, respectively, at van Hiele levels 1, 2 and 3. No learner from this group was at level 4. Of the 71 learners from the SAS subsample that wrote the VHGT, 29 (or 41%) were at level 0, while 16 (or 22%), 17 (or 24%), 2 (or 3%) and 4 (or 6%) were at van Hiele levels 1, 2, 3 and 4, respectively (see section 7.2.3.3). These figures show that, in general, learners from the NS subsample had a poorer knowledge of school geometry than their counterparts from SAS.

- Although performance in Part B of the VHGT was generally poor for learners from both the NS and the SAS subsamples, learners' responses revealed that more grade 10 learners from the SAS subsample were able to handle a triangle problem in geometry that requires two lines of reasoning to the answer than their peers from the NS. Proof writing in geometry proved particularly difficult for the grade 10 and 11 learners in both subsamples, and for the grade 12 learners in the NS subsample, while some grade 12 learners from the SAS subsample were successful in this task (see sections 7.3.1–7.3.3).

Gender differences

- For the entire study sample, and for each of the participating schools, differential gender performance in Part A of the VHGT favoured the male learners. However, the differences between the male mean scores and the female mean scores both for the whole study sample and at each participating school level were not statistically significant (see sections 7.2.1.4–7.2.1.6), consistent with Usiskin (1982).

- In this study, the male and female learners from the SAS subsample obtained higher mean scores for Part A of the VHGT than their respective male and female peers from the NS subsample. The differences between the respective sets of male and female mean scores were statistically significant (see sections 7.2.1.7 and 7.2.1.8).

Learners' scores for each of the other three tests (i.e. the TPGT, GIST and CPGT), as well as their SEM scores, were correlated with their scores in the VHGT for possible relationships. The results of these correlations were presented in Chapter 8, but are summarized in the next section.

10.3.2 Summary of the findings relating to the second research goal and its associated research question

Details of the findings concerning the second research goal were presented and discussed in Chapter 8. The major findings were highlighted in section 8.4 of that chapter. In this section, these findings are summarized as follows:

- For the entire study sample, learners' VHGT scores correlate significantly with their TPGT and CPGT scores at the 0.01 level. No significant correlations were, however, found between learners' VHGT scores and either of their SEM or GIST scores. Given these results, the tentative conclusion reached was that there is a strong relationship between performance in geometry content tests (TPGT and CPGT) and the van Hiele levels (see section 8.2.1).

- For each of the participating schools, learners' VHGT scores correlate significantly with their SEM, TPGT and CPGT scores (see sections 8.2.2 and 8.2.3), consistent with Usiskin (1982). This means that for both subsamples, learners who did well in the VHGT did just as well in the school mathematics examination (SEM) and geometry content tests (TPGT and CPGT), and conversely. For both subsamples, no significant correlation was found between learners' VHGT scores and their scores for the GIST (see section 8.2.3 for a possible reason).

- For the entire study sample, grade level analysis of these correlations tends to suggest that at each grade level, learners' van Hiele levels correlate strongly with their performance in geometry content tests, but not as strongly with their performance in school mathematics examinations (see section 8.2.4).

- At the participating schools level, significant positive correlations were found only between the VHGT scores of grade 10 and 12 learners from the SAS subsample and their respective SEM, TPGT and CPGT scores. Similarly, it was only in grade 11 that there existed a significant positive correlation between learners' VHGT scores and their CPGT scores (see sections 8.2.5 and 8.2.6).

- For the entire study sample, for $n \leq 2$, learners at van Hiele level n obtained higher mean scores for the SEM, TPGT, CPGT and the GIST than learners at level $n-1$. Learners at van Hiele level 3 consistently obtained lower means on nearly all these tests than learners at level 2. However, with the exception of the SEM, learners who were at level 4 achieved higher mean scores in these tests than learners who were at the lower van Hiele levels (see section 8.3.1). These results, therefore, provided support for the hierarchical property of the van Hiele levels in general and levels 0–2 in particular.

- For the entire study sample, there occurred significant differences between the mean scores of learners at the different van Hiele levels only for the TPGT and the CPGT. A *Tukey HSD post-hoc test* indicated that these differences were statistically significant only between learners at van Hiele level 0 and learners at van Hiele level 2, in favour of the latter for each of these two tests (see section 8.3.1).

- For each of the participating schools, for $n \leq 2$, learners at van Hiele level n had higher mean scores in each of the four tests (except for the GIST for the SAS subsample) than their peers at level n–1. As with the whole study sample, the differences between the mean scores of learners at the different van Hiele levels were, however, significant only for the TPGT and the CPGT in each of the subsamples. A *Tukey HSD post-hoc* comparison of means showed that these differences were statistically significant between learners at van Hiele level 0 and learners at van Hiele level 2, in favour of the latter, for the TPGT and the CPGT in both subsamples. However, between learners at van Hiele level 1 and level 2, these differences were statistically significant for the TPGT in the NS subsample alone (see sections 8.3.2 and 8.3.3).

Having determined the van Hiele levels of the participating learners and the relationships between these and the learners' knowledge of school geometry, the study then examined, through the study of videotaped lessons, the classroom instructional practices that may have contributed to the exhibited van Hiele levels of the learners. In addition to attempting to elucidate what learning opportunities the observed teaching methods offered learners in geometry classrooms, the classroom video study also sought to interpret learners' exhibited levels of geometric understanding in terms of their geometry classroom instructional experiences.

10.3.3 Summary of the findings relating to the third research goal and its associated research question

The findings concerning the third research goal were presented and discussed in Chapter 9, with the major findings foregrounded in section 9.5 of that chapter. In this section, these findings are further summarized as follows.

- In terms of the van Hiele model of geometry instruction, observed geometry instructional practices in NS and SAS classrooms taken together offer the learners little opportunity to learn geometry, since observed teaching methods in many of these classrooms deviated significantly from this model (see section 9.3).

- In comparative terms, observed teaching methods in SAS hold greater opportunities for the learners to learn geometry than observed instructional methods in NS classrooms, since in the former, more of the videotaped lessons conformed to the van Hiele model of geometry classroom instruction than in the latter (see section 9.3).

- Besides assessing the videotaped lessons according to the van Hiele model of geometry instruction, the chapter offers a description of certain *images of teaching* evident in the observed classrooms (see sections 9.4.1 and 9.4.2). These images of teaching included the following:
  - Exchange of greetings;
  - Introducing the day's lesson;
  - Presenting the concept for the day;
  - Lesson organization;
  - Who does the (mathematical) work?
  - Review of the day's lesson;
  - Assigning homework.

  On the whole, results of the analysis of these images of teaching favoured the SAS lessons in terms of the learning opportunities that they appeared to offer learners in geometry classrooms (see section 9.5 for the summary of these images of teaching).

The findings presented in this section were found to be consistent with those of Stigler and Hiebert (1999) in that they provide a possible (but not the only) explanation why learners from the SAS subsample performed relatively better than their counterparts from the NS subsample in *all the tests* used in this study.

10.4 Significance of the study

This study is significant in a number of ways:

There are several benefits in knowing the educational practices of others outside of one's own country (Cook, Hite & Epstein, 2004). International comparison of educational systems, practices and pupils' academic performances often point to areas within one's own system and practices that require improvement (Fujita & Jones, 2002). This is one area in which my study is particularly significant. Nigeria has never participated in any of the TIMSS studies, while South African learners ranked among the lowest achievers in mathematics in both of the TIMSS 1995 and TIMSS – Repeat, 1999 studies (Howie, 2001). Hence, using the mathematical performance of the South African school children as the referent, this study is significant in that it provides a benchmark for international comparison between the mathematical performance of Nigerian children and that of their peers from other countries. One significant result of South African participation in TIMSS studies was the comment and debate that followed in the wake of the poor mathematical performances of its representatives, with concomitant suggestions for improvement (Brombacher, 2001). This equally applies to this study. There are several avenues for comment, debate and suggestion regarding the relative strengths and weaknesses of the Nigerian and South African child in school mathematics, given the results of this study.

Although van Hiele (1986) argues that the levels of thought indeed permeate many aspects of school mathematics other than geometry, there appears to be a dearth of empirical evidence in the literature explicitly linking the levels with students' mathematical knowledge in general (see Senk, 1989). By comparing learners' van Hiele levels with their general mathematical achievement, this study has contributed to closing this perceived gap in the existing literature. The finding that for each of the participating schools, learners' van Hiele thought levels correlate significantly with their performance in school mathematics as a whole is significant in that teachers may wish to look for ways of raising their learners' van Hiele levels of geometric thinking.

Furthermore, one of the properties of the van Hiele theory as identified by Usiskin (1982) is its wide applicability (see section 2.8.3). Despite this wide applicability, only a few studies have utilized the van Hiele model to explain students' geometric thinking levels in an African context. As far as I could ascertain, this study is the first to apply the van Hiele model in Nigeria, and one of only a few in the South African context (e.g. De Villiers, 1994; 1998; van der Sandt & Nieuwoudt, 2003; Feza & Webb, 2005; Siyepu, 2005). In general, there appears to be a dearth of published research in the literature concerning the use of the van Hiele theory on instruction to explicate geometry classroom instructional practices (see Hoffer, 1983). This seeming absence makes this study a worthwhile endeavour, particularly in the Nigerian and South African contexts.

Traditionally, many past publications (e.g. Usiskin, 1982; Burger & Shaughnessy, 1986) have communicated the work of the van Hieles primarily in terms of the thought levels, with only a few (e.g. Fuys et al.) explaining the van Hiele theory in terms of the learning phases (Hoffer, 1983). There is little evidence in the literature that both aspects of the van Hiele theory (i.e. the thought levels and the learning phases) have been investigated in a single study, especially in an African context. This study owes its significant and unique attributes to being the first, as far as I have been able to ascertain, that attempts to link learners' exhibited van Hiele thought levels to their geometry instructional experiences in the Nigerian and South African contexts.

Lastly, this study is significant since it represents, as far as I know, the first scholastic attempt simultaneously to compare high school learners' mathematical performance, and instructional methods in geometry classrooms in Nigeria and South Africa using the van Hiele theory. It is of great value, if for no other reason, because it furnishes a baseline of comparison for subsequent studies.

On a final note, other significant areas of strengths of this study relate to its design and include the following:

- The use of a multidimensional approach to the data collection (tests, hands-on manipulatives, interviews and videotapes) yielded dividends in terms of ensuring richness and triangulation (see Denzin, 1988; Cohen et al., 2000). The classroom videos proved especially effective in studying teaching methods in geometry classrooms (see Stoker, 2003).

- The involvement of a consultative panel of independent observers and critical readers (see sections 3.4.2.2 and 9.1) paid dividends in terms of ensuring quality and credibility (Schäfer, 2003).
- The involvement of learners and their teachers from across all three grades of secondary education provided a holistic and comprehensive picture of the teaching-learning processes at that level. This point addressed one of the limitations that Siyepu (2005) identified in his study.

10.5 Limitations of the study

It is acknowledged that many of the findings of this study are limited and tentative in terms of their generalizability and wider application, given the relatively small sample that was involved. This research constituted a case study which, although providing insight into a particular situation, typically can provide only a frame of reference for other similar situations (see Schäfer, 2003).

One shortcoming of this study concerned the number of schools involved in either country. The involvement of only one school each in Nigeria and South Africa particularly limited the scope of the video study component of this study with regard to the findings, even though it was not the objective of the study to generalize. Although the limitation was partly compensated for by videotaping three geometry lessons in each of the participating schools (see section 3.3.4.3), a much broader picture of geometry classroom instructional practices in these countries would have emerged had the study videotaped the lessons from several schools in either country. I did, however, receive endorsement for my sample selection from Stigler and Hiebert (1999), as argued in section 3.3.4.3.

One of the strengths of this study stated in the preceding section is that it made use of a multidimensional approach to the data collection in an attempt to ensure richness and triangulation. This approach could, however, also count as a limitation. With hindsight, I believe that the testing instruments were rather too numerous. The interest of a number of the students declined after they had taken part in two or more of the tests during the fieldwork. I had constantly to hold talks with them in order to encourage and motivate them to sustain their interest. It is possible that this class of students failed to give their 'best' in subsequent test-taking activities, and this may have affected the results in one way or the other.

Inherent in a multidimensional approach to data gathering are the problems of analyzing data collected from a wide variety of sources (Schäfer, 2003). Hamel et al. (as cited in Schäfer, 2003) acknowledge that the strength of a case study lies in its wealth of empirical material. Case studies are based on a rich collection of empirical data, and this particularly resonated with my study. Hamel et al. (as cited in Schäfer, 2003), however, caution that this can sometimes be problematic, particularly, in the analysis of diverse data. I was indeed challenged by the difficulties of analyzing and presenting the almost overwhelming variety of data collected from different sources in this study. Nevertheless, I have attempted to overcome the potential difficulties inherent in the handling of data of different origins by incorporating the numerous diverse findings into a single narrative (see Schäfer, 2003). In this way, by interrelating the diverse findings of the study, I have managed to create a sense of cohesion.

Furthermore, the use of *static* manipulative concept cards for the GIST (see section 3.3.4.1.2) may be viewed as a limitation of this study. De Villiers (1994; 1998) has raised concern about the use of static manipulatives, both for classroom teaching and for research regarding student learning of school geometry. He argues that the use of dynamic software such as the Geometer Sketchpad has the potential to elicit richer responses from learners than static manipulative concept cards of geometric shapes, which tend to limit their thinking about these shapes. In addition to receiving much encouragement from Mayberry (1983), Burger and Shaughnessy (1986), Feza and Webb (2005) and, of course, van Hiele (1999) himself in using these manipulatives, I also incorporated interviews into the GIST in my attempt to respond to De Villiers' empirically informed reservations (see section 3.3.4.1.2).

10.6 Areas for future research

As stated earlier, this study is novel in being the first to compare both Nigerian and South African high school learners' van Hiele geometric thinking levels and geometry instructional practices in these countries, using the van Hiele theory as the lens for research. Given the study's originality and the absence of precedent, its findings can, at best, only be regarded as tentative. Hence, further research may be needed to add credibility to the results of this initiative, even though "no study can be replicated exactly, regardless of the methods and design employed" because of the changing nature of human behaviour (Schäfer, 2003).

The classroom video study component of this study is one of only a few, particularly in an African context, to have attempted to analyse geometry classroom instructional processes using the van Hiele phases of learning. Although the framework (checklist of the van Hiele phase descriptors) used for analysing the videotaped lessons underwent rigorous processes of validation (see sections 3.4.2.2, 9.2 and 9.3), it still cannot be assumed that it is 'excellent' in the form that it was used, particularly as it was neither adopted nor adapted from an existing framework. More research may be needed to further investigate teaching methods in the Nigerian and South African contexts using the van Hiele learning phases, so as to refine and improve the checklist of van Hiele phase descriptors developed in this study.

Given the dearth of research into the van Hiele geometric thought levels of the Nigerian child, and the fact that this study involved only a small sample of schools, further research into the van Hiele levels that could capture a much broader spectrum of the Nigerian educational landscape may be worthwhile.

10.7 Implications and tentative recommendations

The implications and tentative recommendations resulting from this study include the following:

- Geometric terminology comprises the set of linguistic symbols for communicating geometric ideas both inside and outside of the classroom. Van Hiele (1986), for example, stresses the point that when a child has learned how to recognise a figure through direct contact with the figure, the child should develop the appropriate technical term or language with which to communicate his ideas about the figure to others. Learners' generally poor performance in the test of knowledge of geometric terminology in this study holds important implications for geometry classroom teaching and learning. Although a high positive correlation existed between learners' scores for the verbally presented tasks and visually presented tasks of the TPGT, a higher mean score was obtained for the latter than for the former (see section 4.3.1). As important it is to teach the terminology associated with a given content area in geometry, it is equally important for teachers in their instructional design and delivery to ensure that a learner who can give a correct verbal description of a geometric concept also has the correct concept image associated with that concept, and vice versa.

- Many learners in this study were able to recognise shapes only in some standard orientations (see section 5.3.1). The implication here is that these learners do not understand that simple geometric shapes like triangles and quadrilaterals are defined by their properties and not by their orientations in space. Teachers need to provide learners with activities for exploring the properties of simple geometric shapes in different orientations. During these activities, the invariant properties of the shapes should be emphasized.

- As revealed in this study, stating definitions of shapes and formulating conjectures about the relationships between shapes and their properties were particularly difficult for a large majority of the learners (see sections 6.3.1 through 6.3.3). Van Hiele (1986) is strongly critical of an instructional approach in which the teacher requests learners to state definitions of geometric concepts before they have become acquainted with the concepts. In his view, to give a definition of a concept is only feasible if one knows to some extent the thing that is to be defined. The implication of this is that learners need preliminary explorations of the properties of a geometric shape before they can attempt to write a definition of it, and teachers need to be alert to this empirical fact. With appropriate instructional guidance from the teacher, students can be assisted to formulate their own definitions of various shapes using materials and procedures similar to those of worksheet 1 (see Appendix 5.A.1, p.51) used in this study (see also de Villiers, 1998).

- The findings from the classroom video study indicate that only a few of the participating teachers explored their learners' prior knowledge, and of those that did, fewer still attempted to relate learners' prior knowledge to the present topic. Ausubel (1968) and van Hiele (1986; 1999) are unequivocally critical of a teaching approach that neglects learners' prior knowledge. In this regard, teachers need critically to reappraise their classroom instructional processes and look for ways to improve their practices. Regarding improvement, teachers may find the checklist criteria of the van Hiele phase descriptors (see section 2.8.4.1) a dependable companion.

- The findings that learners' van Hiele levels correlate significantly with their mathematical performance in general and with their knowledge of school geometry, specifically, have important teaching implications for teachers' daily classroom practices. As stated earlier in section 2.7.4.3, van Hiele (1986) believes that students' difficulty with school mathematics generally and geometry in particular is caused largely by teachers' failure to deliver instruction that is appropriate to their thinking levels. For teachers to be aware of the levels of thinking that characterize each of the van Hiele levels may help to reduce the mismatch between their teaching methods and learners' cognitive thinking levels. In particular, teachers' familiarity with the instructional cycle of the van Hiele learning phases (see section 2.8.2) should render more effective their efforts to assist learners make progress with their learning.

10.8 A final word of personal reflection

When I undertook this study in July, 2005, it was at first little more than a concession to my never-ending desire for higher academic achievement. No sooner had I started, however, than I realised that for a project such as this, aiming to produce worthwhile and useful research, the desire for personal academic achievement alone cannot be the propelling force. Indeed, the recognition by the participating school principals that my study has the potential to address a major problem in their schools – learners' poor mathematical performance – and the request that a summary of the findings be made available to them in order to assist them to improve the *status quo* (see Chapter 3), was for me inspiring.

Being thus strongly motivated, I was little aware that I had undertaken a project that was going to challenge and completely dominate the next three years of my life. In effect, my experience during these last three years has been quite remarkable. There have been moments of "self-doubt, loss of confidence, exhaustion, anxiety" and even frustration (Schäfer, 2003, p.308) in the course of it all. But today, as I wind up the study, all of that has given way to feelings of enlightenment, growth, satisfaction and accomplishment. Ely et al. (as cited in Schäfer, 2003, p.308) have observed that "the processes of qualitative research also become processes of professional development". This clearly resonates with my own experience in this study. My attempt to research learners' levels of geometric conceptualization and teachers' instructional methods using the van Hiele model has, indeed, opened up new perspectives for my professional development. In retrospect, I am today considerably pleased that on account of my exposition of the van Hiele model in workshops, I received many teaching invitations from schools within the community where I conducted my research in South Africa. The highlight was when the principal and mathematics teacher in one of the schools where I piloted the instruments for my study reported that his learners had begun to evince better mathematical insight as a result of the instruction which he had fashioned after the van Hiele model.

The PhD project may have been concluded, but for me this marks a new beginning in my professional growth. I believe that this study does contribute significantly to the knowledge base within my field. Therefore, I will continue to research into, write about, and refine my contribution (Schäfer, 2003) towards the van Hiele theory and its classroom instructional relevance, in order to keep on contributing to an extensive and invaluable knowledge base.

REFERENCES

Adedayo, O. A. (2000). Availability of basic teaching/learning materials in mathematics in selected secondary schools in Lagos State. *STAN*, *41*, 263–266.

Adele, G. H. (1989). When did Euclid live? An answer plus a short history of geometry. *Mathematics Teacher*, *82*(6), 460–463.

Agwagah, U. N. V. (2000). Mathematics enrichment materials and activities for primary schools. *STAN*, *41*, 270–274.

Allendoerfer, C. B. (1969). The dilemma in geometry. *Mathematics Teacher*, *62*(1), 165–169.

Anderson, G. (1990). *Fundamentals of educational research*. London: Falmer Press.

Andrews, A. G. (1999). Solving geometric problems by using unit blocks. *Teaching Children Mathematics*, *5*(6), 318–326.

Arksey, H., & Knight, P. (1999). *Interviewing for social scientists: An introductory resource with examples*. London: SAGE Publications.

Ashfield, B., & Prestage, S. (2006). Analysing geometric tasks considering hinting support and inscriptions. In D. Hewitt (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *26*(1), 11–16.

Atebe, H. U. (2005). Cultural considerations of mathematics curriculum: Implications for the learner. In N. C. Nwaboku, B. Akinpelu & S. O. Makinde (Eds.), *Education*: *A socialising agent* (pp. 93–102). Lagos: Olu-Akin.

Atebe, H.U., & Schafer, M. (2008). Van Hiele levels of geometric thinking of Nigerian and South African mathematics learners. In M. V. Polaki, T. Mokuku & T. Nyabanyaba (Eds.), *Proceedings of the Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE), Maseru, Lesotho*, 104–116.

Ausubel, D. P.(1968). *Educational psychology: A cognitive view*. New York: Holt, Rinehart and Winston.

Badmus, G. A. (1997). Mathematics education in Nigeria. *Nigerian Academy of Education*, *5*, 54–65.

Barnard, J. J., & Cronjé, L. S. (1996). Euclidean geometry: Cognitive gender differences. *South African Journal of Education*, *16*(1), 1–4.

Battista, M. T. (2002). Learning geometry in a dynamic computer environment. *Teaching Children Mathematics, 8*(6), 333–339.

Bell, F. H. (1978). *Teaching and learning mathematics (in secondary schools)*. Dubuque, Iowa: W. C. Brown Company.

Bennie, K., Blake, P., & Fitton, S. (2006). *Focus on mathematics Grade 11*. Cape Town: Maskew Miller Longman.

Berg, B. L. (2004). *Qualitative research methods for the social sciences*. Boston: Pearson Education.

Betiku, O. F. (1999). Resources for the effective implementation of the 2- and 3-dimensional mathematics topics at the junior and senior secondary school levels in the Federal Capital Territory, Abuja. *Nigerian Journal of Curriculum Studies*, *6*(2), 49–52.

Bishop, A. .J. (1980). Spatial abilities and mathematics achievement: A review. *Educational Studies in Mathematics*, *11*, 257–269.

Bishop, A. .J. (1983). Space and geometry. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 175–203). New York: Academic Press.

Bloom, B. S. (Ed.). (1956). *Taxonomy of educational objectives 1*. London: Longmans, Green.

Borowski, E. J., & Borwein, J. M. (1989). *Dictionary of mathematics*. London: Collins.

Bot, T. D. (2000). Rapid assessment of the competence of undergraduates in the improvisation and utilisation of resources to teach secondary mathematics content. A case study of University of Jos. *STAN*, *41*, 258–262.

Boyer, C. B. (1968). *A history of mathematics*. New York: Wiley.

Brannen, J. (2004). Working qualitatively and quantitatively. In C. Seale, G. Gobo, J.F. Gubrium & D. Silverman (Eds.), *Qualitative research practice* (pp. 312–326). London: SAGE Publications.

Brombacher, A. (2001). How would your students perform on TIMMS? In C. Pournara, M. Graven & M. Dickson (Eds.), *Proceedings of the Seventh National Congress of the Association for Mathematics Education of South Africa (AMESA), University of the Witwatersrand, 2*, 11–19.

Burger, W., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, *17*(1), 31–48.

Burton, D. M. (1985). *The history of mathematics: An introduction*. Massachusetts: Allyn & Bacon.

Cantrell, D. (1993). Alternative paradigms in environmental education research: The interpretive perspective. In R. Mrazek (Ed.), *Alternative paradigms in environmental education research* (pp. 81–104). Ohio: North American Association for Environmental Education.

Casey, J. (1889). *The elements of Euclid*. London: Longmans, Green.

Chambers Dictionary (1998). Edinburgh: Chambers Harrap Publishers.

Choppin, B. H. (1988). Objective test. In J. P. Keeves (Ed.), *Educational research, methodology, and measurement: An international handbook* (pp. 354–358). New York: Pergamon Press.

Clements, D. H. (2004). Perspective on "The child's thought and geometry". In T. P. Carpenter, J. A. Dossey & J. l. Koehler (Eds.), *Classics in mathematics education research* (pp. 60–66). Reston: NCTM.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.

Cogan, L. S., & Schmidt, W. H. (1999). An examination of instructional practices in six countries. In G. Kaiser, E. Luna & I. Huntley (Eds.), *International comparison in mathematics education* (pp. 68–85). London: Falmer Press.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education* (5th ed.). London: Routledge/Falmer.

Connole, H. (1998). 'The research enterprise'. In H. Connole, B. Smith & R. Wiseman (Eds.), *Study guide: Issues and methods in research* (pp. 17–42). Geelong: Deakin University.

Cook, B. J., Hite, S. J., & Epstein, E. H. (2004). Discerning trends, contours, and boundaries in comparative education: A survey of comparativists and their literature. *Comparative Education Review*, *48*(2), 123–149.

Corbin, J., & Strauss, A. (1990). Grounded theory research: procedures, canons, and evaluation criteria. *Qualitative Sociology*, *13*(1), 3–21.

Coxford, A. F. (1995). The case for connection. In A. Peggy & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 3–12). Reston: NCTM.

Crawford, K., & Adler, J. (1996). Teachers as researchers in mathematics education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 1187–1205). Dordrecht: Kluwer Academic Publishers.

Creswell, J. W. (1994). *Research design: Qualitative and quantitative approaches*. Thousand Oaks: SAGE Publications.

Creswell, J. W. (2003). *Research design: Qualitative, quantitative and mixed methods approaches*. Thousand Oaks: SAGE Publications.

D' Ambrosio, U. (1997). Ethnomathematics and its place in the history and pedagogy of mathematics. In A. B. Powell & M, Frankenstein (Eds.), *Ethnomathematics: Challenging eurocentricism in mathematics education*, (pp. 13–24). New York: State University of New York Press.

Daramola, S. O. (1998). *Statistical analysis in education*. Ilorin: Lekan Press.

De Villiers, M. D. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the Learning of Mathematics*, *14*(1), 11–18.

De Villiers, M. D. (1997). The future of secondary school geometry. *Pythagoras*, *44*, 37–54.

De Villiers, M. D. (1998). To teach definitions in geometry or teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the Annual Conference of the Psychology of Mathematics Education (PME), University of Stellenbosch*, *2*, 248–255.

Deighton, H. (1886). *The elements of Euclid*. Cambridge: Deighton, Bell.

Denzin, N. K. (1988). Triangulation. In J. P. Keeves (Ed.), *Educational research, methodology, and measurement: An international handbook* (pp. 511–513). New York: Pergamon Press.

Doyle, W. (1983). Academic work. *Review of Educational Research*, *53*(2), 159–199.

Dreyfus, T. (1999). Why Johnny can't prove: With apology to Morris Kline. *Educational Studies in Mathematics*, *38*(1–3), 85–109.

Driver, R. (1978). When is a stage not a stage? A critique of Piaget's theory of cognitive development and its application to science education. *Educational Research*, *21*(1), 54–61.

Durrheim, K. (1999a). Quantitative measurement. In M. Terre Blanche & K. Durrheim (Eds.), *Research in practice: Applied methods for the social sciences* (pp. 72–95). Cape Town: University of Cape Town Press.

Durrheim, K. (1999b). Quantitative analysis. In M. Terre Blanche & K. Durrheim (Eds.), *Research in practice: Applied methods for the social sciences* (pp. 96–122). Cape Town: University of Cape Town Press.

Durrheim, K., & Wassenaar, D. (1999). Putting design into practice: Writing and evaluating research proposals. In M. Terre Blanche & K. Durrheim (Eds.), *Research in practice: Applied methods for the social sciences* (pp.54–71). Cape Town: University of Cape Town Press.

Euclid (1952). The thirteen books of Euclid's elements (T. L. Heath, Trans.). In R. M. Hutchins (Ed.), *Great books of the Western world* (pp. 1–402). Chicago: University of Chicago Press.

Evans, K. M. (1959). Research on teaching ability. *Educational Research*, *1*(3), 22–36.

Eves, H. (1953). *An introduction to the history of mathematics*. New York: Rinehart.

Eves, H. (1972). *A survey of geometry*. Boston: Allyn & Bacon.

Eves, H. (1976). *An introduction to the history of mathematics*. New York: Holt, Rinehart & Winston.

Ezzy, D. (2002). *Qualitative analysis: Practice and innovation*. London: Routledge.

Federal Republic of Nigeria. Ministry of Education. (1985). *National curriculum for senior secondary schools vol. 5 (Mathematics)*. Lagos: FME Press.

Federal Republic of Nigeria. (1998). *National Policy on Education.* (3rd ed.). Lagos: NERDC Press.

Feza, N., & Webb, P. (2005). Assessment standard, van Hiele levels, and grade seven learners' understandings of geometry. *Pythagoras, 62*, 36–47.

Filimonov, R., & Kreith, K. (1992). Euclidean geometry via programming. *Journal of Computers in Mathematics and Science Teaching*, *11*(3/4), 303–318.

French, D. (2004). *Teaching and learning geometry*. London: Continuum.

Fujita, T., & Jones, K. (2002). Opportunities for the development of geometrical reasoning in current textbooks in the UK and Japan. In K. Jones & T. Fujita (Eds.), *Proceedings of the British Society for Research into Learning Mathematics*, *22*(3), 79–84.

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education Monograph, 3*. Reston: NCTM.

Fuys. D. J., & Liebov, A. K. (1997). Concept learning in geometry: Teaching geometry to elementary students. *Teaching Children Mathematics*, *3*(5), 248–251.

Gardner, H. (1993). *Frames of mind: The theory of multiple intelligences* (2nd ed.). London: Fontana Press.

Gittleman, A. (1975). *History of mathematics*. Columbus, Ohio: C. E. Merrill.

Greenberg, M. J. (1974). *Euclidean and non-Euclidean geometries: Development and history*. San Francisco: W. H. Freeman.

Greenwood, J.J. (1996). On the nature of teaching and using “mathematical power” and “mathematical thinking”. In D. V. Lambdin, P. E. Kehle & R. V. Preston (Eds.), *Emphasis on assessment: Readings from NCTM’s school-based journals* (pp. 18–26). Reston, NCTM.

Hammersley, M., & Gomm, R. (2000). Introduction. In R. Gomm, M. Hammersley & P. Foster (Eds.), *Case study methods* (pp. 1–16). London: SAGE Publications.

Happs, J. C. (1992). Using grade eight students’ existing knowledge to teach about parallel lines. *School Science and Mathematics*, *92*(8), 450–454.

Hewson, P. W. (1999). Research on professional development for school reform. In E. Fennema & K. Taole (Eds.), *Mapping out a research agenda to drive professional development in systemic reform* (pp. 47–62). Pretoria: National Research Foundation.

Hodgson, T. R. (1995). Connections as problem-solving tools. In A. Peggy & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 13–21). Reston: NCTM.

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher*, *74*(1), 11–18.

Hoffer, A. (1983). Van Hiele-based research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 205–227). New York: Academic Press.

Howie, S. (2001). *Mathematics and performance in grade 8 in South Africa 1998/1999: TIMMS-R 1999, South Africa*. Pretoria: Human Sciences Research Council.

Howie, T. (2004). *Interpreting teachers’ mathematical knowledge in the context of learner-centred practice in a selection of grade nine mathematics classrooms in South Africa*. Unpublished master’s thesis, University of the Witwatersrand, Johannesburg.

Igbokwe, D. I. (2000). Dominant factors and error types inhibiting the understanding of mathematics. *STAN*, *41*, 242–249.

Igwue, D. O. (1990). Science teachers’ qualifications and students’ performance in secondary schools in Kano State. *Journal of Science Teachers’ Association of Nigeria*, *26*(2), 47–51.

Ivowi, U. M. O. (1990). The philosophy and objectives of the science and mathematics curricula at the senior secondary school level in Nigeria. *STAN*, *26*(2), 3–8.

Jackson, W. (1995). *Doing social research methods*. Scarborough: Prentice-Hall.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Nisbet, S. (2002). Elementary students’ access to powerful mathematical ideas. In L. D. English (Ed.), *Handbook* *of international research in mathematics education* (pp. 113–141). New Jersey: Lawrence Erlbaun.

Jones, K., Fujita, T., & Ding, L. (2006). Informing the pedagogy for geometry: Learning from teaching approaches in China and Japan. In D. Hewitt (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *26*(2), 109–114.

Justina, W. G. (1991). Teaching science, technology and mathematics in the mother tongue: Implications for the learners. *STAN*, *32*, 118–122.

Kanjee, A. (1999). Assessment research. In M. Terre Blanche & K. Durrheim (Eds.), *Research in practice: Applied methods for the social sciences* (pp. 287–308). Cape Town: University of Cape Town Press.

Kilpatrick, J. (1978). Research on problem solving in mathematics. *School Science and Mathematics*, *70*(3), 187–192.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington: National Academy Press.

Kleiman, G. M. (1995). Seeing and thinking mathematically in the middle school. In A. Peggy & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 153–158). Reston: NCTM.

Krause, E. F. (1986). *Taxicab geometry: An adventure in non-Euclidean geometry*. New York: Dover Publications.

Kvale, S. (1989). To validate is to question. In S. Kvale (Ed.), *Issues of validity in qualitative research* (pp. 73–92). Lund: Studentlitteratur.

LeCompte, M. D., & Preissle, J. (1993). *Ethnography and qualitative design in educational research* (2nd ed.). New York: Academic Press.

Lewis, D. G. (1967). *Statistical methods in education* (4th ed.). London: University of London Press.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. London: SAGE Publications.

Lincoln, Y. S., & Guba, E. G. (2000). Paradigmatic controversies, contradictions, and confluences. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed.) (pp. 163–188). Thousand Oaks: SAGE Publications.

Mansfield, H., & Happs, J. (1996). *Improving teaching and learning in science and mathematics*: *Using student conceptions of parallel lines to plan a teaching program*. New York: Columbia University Press.

Marawu, S. (1997). *A case study of English/Xhosa code switching as a communicative and learning resource in an English medium classroom.* Unpublished master's thesis, Rhodes University, Grahamstown.

Mason, M. (1998). The van Hiele levels of geometric understanding. In L. McDougal (Ed.), *The professional handbook for teachers: Geometry* (pp. 4–8). Boston: McDougal-Littell/Houghton-Mifflin.

Mathematical Association (1923). *The teaching of geometry in schools.* London: G. Bell & Sons.

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate pre-service teachers. *Journal for Research in Mathematics Education*, *14*(1), 58–69.

McInerney, D. M., & McInerney, V. (2002). *Educational psychology: Constructing learning* (3rd ed.). Frenchs Forest: Pearson Education.

Mji, A., & Makgato, M. (2006). Factors associated with high school learners' poor performance: A spotlight on mathematics and physical science. S*outh African Journal of Education*, *26*(2), 253–266.

Mogari, D. (2002). *An ethnomathematical approach to the teaching and learning of some geometrical concepts.* Unpublished doctoral thesis, University of the Witwatersrand, Johannesburg.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston: NCTM.

Neville, P. T. (1969). International association for the Evaluation of Educational Achievement (IEA) mathematics study: Target population, sampling and tests. *International Review of Education*, *5*(1), 136–160.

Nickson, M. (2004). *Teaching and learning mathematics: A guide to recent research and its applications* (2nd ed.). London: Continuum.

Nieuwoudt, H. D., & van der Sandt, S. (2003). How well are mathematics teachers conceptually prepared to teach for learning? A case study of grade 7 geometry. *AMESA*, *1*, 224–235.

Nixon, R. C. J. (Ed.). (1887). *Euclid revised.* Oxford: Clarendon Press.

Oberdorf, C. D., & Taylor-Cox, J. (1999). Shape up! Children's misconceptions about geometry. *Teaching Children Mathematics*, *5*(6), 340–346.

Okonkwo, S. C. (2000). Enriching mathematics education: Implication for the teacher. *STAN*, *41*, 275–278.

Oliver, J. (2003). Fractions at the dawn of history: How did the Babylonians cope with fractions? *Mathematics in School*, *32*(1), 17–18.

Olkun, S., Sinoplu, N. B., & Deryakulu, D. (2005). Geometric exploration with dynamic geometry applications based on van Hiele levels. *International Journal for Mathematics Teaching and Learning*. Retrieved September 7, 2005, from http://www.cimt.plymouth.ac.uk/journal/olkun.pdf

Olson, M., Sakshaug, L., & Olson, J. (1997). How many rectangles? *Teaching Children Mathematics*, *4*(1), 38–42.

Omidiran, P. O., & Sanni, R. I. O. (2001). *Research methods and project writing*. Lagos: Yelead Commercial.

Onabanjo, I. O., & Akinsola, O. S. (2000). An investigation into the utilisation of the available resources in mathematics classroom. *STAN*, *41*, 284–288.

Orton, A., & Frobisher, L. (1996). *Introduction to education: Insights into teaching mathematics*. London: Cassell.

Orton, A. (2004). *Learning mathematics: Issues, theory and practice*. London: Continuum.

Pegg, J. (1995). Learning and teaching geometry. In L. Grimison & J. Pegg (Eds.), *Teaching secondary school mathematics: Theory into practice* (pp. 87–103). London: Harcourt Brace.

Piaget, J., & Inhelder, B. (1969). *The psychology of the child*. London: Routledge and Kegan Paul.

Renne, C. G. (2004). Is a rectangle a square? Developing mathematical vocabulary and conceptual understanding. *Teaching Children Mathematics*, *10*(5), 258–264.

Roux, A. (2003). The impact of language proficiency on mathematical thinking. In S. Jaffer & L. Burgess (Eds.), *Proceedings of the Annual Meeting of the Association for Mathematics Education of South Africa (AMESA)*, *University of Cape Town*, *2*, 362–371.

Schäfer, M. (1996). An assessment and the creative teacher. The Std 7 IEB GEC examination and its effects. *AMESA*, *1*, 217–229.

Schäfer, M. (2003). *The impact of learners' spatial capacity and worldviews on their spatial conceptualisation: A case study*. Unpublished doctoral thesis, Curtin University of Technology, Perth.

Schäfer, M. (2004). Worldview theory and the conceptualisation of space in mathematics education. *Pythagoras*, *59*, 8–17.

Schunk, D. H. (2004). *Learning theories: An educational perspective*. New Jersey: Pearson Education.

Schwandt, T. A. (2000). Three epistemological stances for qualitative inquiry: Interpretivism, hermeneutics, and social constructionism. In N. K. Denzin & Y. S.

Lincoln (Eds.), *Handbook of qualitative research* (pp. 189–213). Thousand Oaks: SAGE Publications.

Seidman, I. E. (1991). *Interviewing as qualitative research: A guide for researchers in education and the social sciences*. New York: Teachers College Press.

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher*, *78*, 448–456.

Senk, S. L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education*, *20*(3), 309–321.

Setati, M. (2002). *Language practices in intermediate multilingual mathematics classrooms*. Unpublished doctoral thesis, University of the Witwatersrand, Johannesburg.

Shannon, P. (2002). Geometry: An urgent case for treatment. *Mathematics Teaching*, *181*, 26–29.

Shaughnessy, J. M., & Burger, W. F. (1985). Spadework prior to deduction in geometry. *Mathematics Teacher*, *78*(6), 419–428.

Sherard, W. H. (1981). Why is geometry a basic skill? *Mathematics Teacher*, *74*(1), 19–21.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.

Simon, D. (2001). Towards a new understanding of code switching in the foreign language classroom. In R. Jacobson (Ed.), *Trends in linguistics: Code switching worldwide* (pp. 311–342). Berlin: Mouton de Gruyter.

Siyepu, S. W. (2005). *The use of van Hiele theory to explore problems encountered in circle geometry: A grade 11 case study*. Unpublished master's thesis, Rhodes University, Grahamstown.

South Africa. Department of Education. (1995). *White Paper on Education and Training*. Cape Town: Government Printer.

South Africa. Department of Education. (2003).*National Curriculum Statement grades 10 – 12 (General): Mathematics*. Pretoria: The Department.

Stake, R. E. (1995). *The art of case study research*. London: SAGE Publications.

Stake, R. E. (2000). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 435–454). London: SAGE Publications.

StatSoft, Inc. (2007).*STATISTICA* (data analysis software system), version 8.0.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Stoker, J. (2003). *An investigation of mathematics teachers' beliefs and practices following a professional development intervention based on constructivist principles*. Unpublished doctoral thesis, Curtin University of Technology, Perth.

Struik, D. J. (1967). *A concise history of mathematics* (3rd ed.). New York: Dover Publications.

Suydam, M. N. (1985). The shape of instruction in geometry: Some highlights from research. *Mathematics Teacher*, *78*(6), 481–486.

Teppo, A. (1991). Van Hiele levels of geometric thought revisited. *Mathematics Teacher*, *84*(3), 210–221.

Terre Blanche, M., & Durrheim, K. (1999). Social constructionist methods. In M. erre Blanche & K. Durrheim (Eds.), *Research in practice: Applied methods for the social sciences* (pp. 147–172). Cape Town: University of Cape Town Press.

Terre Blanche, M., & Kelly, K. (1999). Interpretive methods. In M. Terre Blanche & K. Durrheim (Eds.), *Research in practice: Applied methods for the social sciences* (pp. 123–146). Cape Town: University of Cape Town Press.

Tschudi, F. (1989). Do qualitative and quantitative methods require different approaches to validity? In S. Kvale (Ed.), *Issues of validity in qualitative research* (pp. 109–134). Lund: Studentlitteratur.

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry: Cognitive development and achievement in secondary school geometry project*. Chicago: University of Chicago Press.

Vacc, N. N., & Bright, G. W. (1999). Elementary preservice teachers' changing beliefs and instructional use of children's mathematical thinking. *Journal for Research in mathematics Education*, *30*(1), 89–110.

Van der Sandt, S., & Nieuwoudt, H. D. (2003). Grade 7 teachers' and prospective teachers' content knowledge of geometry. *South African Journal of Education*, *22*(1), 199–205.

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Orlando: Academic Press.

Van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics*, *5*(6), 310–317.

West African Examinations Council. (2003). *Chief Examiner's Report*. Lagos: Megavons (W. A.) Ltd.

Yager, R. E. (1991). The constructivist learning model. *The Science Teacher*, *58*(6), 52–57.

Yin, R. K. (2003). *Case study research: Design and methods*. Thousand Oaks: SAGE Publications.

Zeller, R. A. (1988). Validity. In J.P. Keeves (Ed.), *Educational research, methodology, and measurement: An international handbook* (pp. 322–330). New York: Pergamon Press.

Zuma, J. (2000). *Creating an enabling environment for development*. Opening address at the 2nd South African–Nigerian Bi-National Commission meeting. Union Building, Pretoria.