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APPLICATION OF THE CONTRACTION MAPPING PRINCIPLE TO THE GOURSAT PROBLEM

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ABSTRACT:

This article focuses on developing students' ability to choose the most appropriate method by solving problems in multiple ways, which can increase their interest in mathematics.

Keywords: Hyperbolic equation, Goursat problem, Lipschitz continuity, integral equation, linear space, complete metric space, Banach space, norm, mapping, contraction mapping.

INTRODUCTION:

In this article, on the basis of the Banach theorem, we prove the implementation of the principle of contraction mappings in the proof of the existence of a unique solution to the Goursat problem in a complete metric space.

In a rectangle

$$T=\{(x;y)\in R^2 \,|\, x_0<x<a,\, y_0<y<b\}$$

finding an unknown function $u=u(x,y)$ satisfying the equation

$$u_{xy}=\Phi(x,y,u,u_x,u_y) \qquad (1)$$

and conditions

$$\begin{cases} u|_{x=x_0}=\varphi_1(y),\, y_0\le y\le b \\ u|_{y=y_0}=\varphi_2(x),\, x_0\le x\le a \end{cases} \qquad (2)$$

is called the Goursat problem.

Here

$$\varphi_1(y)\in C^1([y_0;b]), \qquad \varphi_2(x)\in C^1([x_0;a]),$$
$$\varphi_1(y_0)=\varphi_2(x_0). \qquad (3)$$

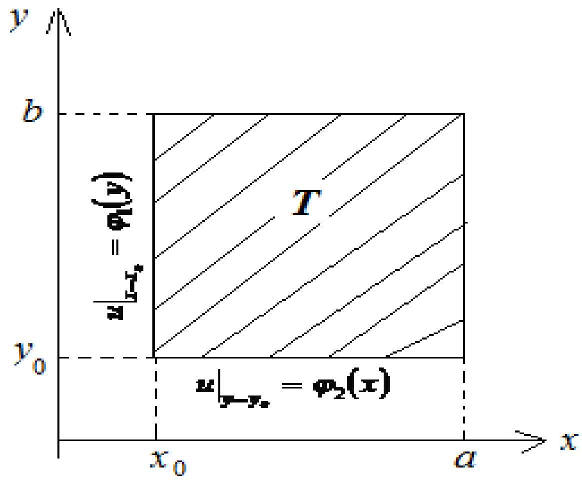

Рисунок 1.

Theorem. Pretending that

$$\Phi(x,y,u,u_x,u_y)\in C([x_0;a]\times[y_0;b]\times R\times R\times R)$$

function and allowing that Lipschitz continuity performs according to the variables u, u_x and u_y т. е.

$$|\Phi(x,y,u,v,w)-\Phi(x,y,u^*,v^*,w^*)|\le L(|u-u^*|+|v-v^*|+|w-w^*|).$$

In this case (1), (2), (3) the Goursat problem has a unique solution

$$u(x,y)\in C^1(\overline{T})\cap C^2(T),$$

where $\overline{T}=\{(x;y)\in R^2 \,|\, x_0\le x\le a,\, y_0\le y\le b\}$.

Proof. Let us reduce the Goursat problem as usual to the solution of a system of integral equations. For this, we introduce the notation $u_x=v$, $u_y=w$. The equality $u_y=w$ will be integrated with respect to the variable y from y_0 to y :

$$u(x,y)=u(x,y_0)+\int_{y_0}^{y} w(x,\eta)d\eta \overset{\left\langle u|_{y=y_0}=u(x,y_0)=\varphi_2(x)\right\rangle}{=} \varphi_2(x)+\int_{y_0}^{y} w(x,\eta)d\eta$$

And from $u_x=v$ equality, we take the derivative with respect to y:

$$v_y=u_{xy}=\Phi(x,y,u,v,w)$$

therefore, the obtained equality is integrable over y from y_0 to y:

$$v(x,y)=v(x,y_0)+\int_{y_0}^{y}\Phi(x,\eta,u(x,\eta),v(x,\eta),w(x,\eta))d\eta \overset{\left\langle v(x,y_0)=u_x|_{y=y_0}=\varphi_2'(x)\right\rangle}{=} \varphi_2'(x)+$$

$$+\int_{y_0}^{y}\Phi(x,\eta,u(x,\eta),v(x,\eta),w(x,\eta))d\eta$$

analogical, from $u_y=w$ equality, we take the derivative with respect to x:

$$w_x=u_{xy}=\Phi(x,y,u,v,w)$$

and the resulting equality is integrated with respect to the variable x from x_0 to x:

$$w(x,y)=w(x_0,y)+\int_{x_0}^{x}\Phi(\xi,y,u(\xi,y),v(\xi,y),w(\xi,y))d\xi \overset{\left\langle w(x_0,y)=u_y|_{x=x_0}=\varphi_1'(y)\right\rangle}{=} \varphi_1'(y)+$$

$$+\int_{x_0}^{x}\Phi(\xi,y,u(\xi,y),v(\xi,y),w(\xi,y))d\xi \ .$$

As a result, we get

$$\begin{cases} u=\varphi_2(x)+\int_{y_0}^{y} w(x,\eta)d\eta \\ v=\varphi_2'(x)+\int_{y_0}^{y}\Phi(x,\eta,u(x,\eta),v(x,\eta),w(x,\eta))d\eta \\ w=\varphi_1'(y)+\int_{x_0}^{x}\Phi(\xi,y,u(\xi,y),v(\xi,y),w(\xi,y))d\xi \end{cases} \quad (4)$$

system of integral equations.

It is clear that if $u(x,y)$ is the solution to the Goursat problem, then u, $v=u_x$, $w=u_y$ functions will be the solution to the system of integral equations (4). And, if it go in reverse, that is u, v, w, continuous functions are the solution to a system of integral equations (4), then the $u(x,y)$ function will be the solution to the Goursat problem. Considering a norm and a mapping in a linear space:

$$\vec{C}_M(T)=\left\{\vec{U}=(u,v,w)\,\middle|\,u\in C(\overline{T}), v\in C(\overline{T}), w\in C(\overline{T})\right\}$$

:

$$\|AU_1-AU_2\|=\max\left\{\sup_{(x,y)\in T}\left|e^{-M(x+y)}(A_1U_1-A_1U_2)\right|,\ \sup_{(x,y)\in T}\left|e^{-M(x+y)}(A_2U_1-A_2U_2)\right|,\ \sup_{(x,y)\in T}\left|e^{-M(x+y)}(A_3U_1-A_3U_2)\right|\right\},$$

(5) $A:\vec{C}_M(T)\to\vec{C}_M(T)$. $\vec{C}_M(T)$ linear space considers as a Banach space.

$$(u,v,w)\overset{A}{\to}(A_1(u,v,w),A_2(u,v,w),A_3(u,v,w))=\left(\varphi_2(x)+\int_{y_0}^{y}wd\eta,\ \varphi_2'(x)+\int_{y_0}^{y}\Phi d\eta,\ \varphi_1'(y)+\int_{x_0}^{x}\Phi d\xi\right)$$

.

if we show that A is a contraction map, then the system of integral equations by the well-known Banach theorem will have a unique solution in the linear space $\vec{C}_M(T)$. To do this

$$\rho(AU_1,AU_2)\le\alpha\rho(U_1,U_2)$$

or

$$\|AU_1-AU_2\|\le\alpha\|U_1-U_2\|$$

(6)

we will evaluate each component of the norm. (5).

For the first component:

$$\left|e^{-M(x+y)}(A_1U_1-A_1U_2)\right|=\left|e^{-M(x+y)}\int_{y_0}^{y}(w_1(x,\eta)-w_2(x,\eta))d\eta\right|=$$

$$=e^{-My}\int_{y_0}^{y}e^{M\eta}\left|e^{-M(x+\eta)}(w_1(x,\eta)-w_2(x,\eta))\right|d\eta\le e^{-My}|U_1-U_2|\left|\int_{y_0}^{y}e^{M\eta}d\eta=\frac{1}{M}\right|U_1-U_2|e^{-My}\left(e^{My}-e^{My_0}\right)=$$

$$=\frac{1}{M}\left(1-e^{M(y_0-y)}\right)|U_1-U_2|\le\frac{1}{M}|U_1-U_2|.$$

For second component:

$$\left|e^{-M(x+y)}(A_2U_1-A_2U_2)\right|\le\left|e^{-M(x+y)}\int_{y_0}^{y}(\Phi(x,\eta,u_1,v_1,w_1)-\Phi(x,\eta,u_2,v_2,w_2))d\eta\right|=$$

$$=e^{-My}\int_{y_0}^{y}e^{M\eta}\left(e^{-M(x+\eta)}|u_1-u_2|+e^{-M(x+\eta)}|v_1-v_2|+e^{-M(x+\eta)}|w_1-w_2|\right)d\eta\le$$

$$\leq 3L|U_1-U_2|e^{-My}\int_{y_0}^{y}e^{M\eta}d\eta=\frac{3L}{M}|U_1-U_2|e^{-My}\left(e^{My}-e^{My_0}\right)=\frac{3L}{M}\left(1-e^{M(y_0-y)}\right)|U_1-U_2|\leq\frac{3L}{M}|U_1-U_2|$$

For third component:

$$\left|e^{-M(x+y)}\left(A_3U_1-A_3U_2\right)\right|\leq\frac{3L}{M}|U_1-U_2|.$$

from these estimates it follows that

$$\|AU_1-AU_2\|\leq\max\left(\frac{1}{M},\frac{3L}{M},\frac{3L}{M}\right)\|U_1-U_2|$$

is equal to

$$\alpha=\max\left(\frac{1}{M},\frac{3L}{M},\frac{3L}{M}\right).$$

If M is large enough, α will be less than one i.e. $\alpha<1$, отображение A will be contracted. It follows from this that the system of integral equations will have a unique solution. Consequently, the Goursat problem will also have a unique solution:

$$u(x,y)\in C^1(\overline{T})\cap C^2(T).$$

<u>The theorem is proved.</u>

Result. If in the first equation (1) the function

$\Phi(x,y,u,u_x,u_y)$

linear in variables u , u_x and u_y , i.e.

$$\Phi(x,y,u,u_x,u_y)=a(x,y)u_x+b(x,y)u_y+c(x,y)u-f(x,y)$$

and

$$\{a(x,y),b(x,y),c(x,y),f(x,y)\}\in C(\overline{T}),$$

then the Goursat problem (1), (2), (3) will have unique solution

$$u(x,y)\in C^1(\overline{T})\cap C^2(T).$$

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