SPECIFIC INTEGRAL MODEL OPERATOR SPECTRUM

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Annotation: The article analyzes the existence of a spectrum of an operator with a special integral model

Keywords: bounded and self-joining operators, compact operator, spectral properties of the operator.

KIRISH. Model operatorni berilgan uzluksiz funksiyaga ko'paytirish operatorining xususiy integralli, ya'ni nokompakt integral operator yordamida qo'zg'alishi sifatida aniqlaymiz. Asosiy maqsadimiz shu operatorning spektrini, aniqrog'i uning muhim spektrini, muhim spektrdan tashqarida yotuvchi xos qiymatlarining operator parametrlaridan bog'liq ravishda mavjudlik masalasini hamda xos qiymatlarning joylashuv o'rnini topishdan iboratdir. Bu masala shuningdek, [1],[2],[3],[5],[6] adabiyotlarda qaralgan. [4] ishda bir o'lchamli Shredinger operatorining parametrning kichik qiymatlaridagi diskret spektri va rezonanslari o'rganilgan. Dastlab kiritilgan operatorning sodda xossalarini, ya'ni uning chiziqli, chegaralangan va o'z-o'ziga qo'shma operatorlar sinfiga tegishli ekanligini bayon qilamiz. Keyinchalik esa berilgan operatorni to'g'ri integralga yig'iluvchanlik masalasiga to'xtalamiz. Bu bilan biz ko'paytirish operatoriga qo'shilgan xususiy integral operatorni to'la integralli kompakt operatorga keltiramiz hamda berilgan operator spectral xossalarini o'rganish masalasini mos Gilbert fazosida anialangan o'z-o'ziga qo'shma chiziqli chegaralangan operatorlar oilasining spektral xossalarini o'rganishga keltiramiz.

Model operatorning kiritilishi va uning sodda xossalari

Biz berilgan uzluksiz funksiyaga ko'paytirish operatorining biror xusuiy integrally integral operator yordamida qo'zg'alishini ifodalovchi bir model mos Gilbert fazosida operatorini kiritamiz. Bu operatorning kiritilgan Gilbert fazosida sodda xossalarini, aniqrog'i uning chiziqli chegaralangan operatorlar sinfiga qarashli ekanligini, o'z-o'ziga qo'shma operator ekanligini isbotlaymiz.

Faraz qilaylik $T^{\nu} \cong (-\pi;\pi]^{\nu}$ - ν-o'lchamli tor, $(T^{\nu})^2 = T^{\nu} \times T^{\nu}$ uning Dekart ko'paytmasi va $L_2((T^{\nu})^2)$ esa $(T^{\nu})^2$ da aniqlangan va kvadrati bilan integrallanuvchi funksiyalarning Gilbert fazosi bo'lsin. Biz T^{ν} torda elementlarni qo'shish va haqiqiy sonmga ko'paytirish amallarini har bir koordinatasi 2π modul (taqqoslama) bo'yicha xuddi R^{ν} da mos ravishda elementlarni qo'shish va haqiqiy songa ko'paytirish amallari kabi kiritamiz. Masalan $\nu=1$ o'lchamli holda quyidagi misollarni keltirishimiz mumkin:

$$\frac{3\pi}{4}+\pi=\frac{7\pi}{4}=-\frac{\pi}{4}\,\mathrm{mod}(2\pi),\quad 10\cdot\frac{\pi}{2}=5\pi=\pi\ \mathrm{mod}(2\pi).$$

Xuddi shunnday $\nu=2$ o'lchamli holda quyidagi misollarni keltirishimiz mumkin:

$$\left(\frac{3\pi}{4};-\frac{2\pi}{3}\right)+\left(\pi;-\frac{\pi}{2}\right)=\left(\frac{7\pi}{4};-\frac{7\pi}{6}\right)=\left(-\frac{\pi}{4};\frac{5\pi}{6}\right)\mathrm{mod}(2\pi,2\pi),$$

$$10\cdot\left(\frac{\pi}{2},-\pi\right)=(5\pi;-10\pi)=(\pi;0)\ \mathrm{mod}(2\pi;2\pi)$$

Shu kabi ixtiyoriy o'lchamli torda qo'shish va songa ko'paytirish amallarini aniqlaymiz. Demak $L_2(T^\nu)$ Gilbert fazosining elementlarini R^ν da aniqlangan va har bir argumenti bo'yicha 2π davrli hamda $(-\pi;\pi]^\nu$ da modulining kvadrati bilan integrallanuvchi ekvivalent funksiyalar sinfi deb qarash mumkin. Bu fazoning elementlari sifatida quyidagi funksiyalarni misol sifatida keltish mumkin:

$$f(q)=\cos q_1+\cos q_2+\cdots+\cos q_\nu,\quad g(q)=\sin q_1\sin q_2\cdots\sin q_\nu.$$

$L_2\left((T^\nu)^2\right)$ Gilbert fazosida

$$h_\mu=h_0-\mu V \tag{1}$$

ko'rinishda aniqlangan model operatorni qaraymiz. Bunda h_0 berilgan uzluksiz funksiyaga ko'paytirish operatori, V esa xususiy integralli integral operator bo'lib, ular $L_2\left((T^\nu)^2\right)$ fazoda mos ravishda quyidagi ta'sir formulalariga ega:

$$\begin{aligned}&(h_0 f)(p,q)=u(p,q)f(p,q),\quad f\in L_2\left((T^\nu)^2\right)\\&(Vf)(p,q)=\varphi(q)\int_{T^\nu}\varphi(t)f(p,t)dt,\quad f\in L_2\left((T^\nu)^2\right)\end{aligned} \tag{2}$$

Buyerda $u(\cdot,\cdot)$– $(T^\nu)^2$ da aniqlangan haqiqiy qiymatli berilgan analitik funksiya, $\varphi(\cdot)$ esa T^ν da aniqlangan haqiqiy qiymatli berilgan analitik funksiya va $\mu>0$- biror musbat haqiqiy qiymatli parameter.

1-Lemma. (1) formula bilan berilgan h_μ *operator* $L_2\left((T^\nu)^2\right)$ *fazoda aniqlangan chiziqli chegaralangan, o'z-o'ziga qo'shma operatorlar sinfiga qarashli bo'ladi.*

Isbot. Dastlab (1) formula bilan berilgan operatorni chiziqli operator bo'lishini tekshiramiz. Buning uchun oldinbgi bobdagi berilgan ta'rifga binoan ixtiyoriy $\forall f,g\in L_2\left((T^\nu)^2\right)$ elementlar va ixtiyoriy $\alpha,\beta\in C$ sonlar uchun

$$\left(h_\mu(\alpha f+\beta g)\right)(p,q)=\alpha(h_\mu f)(p,q)+\beta(h_\mu g)(p,q) \tag{3}$$

tenglik o'rinli ekanligini ko'rsatish kerak. (3) tenglikning chap tomonidan uning o'ng tomonini keltirib chiqaramiz. Haqiqatan ham

$$\left(h_\mu(\alpha f+\beta g)\right)(p,q)=u(p,q)(\alpha f+\beta g)(p,q)-\mu\varphi(q)\int_{T^\nu}\varphi(t)(\alpha f+\beta g)(p,t)dt.$$

Bu tenglikning o'ng qismida funksiyalarning yigindisi va songa ko'paytmasi ta'rifini tatbiq qilsak quyidagi tenglikni olamiz:

$$\left(h_\mu(\alpha f+\beta g)\right)(p,q)=u(p,q)(\alpha f(p,q)+\beta g(p,q))-\mu\varphi(q)\int_{T^\nu}\varphi(t)(\alpha f(p,t)+\beta g(p,t))dt\,.$$

Agar bunda integrallanuvchi funksiyalar yig'indisining integrali integrallar yig'indisiga teng eknaligini hisobga olsak va hosil bo'lgan ifodada guruhlashni tatbiq qilib quyidagi tenglikka kelamiz:

$$\left(h_\mu(\alpha f+\beta g)\right)(p,q)=\alpha\left[u(p,q)f(p,q)-\mu\varphi(q)\int_{T^\nu}\varphi(t)f(p,t)dt\right]+$$

$$+\beta\left[u(p,q)f(p,q)-\mu\varphi(q)\int_{T^\nu}\varphi(t)f(p,t)dt\right]=\alpha(h_\mu f)(p,q)+\beta(h_\mu g)(p,q).$$

Bu esa (1) tenglikni, ya'ni h_μ operatorning $L_2\left((T^\nu)^2\right)$ fazoda aniqlangan chiziqli operator ekanligini anglatadi.

Endi h_μ operatorning chegaralangan operatorlar sinfiga qarashli ekanligini tekshiramiz. Oldingi bobda berilgan ta'rifga ko'ra ixtiyoriy $\forall f\in L_2\left((T^\nu)^2\right)$ uchun

$$\|h_\mu f\|\le C\|f\|$$

tengsizlikni qanoatlantiruvchi chekli musbat $C>0$ haqiqiy sonning mavjudligini ko'rsatish yetarli. (3.1.1) va normaning xossasidan quyidagi tengsizlik o'rinli

$$\|h_\mu f\|=\|h_0 f-\mu Vf\|\le\|h_0 f\|+\mu\|Vf\|. \tag{4}$$

Demak (2) formulalar bilan aniqlangan h_0 va V operatorlarning chegaralangan ekanligini ko'rsatish yetarli bo'lar ekan. Ularning normalarini baholaymiz. Haqiqatan ham $L_2\left((T^\nu)^2\right)$ fazoda normaning aniqlanishidan $\forall f\in L_2\left((T^\nu)^2\right)$ uchun quyidagiga egamiz:

$$\|h_0 f\|^2=\int_{(T^\nu)^2}\left|(h_0 f)(p,q)\right|^2dpdq\,.$$

Bunda h_0 operator ta'sir ifodasi uchun (2) formulani e'tiborga olsak yuqoridagi ifoda quyidagi ko'rinishga ega bo'ladi:

$$\|h_0 f\|^2=\int_{(T^\nu)^2}|u(p,q)|^2\,|f(p,q)|^2dpdq\,.$$

Shartga asosan $u(\cdot,\cdot)-$ $(T^\nu)^2$ da aniqlangan haqiqiy qiymatli analitik funksiya bo'lganligi uchun u $(T^\nu)^2$ kompaktda o'zining eng katta qiymatiga erishadi. U holda yuqoridagi normani quyidagicha baholash mumkin:

$$\|h_0 f\|^2\le M^2\int_{(T^\nu)^2}|f(p,q)|^2dpdq=M^2\|f\|^2,\ \forall f\in L_2\left((T^\nu)^2\right).$$

Demak $\forall f \in L_2\left((T^\nu)^2\right)$ uchun $\| h_0 f \| \le |M| \, \| f \|, \ \forall f \in L_2\left((T^\nu)^2\right)$ tengsizlik o'rinli bo'lib, h_0 chegaralangan operator bo'lar ekan.

Enda (2) ning ikkinchi formulasi bilan aniqlangan V operatorni chegaralanganlikka tekshiramiz. Haqiqatan ham $L_2\left((T^\nu)^2\right)$ fazoda normaning aniqlanishidan $\forall f \in L_2\left((T^\nu)^2\right)$ uchun quyidagiga egamiz:

$$\| Vf \|^2 = \int_{(T^\nu)^2} |(Vf)(p,q)|^2 \, dpdq .$$

Bunda V operator ta'sir ifodasi uchun (2) formulani e'tiborga olsak yuqoridagi ifoda quyidagi ko'rinishga ega bo'ladi:

$$\| Vf \|^2 = \int_{(T^\nu)^2} \left| \varphi(q) \int_{T^\nu} \varphi(t) f(p,t) dt \right|^2 dpdq .$$

O'ng tomondagi ichki integralga Koshi-Bunyakovskiy tengsizligini qo'llasak quyidagi tengsizlikka kelamiz:

$$\| Vf \|^2 \le \int_{(T^\nu)^2} |\varphi(q)|^2 \left\{ \int_{T^\nu} |\varphi(t)|^2 dt \cdot \int_{T^\nu} |f(p,t)|^2 dt \right\} dpdq =$$

$$= \left(\int_{(T^\nu)^2} |\varphi(q)|^2 dq \right)^2 \int_{(T^\nu)^2} |f(p,t)|^2 dpdt = \|\varphi\|^4 \|f\|^2 .$$

Bunda biz takroriy va karrali integrallarning tengligi haqidagi Fubini teoremasi (qarang [9]) tasdig'idan foydalandik. Shunday qilib, biz V operator normasi uchun

$$\|Vf\| \le \|\varphi\|^2 \|f\|, \ \forall f \in L_2\left((T^\nu)^2\right)$$

bahoning o'rinliligini oldik. Bu esa V chegaralangan operator ekanligini bildiradi. Natijada (4) ga asosan h_μ operator normasi uchun

$$\| h_\mu f \| \le \left(|M| + \|\varphi\|^2\right) \|f\|, \ \forall f \in L_2\left((T^\nu)^2\right)$$

tengsizlik bajariladi. Bu esa ta'rifga asosan h_μ ning $L_2\left((T^\nu)^2\right)$ fazoda chegaralangan operatorlar oilasiga qarashli ekanligini anglatadi.

Endi 1-Lemmaninig isbotini yakunlash maqsadida h_μ operatorning o'z-o'ziga qo'shma operator ekanligini kp'rsatish bilan shug'ullanamiz. Buning uchun oldingi bobda berilgan ta'rifga binoan $\forall f, g \in L_2\left((T^\nu)^2\right)$ elemaentlar uchun

$$(h_\mu f, g) = (f, h_\mu g)$$

tenglikning bajarilishini ko'rsatish yetarli. Agar (1) formulani hisobga olsak yuqoridagi tenglik chap qismi quyidagicha yoziladi:

$$(h_\mu f, g) = (h_0 f, g) - \mu(Vf, g).$$

Demak h_0 va V operatorlarning o'z-o'ziga qo'shmaligini ko'rsatish yetarli. Dastlab h_0 operatorni o'z-o'ziga qo'shmalikka tekshiramiz. Ixtiyoriy $\forall f, g \in L_2\left((T^\nu)^2\right)$ uchun bu fazoda skalyar ko'paytmaning aniqlanishidan quyidagiga egamiz:

$$(h_0 f, g) = \int_{(T^\nu)^2} (h_0 f)(p,q)\overline{g(p,q)}dpdq\,.$$

Bunda (2) formulani nazarga olsak, quyidagi tenglikka kelamiz:

$$(h_0 f, g) = \int_{(T^\nu)^2} u(p,q) f(p,q)\overline{g(p,q)}dpdq$$

yoki

$$(h_0 f, g) = \int_{(T^\nu)^2} f(p,q)\overline{\overline{u(p,q)}g(p,q)}dpdq\,.$$

Shartga asosan $u(\cdot,\cdot)-$ $(T^\nu)^2$ da aniqlangan haqiqiy qiymatli funksiya bo'lganligi uchun yuqoridagi ifoda quyidagiga teng kuchli bo'ladi:

$$(h_0 f, g) = \int_{(T^\nu)^2} f(p,q)\overline{u(p,q)g(p,q)}dpdq = (f, h_\mu g).$$

Demak h_0 o'z-o'ziga qo'shma operator ekan. Endi xususiy integralli V operatorni o'z-o'ziga qo'shma operator bo'lishlikka tekshiramiz. Ixtiyoriy $\forall f, g \in L_2\left((T^\nu)^2\right)$ uchun bu fazoda skalyar ko'paytmaning aniqlanishidan quyidagiga egamiz:

$$(Vf, g) = \int_{(T^\nu)^2} (Vf)(p,q)\overline{g(p,q)}dpdq\,.$$

Bunda (2) formulani nazarga olsak, yuqoridagi ifodani quyidagicha yozishimiz mumkin bo'ladi:

$$(Vf, g) = \int_{(T^\nu)^2} \left\{\varphi(q)\int_{T^\nu}\varphi(t) f(p,t)dt\right\}\overline{g(p,q)}dpdq$$

Takroriy integrallarning tenglifi haqidagi Fubini teoremasini qo'llasak va $\varphi(\cdot)$ ning haqiqiy qiymatli finksiya ekanligini e'tiborga olsak oxirgi ifoda quyidagi ko'rinishda tasvirlanadi:

$$(Vf, g) = \int_{(T^\nu)^2} f(p,t)\left\{\overline{\varphi(t)\int_{T^\nu}\varphi(q)\, g(p,q)dq}\right\}dpdt\,.$$

Bu integrallarda o'zgaruvchilarni $q := t;\, t := q$ kabi almashtirsak u quyidagicha yoziladi:

$$(Vf, g) = \int_{(T^\nu)^2} f(p,q)\left\{\overline{\varphi(q)\int_{T^\nu}\varphi(t)\, g(p,t)dt}\right\}dpdq = (f, Vg).$$

Bu esa $V : L_2\left((T^\nu)^2\right) \to L_2\left((T^\nu)^2\right)$ operatorning va o'z navbatida h_μ operatorning $L_2\left((T^\nu)^2\right)$ fazoda aniqlangan o'z-o'ziga qo'shma operatorlar oilasiga qarashli ekanligini anglatadi. *1-Lemma isbot bo'ldi.*

2-Lemma. (1) formula bilan berilgan h_μ operator $L_2((T^\nu)^2)$ fazoda $p \in T^\nu$ o'zgaruvchining ixtiyoriy funksiyasi $w(p)$ funksiyaga ko'paytirish operatori $W : L_2((T^\nu)^2) \to L_2((T^\nu)^2)$ bilan o'rin almashtiradi.

Eslatma. Yuqoridagi ta'kidlangan tasdiqlarga va Lebegning yaqinlashish haqidagi teoremasiga ko'ra (qarang [7],[8]) ixtiyoriy $z < m_u = 0$ uchun $\Lambda(\cdot, z)$ funksiyaning barcha ikkinchi tartibli xususiy hosilalari $C^{(2)}(T^4)$ fazoning elementi bo'ladi.

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