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Recurrence Decompositions in Finsler Space

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Abstract

Finsler geometry is a kind of differential geometry originated by P. Finsler. Indeed, Finsler geometry has several uses in a wide variety and it is playing an important role in differential geometry and applied mathematics of problems in physics relative, manual footprint. It is usually considered as a generalization of Riemannian geometry. In the present paper, we introduced some types of generalized W^h -birecurrent Finsler space , generalized W^h -birecurrent affinely connected space and we defined a Finsler space F_n for Weyl's projective curvature tensor W^i_{jkh} satisfies the generalized-birecurrence condition with respect to Cartan's connection parameters Γ^{*i}_{kh} , such that given by the condition (2.1), where $|m\ |n$ is h-covariant derivative of second order (Cartan's second kind covariant differential operator) with respect to x^m and x^n , successively, λ_{mn} and μ_{mn} are non-null covariant vectors field and such space is called as a generalized W^h -birecurrent space and denoted briefly by GW^h - BRF_n . We have obtained some theorems of generalized W^h -birecurrent affinely connected space for the h-covariant derivative of the second order for Wely's projective torsion tensor W^i_{kh} , Wely's projective deviation tensor W^i_h in our space. We have obtained the necessary and sufficient condition forsome tensors in our space.

Keywords: Generalized W^h -birecurrent affinely connected space, Generalized W^h - Birecurrent space, Weyl's projective curvature tensor W^i_{jkh}, Finsler space F_n.
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1. Introduction

The recurrent for different curvature tensors have been discussed by Al-Qashbari [1], Dikshit [2], Matsumoto [3], Pandey [4], Pandey, et al. [5], Qasem [6], Qasemet al. [7, 8] and others. Ahsanand Ali [9] who discussed a recurrent curvature tensor on some properties of W-curvature tensor of Weyl's projective curvature tensor W^i_{jkh} is skew-symmetric, it is indices k and h. The generalized birecurrent space was studied by Hadi [10], Qasem and Hadi [11]. Also, W-generalized birecurrent space studied by Qasem and Saleem [12] and others.

An n-dimensional Finsler space, Figure 1, equipped with the metric function f satisfies the requisite conditions [13]. Consider the components of the corresponding metric tensor

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g_{ij} , Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^{i}_{jk} (the indices $i, j, k, \ldots$ assume positive integral values from 1 to n). These are symmetric in their lower indices.

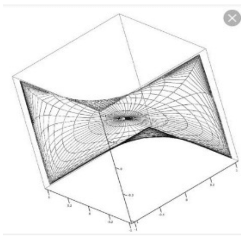

Figure 1: Finsler Space as a Locally Minkowskian Space

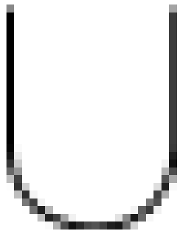

Figure 2: Metric Tensor g_{ij}

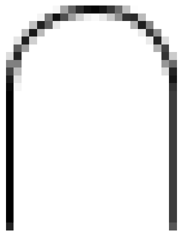

Figure 3: Metric Tensor g^{ij}

The vectors y_i and y^i satisfy the following relations [13]:

$$y_i = g_{ij}\ y^j, \tag{1.1}$$

$$y_i\, y^i = F^2, \tag{1.2}$$

$$\dot{\partial}_i y_j = g_{ij}, \tag{1.3}$$

$$\dot{\partial}_j y^i = \delta^i_j. \tag{1.4}$$

The metric tensor g_{ij} Figure 2, the metric tensor g^{ij} Figure 3 and the vector y^i are covariant constant with respect to the above process.

$$g_{ij|k} = 0, \tag{1.5}$$

$$y^i_{|k} = 0, \tag{1.6}$$

$$g^{ij}_{|k} = 0., \tag{1.7}$$

The h-covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j , successively ,we get

$$X^i_{|k\,|j} = \dot{\partial}_j \left(X^i_{|k}\right) - \left(X^i_{|r}\right)\Gamma^{*r}_{kj} + \left(X^r_{|k}\right)\Gamma^{*i}_{rj} - \dot{\partial}_r \left(X^i_{|k}\right)\Gamma^{*r}_{js}\, y^s \tag{1.8}$$

Taking skew-symmetric part with respect to the indices *k* and *j*, we get the commutation formula for h-covariant differentiation as follows [13]:

$$X^i_{|k\,|j} - X^i_{|j\,|k} = X^r K^i_{rkj} - \left(\dot{\partial}_r X^i\right) K^r_{skj}\, y^s, \tag{1.9}$$

where

$$K^i_{rkj} := \partial_j \Gamma^{*i}_{kr} + \left(\dot{\partial}_l \Gamma^{*i}_{rj}\right) G^l_k + \Gamma^{*i}_{mj} \Gamma^{*m}_{kr} - \frac{j}{k}. \tag{1.10}$$

where the indices i, j, k ,... are positive integral values from 1 to n, and $-\frac{j}{k}$ means the subtraction from the former term by interchanging the indices k and j.

The tensor K^i_{rkj} as defined above is called Cartan's fourth curvature tensor.

The process of h-covariant differentiation, with respect to x^k , commute with partial differentiation with respect to y^j for arbitrary vector filed X^i , according to [13],

$$\dot{\partial}_j \left(X^i_{|k}\right) = \left(\dot{\partial}_j X^i\right)_{|k} + X^r \left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk} \tag{1.11}$$

and

$$\dot{\partial}_j \Gamma^{*i}_{rk} = \Gamma^{*i}_{rkj}. \tag{1.12}$$

The tensor W^i_{jkh} is known as projective curvature tensor (generalized Wely's projective curvature tensor), the tensor W^i_{jk} is known as projective torsion tensor (Wely's torsion tensor) and the tensor W^i_j is known as projective deviation tensor (Wely's deviation tensor are defined by

$$\begin{aligned} W^i_{jkh} \quad = \quad & H^i_{jkh} + \frac{2\delta^i_j}{n+1} H_{hk} + \frac{2y^i}{n+1} \dot{\partial}_j H_{kh} + \frac{\delta^i_k}{n^2-1} \left(nH_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right) \\ & - \frac{\delta^i_h}{n^2-1} \left(nH_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\right), \end{aligned} \tag{1.13}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H_{jk} + \frac{2\dot{\partial}_j}{n^2-1} \left(nH_{|k} - y^r H_{k|r}\right), \tag{1.14}$$

and

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{n+1} \left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right) y^i, \tag{1.15}$$

respectively.

The tensors W^i_{jkh} , W^i_{jk} and W^i_j are satisfying the following identities [13]

$$W^i_{jkh}\, y^j = W^i_{kh}\, , \tag{1.16}$$

$$W^i_{jk}\, y^j = W^i_k\, , \tag{1.17}$$

$$\dot{\partial}_j W^i_{kh} = W^i_{jkh}, \tag{1.18}$$

$$\dot{\partial}_k W^i_h = W^i_{kh}. \tag{1.19}$$

The projective curvature tensor W^i_{jkh} is skew-symmetric in its indices k and h see the definition in (1.13).

An affinely connected space has some properties as follows:

$$G^i_{jkh} \; = \; 0, \text{ and } \; C_{ijk|h} = 0 \tag{1.20}$$

Remark 1.1. An affinely connected space or Berwald space characterized by any one of the above two equivalent conditions. Also, we have the following properties.

The connection parameters Γ^{*i}_{jk} of Cartan and G^{i}_{jk} of Berwald coincide in an affinely connected space and they are independent of the directional arguments [13], i.e.

$$G^{i}_{jkh} = \dot{\partial}_j \mathbf{G}^{\mathbf{i}}_{\mathbf{kh}} = 0, \tag{1.21}$$

$$\dot{\partial}_j \Gamma^{*i}_{kh} = 0, \tag{1.22}$$

$$y_r\, G^{r}_{ijk} = -2\, C_{ijk|h}\, y^{h} = -2\, P_{ijk} = 0. \tag{1.23}$$

2. On Necessary and Sufficient Condition of Generalized W^h -Birecurrent

A Finsler space F_n for which Weyl's projective curvature tensor W^{i}_{jkh} satisfies the recurrence property withrespect to Cartan's coefficient connection parameters Γ^{*i}_{jk} which is characterized by the condition [13]

$$W^{i}_{jkh\,|m} = a_m\, W^{i}_{jkh} + b_m\, \left(\delta^{i}_{h} g_{jk} - \delta^{i}_{k} g_{jh}\right),\ W^{i}_{jkh} \neq 0, \tag{2.1}$$

where $|m$ is h-covariant derivative of first order (Cartan's second kind covariant differential operator) with respect to x^m , the quantities a_m and b_m are non-null covariant vectors field. It's space is called as a generalized W^h -recurrent space and is denoted briefly by GW^h - RF_n.

Taking the h-covariant derivative for (2.1) with respect to x^n and using (1.5), we get

$$W^{i}_{jkh|m|n} = a_{m|n}\, W^{i}_{jkh} + a_m\, W^{i}_{jkh|n} + b_{m|n}\, \left(\delta^{i}_{h} g_{jk} - \delta^{i}_{k} g_{jh}\right)$$

where $g_{jk|n} = 0$.

In view of (2.1), the above equation yields

$$W^{i}_{jkh|m|n} = \lambda_{mn} W^{i}_{jkh} + \mu_{mn}\, \left(\delta^{i}_{h} g_{jk} - \delta^{i}_{k} g_{jh}\right), \tag{2.2}$$

where $\lambda_{mn} = a_{m|n} + a_m\, a_n$, $\mu_{mn} = a_m\, a_n + b_{m|n}$ and $|m\ |n$ is h-covariant derivative of second order (Cartan's second kind covariant differential operator) with respect to x^m and x^n, successively, λ_{mn} and μ_{mn} are non-null covariant vectors field and such space is called as a generalized W^h -birecurrent spaceand denoted briefly by GW^h - $BIRF_n$.

Result Every generalized W^h -recurrent space is generalized W^h -birecurrent space.

Transvecting the condition (2.2) by y^j and by y^k , successively, using (1.6), (1.16), (1.17), (1.1) and (1.2), we get

$$W^{i}_{kh|m\,|n} = \lambda_{mn} W^{i}_{kh} + \mu_{mn}\, \left(\delta^{i}_{h} y_k - \delta^{i}_{k} y_h\right), \tag{2.3}$$

$$W^{i}_{h|m\,|n} = \lambda_{mn}\, W^{i}_{h} + \mu_{mn}\, \left(\delta^{i}_{h} F^2 - y_h\, y^{i}\right), \tag{2.4}$$

Theorem 2.1. *In GW^h - $BIRF_n$, the h-covariant derivative of the second order forWely's projective torsion tensor W^i_{kh} and Wely's projective deviation tensor W^i_h are given by (2.3) and (2.4), respectively.*

Differentiating partially of (2.3) with respect to y^j, using (1.23) and (1.3), we get

$$\begin{aligned}\dot{\partial}_r\left(W^i_{kh|m|n}\right) &= (\dot{\partial}_j\lambda_{mn})\ W^i_{kh} + \lambda_{mn}\ W^i_{jkh} + (\dot{\partial}_j\mu_{mn})\ (\delta^i_h y_k - \delta^i_k y_h) \\ &\quad +\mu_{mn}\ (\delta^i_h g_{jk} - \delta^i_k g_{jh}).\end{aligned}$$

Using the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor $W^i_{kh|m}$, in the above equation, we get

$$\begin{aligned}&\left\{\dot{\partial}_r W^i_{kh|m}\right\}_{|n} + W^r_{kh|m}\ (\dot{\partial}_j\Gamma^{*i}_{rn}) - W^i_{rh|m}\ (\dot{\partial}_j\Gamma^{*r}_{kn}) \\ &-W^i_{kr|m}\ (\dot{\partial}_j\Gamma^{*r}_{hn}) - W^i_{kh|r}\ (\dot{\partial}_j\Gamma^{*r}_{mr}) - \left(\dot{\partial}_r W^i_{kh|m}\right)\ P^r_{jn} \\ = &\ (\dot{\partial}_j\lambda_{mn})\ W^i_{kh} + \lambda_{mn}\ W^i_{jkh} + (\dot{\partial}_j\mu_{mn})\ (\delta^i_h y_k - \delta^i_k y_h) \\ &+\mu_{mn}\ (\delta^i_h g_{jk} - \delta^i_k g_{jh}).\end{aligned} \tag{2.5}$$

Again, applying the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor (W^i_{kh}), in the equation (2.5) and using (1.18), we obtain

$$\begin{aligned}&\left\{W^i_{jkh|m} + W^r_{kh}\ (\dot{\partial}_j\Gamma^{*i}_{rm}) - W^i_{rh}\ (\dot{\partial}_j\Gamma^{*r}_{km}) - W^i_{kr}\ (\dot{\partial}_j\Gamma^{*r}_{hm}) - (\dot{\partial}_r W^i_{kh})\ P^r_{jm}\right\}_{|n} \\ &+W^r_{kh|m}\ (\dot{\partial}_j\Gamma^{*i}_{rn}) - W^i_{rh|m}\ (\dot{\partial}_j\Gamma^{*r}_{kn}) - W^i_{kr|m}\ (\dot{\partial}_j\Gamma^{*r}_{hn}) \\ &-W^i_{kh|r}\ (\dot{\partial}_j\Gamma^{*r}_{mn}) - \left(\dot{\partial}_r W^i_{kh|m}\right) P^r_{jn} \\ = &\ (\dot{\partial}_j\lambda_{mn})\ W^i_{kh} + \lambda_{mn} W^i_{jkh} + (\dot{\partial}_j\mu_{mn})\ (\delta^i_h y_k - \delta^i_k y_h) \\ &+\mu_{mn}\ (\delta^i_h g_{jk} - \delta^i_k g_{jh}),\end{aligned}$$

which can be written as

$$\begin{aligned}&W^i_{jkh|m|n} + \left\{W^r_{kh}\ (\dot{\partial}_j\Gamma^{*i}_{rm}) - W^i_{rh}\ (\dot{\partial}_j\Gamma^{*r}_{km}) - W^i_{kr}\ (\dot{\partial}_j\Gamma^{*r}_{hm}) - (\dot{\partial}_r W^i_{kh})\ P^r_{jm}\right\}_{|n} \\ &+W^r_{kh|m}\ (\dot{\partial}_j\Gamma^{*i}_{rn}) - W^i_{rh|m}\ (\dot{\partial}_j\Gamma^{*r}_{kn}) - W^i_{kr|m}\ (\dot{\partial}_j\Gamma^{*r}_{hn}) \\ &-W^i_{kh|r}\ (\dot{\partial}_j\Gamma^{*r}_{mn}) - \left(\dot{\partial}_r W^i_{kh|m}\right) P^r_{jn} \\ = &\ (\dot{\partial}_j\lambda_{mn})\ W^i_{kh} + \lambda_{mn} W^i_{jkh} + (\dot{\partial}_j\mu_{mn})\ (\delta^i_h y_k - \delta^i_k y_h) \\ &+\mu_{mn}\ (\delta^i_h g_{jk} - \delta^i_k g_{jh}).\end{aligned} \tag{2.6}$$

This shows that

$$W^i_{jkh|m|n} = \lambda_{mn} W^i_{jkh} + \mu_{mn}\ (\delta^i_h g_{jk} - \delta^i_k g_{jh})$$

if and only if

$$\begin{aligned}
&\{W^r_{kh}\left(\dot\partial_j\Gamma^{*i}_{rm}\right)-W^i_{rh}\left(\dot\partial_j\Gamma^{*r}_{km}\right)-W^i_{kr}\left(\dot\partial_j\Gamma^{*r}_{hm}\right)-\left(\dot\partial_r W^i_{kh}\right)\ P^r_{jm}\ \}_{|n}\\
&+W^r_{kh|m}\left(\dot\partial_j\Gamma^{*i}_{rn}\right)-W^i_{rh|m}\left(\dot\partial_j\Gamma^{*r}_{kn}\right)-W^i_{kr|m}\left(\dot\partial_j\Gamma^{*r}_{hn}\right)\\
&-W^i_{kh|r}\left(\dot\partial_j\Gamma^{*r}_{mn}\right)-\left(\dot\partial_r W^i_{kh|m}\right)P^r_{jn}\ -\left(\dot\partial_j\lambda_{mn}\right)\ W^i_{kh}\\
&-\left(\dot\partial_j\mu_{mn}\ \right)\left(\delta^i_h y_k-\delta^i_k y_h\ \right)=0. \qquad (2.7)
\end{aligned}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 2.2. *In* GW^h - $BIRF_n$ *, Weyl's projective curvature tensor* W^i_{jkh} *is generalized birecurrent, if and only if the equation (2.7) is equal to zero.*

Differentiating partially of (2.4) with respect to y^k, using (1.19) and (1.3), we obtain

$$\begin{aligned}
\dot\partial_k\left(W^i_{h|m|n}\right) &= \left(\dot\partial_k\lambda_{mn}\right)\ W^i_h+\lambda_{mn}W^i_{kh}+\left(\dot\partial_k\mu_{mn}\ \right)\left(\delta^i_h F^2-y_h\ y^i\right)\\
&\quad+\mu_{mn}\ \left(\dot\partial_k\delta^i_h F^2-\delta^i_k y_h\ \right).
\end{aligned}$$

Using the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor $W^i_{h|m}$, in the above equation, we have

$$\begin{aligned}
&\left(\dot\partial_k W^i_{h|m}\right)_{|n}+W^r_{h|m}\left(\dot\partial_k\Gamma^{*i}_{rn}\right)-W^i_{r|m}\left(\dot\partial_k\Gamma^{*r}_{hn}\right)\\
&-W^i_{h|r}\left(\dot\partial_k\Gamma^{*r}_{mn}\right)-\left(\dot\partial_r W^i_{h|m}\right)P^r_{kn}\\
=\ &\left(\dot\partial_k\lambda_{mn}\right)\ W^i_h+\lambda_{mn}W^i_{kh}+\left(\dot\partial_k\mu_{mn}\right)\left(\delta^i_h F^2-y_h\ y^i\right)\\
&+\mu_{mn}\ \left(\dot\partial_k\delta^i_h F^2-\delta^i_k y_h\right). \qquad (2.8)
\end{aligned}$$

Again, apply the commutation formula exhibited by (1.11), for the $h(v)$ torsion tensor $(\ W^i_h\ )$, in the equation (2.8) and using (1.19), we get

$$\begin{aligned}
&\left\{W^i_{kh|m}+W^r_h\left(\dot\partial_j\Gamma^{*r}_{hm}\right)-W^i_r\left(\dot\partial_k\Gamma^{*r}_{hm}\right)-\left(\dot\partial_r W^i_h\right)\ P^r_{km}\ \right\}_{|n}+W^r_{h|m}\left(\dot\partial_k\Gamma^{*r}_{rn}\right)\\
&-W^i_{r|m}\left(\dot\partial_k\Gamma^{*r}_{hn}\right)-W^i_{h|r}\left(\dot\partial_k\Gamma^{*r}_{mn}\right)-\left(\dot\partial_r W^i_{h|m}\right)P^r_{kn}\\
=\ &\left(\dot\partial_k\lambda_{mn}\right)\ W^i_h+\lambda_{mn}W^i_{kh}\\
&+\left(\dot\partial_k\mu_{mn}\ \right)\left(\delta^i_h F^2-y_h\ y^i\right)+\mu_{mn}\ \left(\dot\partial_k\delta^i_h F^2-\delta^i_k y_h\ \right),
\end{aligned}$$

which can be written as

$$\begin{aligned}
&W^i_{kh|m|n}+\{W^r_h\left(\dot\partial_j\Gamma^{*r}_{hm}\right)-W^i_r\left(\dot\partial_k\Gamma^{*r}_{hm}\right)-\left(\dot\partial_r W^i_h\right)\ P^r_{km}\ \}_{|n}+W^r_{h|m}\left(\dot\partial_k\Gamma^{*r}_{rn}\right)\\
&-W^i_{r|m}\left(\dot\partial_k\Gamma^{*r}_{hn}\right)-W^i_{h|r}\left(\dot\partial_k\Gamma^{*r}_{mn}\right)-\left(\dot\partial_r W^i_{h|m}\right)P^r_{kn}\\
=\ &\left(\dot\partial_k\lambda_{mn}\right)\ W^i_h+\lambda_{mn}W^i_{kh}\\
&+\left(\dot\partial_k\mu_{mn}\ \right)\left(\delta^i_h F^2-y_h\ y^i\right)+\mu_{mn}\ \left(\dot\partial_k\delta^i_h F^2-\delta^i_k y_h\ \right). \qquad (2.9)
\end{aligned}$$

This shows that

$$W^i_{kh|m|n} = \lambda_{mn} W^i_{kh} + \mu_{mn} \left(\dot\partial_k \delta^i_h F^2 - \delta^i_k y_h \right) \tag{2.10}$$

if and only if

$$\begin{aligned} &\left\{ W^r_h \left(\dot\partial_j \Gamma^{*r}_{hm} \right) - W^i_r \left(\dot\partial_k \Gamma^{*r}_{hm} \right) - \left(\dot\partial_r W^i_h \right) \; P^r_{km} \right\}_{|n} + W^r_{h|m} \left(\dot\partial_k \Gamma^{*r}_{rn} \right) \\ &- W^i_{r|m} \left(\dot\partial_k \Gamma^{*r}_{hn} \right) - W^i_{h|r} \left(\dot\partial_k \Gamma^{*r}_{mn} \right) - \left(\dot\partial_r W^i_{h|m} \right) P^r_{kn} \\ = \; & \left(\dot\partial_k \lambda_{mn} \right) \; W^i_h + \left(\dot\partial_k \mu_{mn} \right) \left(\delta^i_h F^2 - y_h \, y^i \right). \end{aligned} \tag{2.11}$$

Therefore, it is concluded the following.

Theorem 2.3. *In GW^h - $BIRF_n$, Wely's projective torsion tensor W^i_{kh} is given by (2.10), if and only if the equation (2.11) holds good.*

3. Affinely Connected Space on Generalized W^h -Birecurrent

Let us introduce definition of $GW^{\mathbf{h}} - BIRF_{\mathbf{n}}$ to be affinely connected space as follows:

Definition 3.1. The generalized W^h -birecurrent space which is an affinely connected space [satisfies any one of the conditions (1.21), (1.22) and (1.23)] will be called generalized W^h -birecurrentaffinely connected space and will denote it briefly by GW^h - BIR -affinely connected space.

Remark 3.2. It will be sufficient to call the tensor which satisfies the condition of GW^h - BIR -affinely connected space as generalized h -birecurrent tensor (briefly Gh - BIR) .

By using the conditions (1.21), (1.22) and (1.23), the equation (2.6) reduce to

$$\begin{aligned} W^i_{jkh|m|n} = \; & \left(\dot\partial_j \lambda_{mn} \right) \; W^i_{kh} + \lambda_{mn} W^i_{jkh} + \left(\dot\partial_j \mu_{mn} \right) \left(\delta^i_h y_k - \delta^i_k y_h \right) \\ & + \mu_{mn} \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right). \end{aligned} \tag{3.1}$$

This shows that

$$W^i_{kh|m|n} = \lambda_{mn} W^i_{jkh} + \mu_{mn} \left(\delta^i_h g_{jk} - \delta^i_k g_{jh} \right) \tag{3.2}$$

if and only if

$$\left(\dot\partial_j \lambda_{mn} \right) \; W^i_{kh} + \left(\dot\partial_j \mu_{mn} \right) \left(\delta^i_h y_k - \delta^i_k y_h \right) = 0. \tag{3.3}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.3. *In GW^h - BIR - affinely connected space, Weyl's projective curvature tensor W^i_{jkh} is Gh - BIR if and only if the equation (3.3) is equal to zero.*

Transvecting the equation (3.1) by y^j , using (1.6), (1.16) and (1.1), we get

$$\begin{aligned} W^i_{kh|m|n} &= (\dot\partial_j\lambda_{mn})\ W^i_{kh}\ y^j + \lambda_{mn}W^i_{kh} + (\dot\partial_j\mu_{mn}\ )\ (\delta^i_h y_k - \delta^i_k y_h\ )\ y^j \\ &\quad + \mu_{mn}\ (\delta^i_h y_k - \delta^i_k y_h\ ) . \end{aligned} \tag{3.4}$$

This shows that

$$W^i_{kh|m|n} = \lambda_{mn}W^i_{kh} + \mu_{mn}\ (\delta^i_h y_k - \delta^i_k y_h\ ) \tag{3.5}$$

if and only if

$$(\dot\partial_j\lambda_{mn})\ W^i_{kh} + (\dot\partial_j\mu_{mn}\ )\ (\delta^i_h y_k - \delta^i_k y_h\ ) = 0,\ \text{since } y^j \neq 0. \tag{3.6}$$

Transvecting the equation (3.4) by y^k , using (1.6), (1.16), (1.2) and (1.1), we obtain

$$\begin{aligned} W^i_{h|m|n} &= (\dot\partial_j\lambda_{mn})\ W^i_h\ y^j + \lambda_{mn}W^i_h + (\dot\partial_j\mu_{mn}\ )\ (\delta^i_h F^2 - y_h\ y^i)\ y^j \\ &\quad + \mu_{mn}\ (\delta^i_h F^2 - y_h\ y^i) . \end{aligned}$$

This shows that

$$W^i_{h|m|n} = \lambda_{mn}W^i_h + \mu_{mn}\ (\delta^i_h F^2 - y_h\ y^i) \tag{3.7}$$

if and only if

$$(\dot\partial_j\lambda_{mn})\ W^i_h y^j + (\dot\partial_j\mu_{mn}\ )\ (\delta^i_h F^2 - y_h\ y^i)\ y^j = 0. \tag{3.8}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.4. *In* GW^h - BIR *-affinely connected space,Wely's projective torsion tensor* W^i_{kh} *and Wely's projective deviation tensor* W^i_h *are given by (3.5) and (3.7), respectively, if and only if the equations (3.6) and (3.8), respectively are equals to zero.*

By using the conditions (1.21), (1.22) and (1.23), the equation (2.9) reduce to

$$\begin{aligned} W^i_{kh|m|n} &= (\dot\partial_k\lambda_{mn})\ W^i_h + \lambda_{mn}W^i_{kh} + (\dot\partial_k\mu_{mn}\ )\ (\delta^i_h F^2 - y_h\ y^i) \\ &\quad + \mu_{mn}\ \ (\dot\partial_k\ \delta^i_h F^2 - \delta^i_k y_h\ ) . \end{aligned} \tag{3.9}$$

This shows that

$$W^i_{kh|m|n} = \lambda_{mn}W^i_{kh} + \mu_{mn}\ \ (\dot\partial_k\ \delta^i_h F^2 - \delta^i_k y_h\ ) \tag{3.10}$$

if and only if

$$(\dot\partial_k\lambda_{mn})\ W^i_h + (\dot\partial_k\mu_{mn}\ )\ (\delta^i_h F^2 - y_h\ y^i) = 0. \tag{3.11}$$

Therefore, it is concluded the following.

Theorem 3.5. *In* GW^h - BIR - *affinely connected space, Wely's projective torsion tensor* W^i_{kh} *is given by (3.10) if and only if the equation (3.11) hold good.*

Transvecting the equation (3.9) by y^k , using (1.6), (1.17), (1.2) and (1.1), we get

$$\begin{aligned} W^i_{h|m|n} &= \left(\dot{\partial}_k\lambda_{mn}\right) W^i_h y^k + \lambda_{mn} W^i_h + \left(\dot{\partial}_k\mu_{mn}\right)\left(\delta^i_h F^2 - y_h y^i\right) y^k \\ &\quad + \mu_{mn}\ \left(\delta^i_h F^2 - y_h\ y^i\right). \end{aligned}$$

This shows that

$$W^i_{h|m|n} = \lambda_{mn} W^i_h + \mu_{mn}\left(\delta^i_h F^2 - y_h\ y^i\right) \tag{3.12}$$

if and only if

$$\left(\dot{\partial}_k\lambda_{mn}\right)\ W^i_h y^k + \left(\dot{\partial}_k\mu_{mn}\ \right)\left(\delta^i_h F^2 - y_h\ y^i\right) y^k = 0. \tag{3.13}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.6. *In* GW^h - BIR *-affinely connected space, Wely's projective deviation tensor* W^i_h *is given by (3.12) if and only if the equation (3.13) hold good.*

4. Conclusions and Recommendations

A Finsler space is called generalized W^h -birecurrent spaceif it satisfies the condition (2.1). In GW^h - BIR , the h-covariant derivative of the second order for Wely's projective torsion tensor W^i_{kh} and Wely's projective deviation tensor W^i_h are given by (2.3) and (2.4), respectively. In GW^h - BIR , the necessary and sufficient condition of Weyl's projective curvature tensor W^i_{jkh} is generalized birecurrent, if and only if the equation (2.7) is equal to zero and Wely's projective torsion tensor W^i_{kh} is given by the equation (2.10), if and only if the equation (2.11) is equal to zero.

In GW^h - BIR - affinely connected space, Weyl's projective curvature tensor W^i_{jkh} is Gh - BIR if and only if the equation (3.3) is equal to zero, GW^h - BIR - affinely connected space, Wely's projective torsion tensor W^i_{kh} and Wely's projective deviation tensor W^i_h are given by (3.5) and (3.7), respectively, if and only if the equations (3.6) and (3.8), respectively are equals to zero and in GW^h - BIR - affinely connected space, Wely's projective torsion tensor W^i_{kh} and Wely's projective deviation tensor W^i_h are given by equations (3.10) and (3.12), if and only if the equations (3.11) and (3.13) are hold good.

Authors recommend the need for continuing research and development in generalized W^h -birecurrent Finsler spaces and interlard it with the properties of special spaces for Finsler space.

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