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Some existence and stability results for ψ-Hilfer fractional implicit differential equation with periodic conditions

Mohammed A. Almalahi[a,*], Satish K. Panchal[a]

[a]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India.

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Abstract

In this paper, we study the class of boundary value problems for a nonlinear implicit fractional differential equation with periodic conditions involving a ψ-Hilfer fractional derivative. With the help of properties Mittag-Leffler functions, and fixed-point techniques, we establish the existence and uniqueness results, whereas the generalized Gronwall inequality is applied to get the stability results. Also, an example is provided to illustrate the obtained results.

Keywords: Fractional differential equations, Fractional derivatives, Mittag-Leffler functions, Ulam stability, Fixed point theorem.
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1. Introduction

The theory of fractional differential equations is very important since their nonlocal property is appropriate to describe memory phenomena such as nonlocal elasticity, propagation in complex medium, polymers, biological tissues, earth sediments, expansion of universe too, and they have been emerging as an important area of investigation in the last few decades. For details, we refer the reader to monographs of Kilbas et al. [1], Samko et al. [2], Hilfer [3], Podlubny [4]. Over the last years, the stability results of functional differential equations have been robustly developed. Very significant contributions about concept of stability were introduced by Ulam [5], Hyers [6] and this type of stability called Ulam-Hyers (UH) stability. Thereafter improvement of Ulam-Hyers stability provided by Rassias [7] in 1978 so-called Ulam-Hyers-Rassias (UHR). For some recent results of stability analysis by different types of fractional derivative operators (FDOs), we refer the reader to a series of papers [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

Recently, Kilbas et al., in [1] introduced the properties of fractional integrals and fractional derivatives with respect to another function. On the other hand, Furati and Kassim

*Corresponding author: aboosama736242107@gmail.com

[24] studied the existence, uniqueness and stability results for a Cauchy-type problem involving Hilfer FD. Sousa and Oliveira [25] proposed a ψ- Hilfer FD and extended few previous works dealing with the Hilfer [3, 24]. Teodoro et al., [26] analyzed a list of expressions to have a general overview of the concept of fractional integrals and derivatives. Sousa and Oliveira [27] presented a Leibniz type rule for the ψ-Hilfer fractional derivative operator in two forms. In [28], the authors discussed the existence, uniqueness and UHR stability results for ψ-Hilfer problem via a generalized Gronwall inequality. Moreover, they discussed some important qualitative properties of solutions such as existence, uniqueness, and stability results in the following chain [15, 18, 19, 25, 28, 29, 30, 31, 32, 33, 34]. Recently Gao et al., in [35] established the existence and uniqueness of solutions to the Hilfer nonlocal boundary value problem

$$\begin{aligned} D_{0^+}^{p,\beta} y(t) - c y(t) &= f(t, y(t)),\ c < 0, 0 < p < 1, 0 \leqslant \beta \leqslant 1, t \in (0, T], \\ I_{0^+}^{1-r} y(0) &= \sum_{i=1}^{m} \lambda_i I_{0^+}^{\zeta} y(\tau_i),\ p \leqslant r = p + \beta - p\beta, \tau_i \in (0, T], \end{aligned}$$

where $D_{0^+}^{p,\beta}$ denotes the Hilfer FD of order $p \in (0,1)$ and type $\beta \in [0,1]$, $I_{0^+}^{1-r}$ is the Reimann Liouville fractional integral of order $1-r$, $r = p + \beta(1-p)$, $c < 0$ by using some properties of Mittag-Leffler functions, and fixed point methods. Almalahi et al., in [36] studied the existence, uniqueness and different types of stabilities of solutions for the following problem:

$$\begin{aligned} D_{0^+}^{p,\beta} y(t) - \lambda y(t) &= f(t, y(t), D_{0^+}^{p,\beta} y(t)), t \in (0, T], \\ I_{0^+}^{1-r} y(0) &= I_{0^+}^{1-r} y(T) \end{aligned}$$

where $D_{0^+}^{p,\beta}$ denotes the Hilfer FD of order $p \in (0,1)$ and type $\beta \in [0,1]$, $I_{0^+}^{1-r}$ is the Reimann Liouville fractional integral of order $1-r$, $r = p + \beta(1-p)$, $\lambda < 0$.

Motivated by [35, 36], the objective of this study is to investigate the existence, uniqueness as well as the HU and HUR stabilities of the solutions of the proposed problem involving ψ-Hilfer FD of the form:

$$ {}^{H}D_{0^+}^{p,\beta;\psi} v(t) - \lambda v(t) = f(t, v(t), {}^{H}D_{0^+}^{p,\beta;\psi} v(t)), \quad t \in J := (0, T], \tag{1.1}$$

$$\lim_{t \to 0^+} I_{0^+}^{1-r;\psi} v(t) = \lim_{t \to T^-} I_{0^+}^{1-r;\psi} v(t), \tag{1.2}$$

where ${}^{H}D_{0^+}^{p,\beta,\psi}$ denotes the ψ-Hilfer FD of order $p \in (0,1)$ and type $\beta \in [0,1]$, $I_{0^+}^{1-r,\psi}$ is the ψ-Reimann Liouville fractional integral of order $1-r$, $r = p + \beta(1-p)$, $\lambda < 0$, and $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is given function .

This paper is organized as following: In Section 2, we recall the basic definitions and lemmas which are used throughout this paper, also we present the concepts of some fixed point theorems. In Section 3, we study the existence and uniqueness results of ψ-Hilfer fractional implicit differential equation (FIDEs) by using Schaefer's fixed point theorem, Banach contraction principle and properties of Mittag-Leffler function. In Section 4, we discuss four different types stability of solutions to a given problem. At the end, an example to illustrate our results in Section 5.

2. Preliminaries

In this section, we give some basic definitions and lemmas which are essential for the proofs of our results.

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions on J into $\mathbb{R}$ with $\|v\| = \max\{|v(t)| : t \in J\}$. For $0 \leqslant r < 1$, the weighted space $C_{1-r;\psi}(J,\mathbb{R})$ of continuous function $v : J \to \mathbb{R}$ is defined by

$$C_{1-r;\psi}(J,\mathbb{R}) = \left\{v : J \to \mathbb{R}; (\psi(t)-\psi(0))^{1-r}v(t) \in C(J,\mathbb{R})\right\}, 0 \leqslant r < 1$$

with norm

$$\|v\|_{C_{1-r;\psi}(J,\mathbb{R})} = \max_{v\in J}\left|(\psi(t)-\psi(0))^{1-r}v(t)\right|.$$

Obviously $C_{1-r,\psi}(J,\mathbb{R})$ is Banach space with the above norm.

Definition 2.1. [25] Let $p > 0$, $v \in L_1(J,\mathbb{R})$ and $\psi \in C^1(J,\mathbb{R})$ be an increasing function with $\psi'(t) \neq 0$, for all $t \in J$. Then, the left-sided ψ-Riemann-Liouville fractional integral of a function v is defined by

$$I_{0^+}^{p,\psi}v(t) = \frac{1}{\Gamma(p)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}v(s)ds, t \in J.$$

Definition 2.2. [25] Let $n-1 < p < n$ with $n \in \mathbb{N}$, and $v, \psi \in C^n(J,\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided ψ-Hilfer FD of a function v of order p and type $0 \leqslant \beta \leqslant 1$ is defined by

$${}^{H}D_{0^+}^{p,\beta,\psi}v(t) = I_{0^+}^{\beta(n-p);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{0^+}^{(1-\beta)(n-p);\psi}v(t).$$

One has,

$${}^{H}D_{0^+}^{p,\beta,\psi}v(t) = I_{0^+}^{\beta(n-p);\psi}D_{0^+}^{r;\psi}v(t).$$

Theorem 2.3. *[25] Let $v \in C^1(J,\mathbb{R})$, $0 < p < 1$, and $0 \leqslant \beta \leqslant 1$. Then*

$${}^{H}D_{0^+}^{p,\beta,\psi}I_{0^+}^{p,\psi}v(t) = v(t).$$

Lemma 2.4. *[25] Let $v \in C_{1-r,\psi}[0,T]$, $0 < p < 1$. Then*

$$I_{0^+}^{p,\psi}\,v(0) = \lim_{t\longrightarrow 0^+} I_{0^+}^{p,\psi}v(t) = 0,\ 0 \leqslant r < p$$

Lemma 2.5. *[12] (Lemma 20) Let $p > 0$, $\beta > 0$ and $r = p+\beta-p\beta$. If $v(t) \in C_{1-r,\psi}^{r}(J,\mathbb{R})$, then*

$$I_{0^+}^{r,\psi}\ {}^{H}D_{0^+}^{r,\psi}v(t) = I_{0^+}^{p,\psi}\ {}^{H}D_{0^+}^{p,\beta,\psi}v(t),$$

and

$${}^{H}D_{0^+}^{r,\psi}I_{0^+}^{p,\psi}v(t) = {}^{H}D_{0^+}^{\beta(1-p)}v(t).$$

Lemma 2.6. *[1] Let $p, r > 0$, then*

$$I_{0^+}^{p,\psi}(\psi(t)-\psi(s))^{r-1}=\frac{\Gamma(r)}{\Gamma(p+r)}(\psi(t)-\psi(s))^{p+r-1}$$

and

$$D_{0^+}^{r,\psi}(\psi(t)-\psi(s))^{r-1}=0,$$

Lemma 2.7. *[25] Let* $p\in(0,1)$, $0\leqslant r\leqslant 1$, $v\in C_{1-\gamma,\psi}(J,\mathbb{R})$ *and* $I_{0^+}^{1-r;\psi}v\in C^1_{1-r,\psi}(J,\mathbb{R})$. *Then we have*

$$I_{0^+}^{p;\psi}\,{}^{H}D_{0^+}^{p,\beta;\psi}v(t)=v(t)-\frac{(\psi(t)-\psi(0))^{r-1}}{\Gamma(r)}I_{0^+}^{1-r;\psi}v(0).$$

Lemma 2.8. *([37], Lemma 2) Let* $p\in(0,1)$ *and* $\beta>0$ *be arbitrary. The function* $E_p(\cdot)$, $E_{p,p}(\cdot)$ *and* $E_{p,\beta}(\cdot)$ *are nonnegative, and for all* $z<0$

$$E_\rho(z):=E_{\rho,1}(z)\leqslant 1,\ \ E_{\rho,\rho}(z)\leqslant\frac{1}{\Gamma(\rho)},\ \ E_{\rho,\beta}(z)\leqslant\frac{1}{\Gamma(\beta)}.$$

Lemma 2.9. *Let* $\rho\in(0,1)$ *and* $\beta>0$. *Then, for* $t_1,t_2\in J$, *we have*

$$E_{\rho,\rho+\beta}\left(\lambda(\psi(t_2)-\psi(0))^\rho\right)\to E_{\rho,\rho+\beta}\left(\lambda(\psi(t_1)-\psi(0))^\rho\right)\ \ \textit{as } t_2\to t_1, \tag{2.1}$$

where $E_{\rho,\rho+\beta}$ *the Mittag-Leffler function defined by*

$$E_{\rho,\rho+\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\rho+\beta)},\quad z\in\mathbb{R}.$$

Proof. By definition of Mittag-Leffler function, we get

$$\begin{aligned}&E_{\rho,\rho+\beta}\left(\lambda(\psi(t_2)-\psi(0))^\rho\right)-E_{\rho,\rho+\beta}\left(\lambda(\psi(t_1)-\psi(0))^\rho\right)\\ =&\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\rho k+\rho+\beta)}\left[(\psi(t_2)-\psi(0))^{\rho k}-(\psi(t_1)-\psi(0))^{\rho k}\right].\end{aligned}$$

Now, let $h(t)=(\psi(t)-\psi(0))^{\rho k}$. By Lagrange Mean value theorem, there exist $\epsilon\in[t_1,t_2]\subset J$, such that

$$\frac{h(t_2)-h(t_1)}{t_2-t_1}\leqslant h'(\epsilon).$$

Then, for $\epsilon\leqslant t_2\leqslant T$, we get

$$\begin{aligned}&(\psi(t_2)-\psi(0))^{\rho k}-(\psi(t_1)-\psi(0))^{\rho k}\\ \leqslant\ &\rho k\psi'(\epsilon)(\psi(\epsilon)-\psi(0))^{\rho k-1}|t_2-t_1|\\ \leqslant\ &\rho k\psi'(T)(\psi(T)-\psi(0))^{\rho k-1}|t_2-t_1|.\end{aligned}$$

Thus

$$\begin{aligned}&\left|E_{\rho,\rho+\beta}\left(\lambda(\psi(t_2)-\psi(0))^\rho\right)-E_{\rho,\rho+\beta}\left(\lambda(\psi(t_1)-\psi(0))^\rho\right)\right|\\ \leqslant\ &\sum_{k=0}^{\infty}\frac{\left|\lambda^k\right|\rho k\psi'(T)(\psi(T)-\psi(0))^{\rho k-1}}{\Gamma(\rho k+\rho+\beta)}|t_2-t_1|\\ \to\ &0 \text{ as } t_2\to t_1.\end{aligned}$$

Hence $E_{\rho,\rho+\beta}\left(\lambda(\psi(t_2)-\psi(0))^\rho\right)\to E_{\rho,\rho+\beta}\left(\lambda(\psi(t_1)-\psi(0))^\rho\right)$ as $t_2\to t_1$. □

Lemma 2.10. *Let* $p \in (0,1)$, $\beta > 0$, $r > 0$ *and* $\lambda \in \mathbb{R}$. *Then*

$$\begin{aligned} & I_{0+}^{p,\psi}(\psi(t)-\psi(0))^{\beta-1}E_{\rho,\beta}\left(\lambda(\psi(t)-\psi(0))^{\rho}\right) \\ = \ & (\psi(t)-\psi(0))^{p+\beta-1}E_{\rho,p+\beta}(\lambda(\psi(t)-\psi(0))^{\rho}). \end{aligned}$$

Proof. By definition 2.1, we have

$$\begin{aligned} & I_{0+}^{p,\psi}(\psi(t)-\psi(0))^{\beta-1}E_{\rho,\beta}\left(\lambda(\psi(t)-\psi(0))^{\rho}\right) \\ = \ & \frac{1}{\Gamma(p)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}(\psi(s)-\psi(0))^{\beta-1} \\ & \sum_{n=0}^{\infty}\frac{\lambda^n(\psi(s)-\psi(0))^{n\rho}}{\Gamma(\rho n+\beta)}ds \\ = \ & \sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(\rho n+\beta)}\frac{1}{\Gamma(p)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}(\psi(s)-\psi(0))^{\rho n+\beta-1}, \end{aligned}$$

by lemma 2.6, we get

$$\begin{aligned} & I_{0+}^{p,\psi}(\psi(t)-\psi(0))^{\beta-1}E_{\rho,\beta}\left(\lambda(\psi(t)-\psi(0))^{\rho}\right) \\ = \ & (\psi(t)-\psi(0))^{p+\beta-1}E_{\rho,p+\beta}(\lambda(\psi(t)-\psi(0))^{\rho}) \end{aligned}$$

□

Lemma 2.11. *Let* $p \in (0,1)$, $\beta > 0$, $\rho > 0$, $\lambda \in \mathbb{R}$, $z \in \mathbb{R}$ *and* $f \in C(J,\mathbb{R})$, *then*

$$\begin{aligned} & I_{0+}^{p,\psi}\int_0^z \psi'(s)(\psi(z)-\psi(s))^{\rho-1}E_{\rho,\rho}\left(\lambda(\psi(z)-\psi(s))^{\rho}\right)f(s)ds \\ = \ & \int_0^z \psi'(s)(\psi(z)-\psi(s))^{\rho+p-1}E_{\rho,\rho+p}\left(\lambda(\psi(z)-\psi(s))^{\rho}\right)f(s)ds. \end{aligned}$$

Proof. According to definition 2.1, we obtain

$$\begin{aligned} & I_{0+}^{p,\psi}\int_0^z \psi'(s)(\psi(z)-\psi(s))^{\rho-1}E_{\rho,\rho}\left(\lambda(\psi(z)-\psi(s))^{\rho}\right)f(s)ds \\ = \ & \frac{1}{\Gamma(p)}\int_0^z \psi'(u)(\psi(z)-\psi(u))^{p-1} \\ & \left\{\int_0^u \psi'(s)(\psi(u)-\psi(s))^{\rho-1}E_{\rho,\rho}\left(\lambda(\psi(u)-\psi(s))^{\rho}\right)f(s)ds\right\}du \\ = \ & \frac{1}{\Gamma(p)}\int_0^z\int_s^z \psi'(s)(\psi(u)-\psi(s))^{\rho-1} \\ & E_{\rho,\rho}\left(\lambda(\psi(u)-\psi(s))^{\rho}\right)\psi'(u)(\psi(z)-\psi(u))^{p-1}f(s)duds \\ = \ & \frac{1}{\Gamma(p)}\int_0^z f(s)\Gamma(p)\psi'(s)(\psi(z)-\psi(s))^{\rho+p-1}E_{\rho,\rho+p}\left(\lambda(\psi(z)-\psi(s))^{\rho}\right)ds \\ = \ & \int_0^z \psi'(s)(\psi(z)-\psi(s))^{\rho+p-1}E_{\rho,\rho+p}\left(\lambda(\psi(z)-\psi(s))^{\rho}\right)f(s)ds. \end{aligned}$$

□

Lemma 2.12. *[28] Let $p > 0$ and x, v be two nonnegative function locally integrable on J. Assume that g is nonnegative and nondecreasing, and let $\psi \in C^1(J, \mathbb{R})$ an increasing function such that $\psi'(t) \neq 0$ for all $t \in J$. If*

$$x(t) \leqslant v(t) + g(t) \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} x(s) ds, \quad t \in J,$$

then

$$x(t) \leqslant v(t) + \int_0^t \sum_{n=1}^{\infty} \frac{[g(t)\Gamma(p)]^n}{\Gamma(np)} \psi'(s)(\psi(t) - \psi(s))^{np-1} v(s) ds, \quad t \in J.$$

If v be a nondecreasing function on J. then, we have

$$x(t) \leqslant v(t) E_p \{g(t)\Gamma(p)(\psi(t) - \psi(0))^p\}, \quad t \in J.$$

Lemma 2.13. *Let $f \in C_{1-r;\psi}(J, \mathbb{R})$ and $\lambda \in \mathbb{R}, p \in (0,1), \beta \in [0,1]$. Then the ψ-Hilfer problem*

$$\begin{array}{c} {}^H D_{0^+}^{p,\beta;\psi} v(t) - \lambda v(t) = f(t), \qquad t \in J, \\ \lim_{t \to 0^+} I_{0^+}^{1-r;\psi} v(t) = \lim_{t \to T^-} I_{0^+}^{1-r;\psi} v(t), p \leqslant r = p + \beta - p\beta \end{array} \tag{2.2}$$

is equivalent to integral equation

$$\begin{aligned} v(t) &= (\psi(t) - \psi(0))^{r-1} \mathcal{R} \\ &\int_0^T \psi'(s)(\psi(T) - \psi(s))^{p-r} E_{p,p-r+1}(\lambda(\psi(T) - \psi(s))^p) f(s) ds \\ &+ \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} E_{p,p}(\lambda(\psi(t) - \psi(s))^p) f(s) ds, \end{aligned}$$

where $E_{p,1}(\lambda(\psi(T) - \psi(0))^p) \neq 1$ and $\mathcal{R} := \frac{E_{p,r}(\lambda(\psi(T) - \psi(0))^p)}{1 - E_{p,1}(\lambda(\psi(T) - \psi(0))^p)}$.

Proof. By [38], the solution of the following problem

$$\begin{array}{ll} {}^H D_{0^+}^{p,\beta;\psi} v(t) - \lambda v(t) = f(t), & t \in J, \\ I_{0^+}^{1-r;\psi} v(0) = v_0, & p \leqslant r = p + \beta - p\beta \end{array}$$

is given by

$$\begin{aligned} v(t) &= (\psi(t) - \psi(0))^{r-1} E_{p,r}(\lambda(\psi(t) - \psi(0))^p) I_{0^+}^{1-r;\psi} v(0) \\ &+ \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-1} E_{p,p}(\lambda(\psi(t) - \psi(s))^p) f(s) ds. \end{aligned} \tag{2.3}$$

Next, by multiplying the operator $I_{0^+}^{1-r,\psi}$ to both sides of (2.3), with the help of lemmas 2.10, 2.11, we get

$$\begin{aligned} I_{0^+}^{1-r,\psi} v(t) &= E_{p,1}(\lambda(\psi(t) - \psi(0))^p) I_{0^+}^{1-r} v(0) \\ &+ \int_0^t \psi'(s)(\psi(t) - \psi(s))^{p-r} E_{p,p-r+1}(\lambda(\psi(t) - \psi(s))^p) f(s) ds. \end{aligned} \tag{2.4}$$

Taking the limit $t \longrightarrow T^-$ on both sides of (2.4), it follows that

$$I_{0^+}^{1-\gamma,\psi} v(0) = \frac{I_{0^+}^{1-r,\psi} v(T)}{E_{p,1}\left(\lambda(\psi(T)-\psi(0))^p\right)} - \frac{1}{E_{p,1}\left(\lambda(\psi(T)-\psi(0))^p\right)} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) f(s)ds.$$

Since $\lim_{t\to 0^+} I_{0^+}^{1-r;\psi} v(t) = \lim_{t\to T^-} I_{0^+}^{1-r;\psi} v(t)$, we obtain

$$I_{0^+}^{1-\gamma,\psi} v(0) = \frac{1}{1-E_{p,1}\left(\lambda(\psi(T)-\psi(0))^p\right)} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) f(s)ds. \quad (2.5)$$

From (2.2) and (2.5), it follwos that

$$v(t) = (\psi(t)-\psi(0))^{r-1}\mathcal{R} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) f(s)ds + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) f(s)ds. \quad (2.6)$$

Conversely, applying $I_{0^+}^{1-r,\psi}$ to both sides of (2.6), with the help of lemmas 2.10 and 2.11, we have

$$I_{0^+}^{1-r,\psi} v(t) = \frac{E_{p,1}\left(\lambda(\psi(t)-\psi(0))^p\right)}{1-E_{p,1}\left(\lambda(\psi(T)-\psi(0))^p\right)} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) f(s)ds + \int_0^t \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(t)-\psi(s))^p\right) f(s)ds. \quad (2.7)$$

By lemma 2.4 and taking the limit as $t \to 0$,

$$I_{0^+}^{1-r,\psi} v(0) = \frac{1}{1-E_{p,1}\left(\lambda(\psi(T)-\psi(0))^p\right)} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) f(s)ds, \quad (2.8)$$

Similarly, taking the limit as $t \to T$ of (2.7), we have

$$I_{0^+}^{1-r,\psi} v(T) = \frac{1}{1-E_{p,1}\lambda(\psi(T)-\psi(0))^p} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) f(s)ds. \quad (2.9)$$

From (2.8) and (2.9), it follows that

$$I_{0^+}^{1-r,\psi} v(0) = I_{0^+}^{1-r,\psi} v(T).$$

On the other hand, applying $D_{0^+}^{r,\psi}$ to both sides of (2.6), and using lemmas 2.5, 2.6, then applying $I_{0^+}^{\beta(1-p),\psi}$ on result, it follows that

$$^{H}D_{0^+}^{p,\beta;\psi}v(t) - \lambda v(t) = f(t).$$

□

3. Existence of solution

The existence and uniqueness theorems of solutions to ψ-Hilfer equation (1.1) with period condition (1.2) are presented in this section. For our analysis, the following assumptions should be valid.

(H_1) Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C_{1-r,\psi}(J,\mathbb{R})$, for any $v \in C_{1-r,\psi}(J,\mathbb{R})$, and there exist positive constant $\vartheta > 0$ and $\varpi \in (0,1)$, such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leqslant \vartheta |u_1 - u_2| + \varpi |v_1 - v_2|,$$

for any $u_i, v_i \in \mathbb{R}, i = 1,2$ and $t \in J$.

(H_2) There exist $\sigma, \mu, \kappa \in C(J,\mathbb{R})$ such that

$$|f(t,u,v)| \leqslant \sigma(t) + \mu(t)|u| + \kappa(t)|v|,$$

with $\kappa^* = \sup_{t\in J}\kappa(t)$, $\mu^* = \sup_{t\in J}\mu(t)$ and $\sigma^* = \sup_{t\in J}\sigma(t) < 1$, for all $t \in J$, $u,v \in \mathbb{R}$.

(H_3) The following inequality holds

$$\Omega := \frac{(\lambda+\vartheta)}{1-\varpi}\left\{\frac{\mathcal{R}\Gamma(r)}{\Gamma(p+1)}(\psi(T)-\psi(0))^p + \frac{\mathcal{B}(p,r)}{\Gamma(p)}(\psi(T)-\psi(0))^{1-r+p}\right\} < 1.$$

Theorem 3.1. *Assume that* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous, satisfies* (H_2). *If*

$$\sigma := \left(\frac{\mathcal{R}\Gamma(r)}{\Gamma(p+1)} + \frac{\mathcal{B}(p,r)}{\Gamma(p)}\right)\frac{(\lambda+\mu^*)(\psi(T)-\psi(0))^p}{(1-\kappa^*)} < 1, \tag{3.1}$$

then the ψ-Hilfer problem (1.1)-(1.2) has at least one solution in $C_{1-r,\psi}(J,\mathbb{R})$.

Proof. According to lemma 2.13, the solution of the ψ-Hilfer problem (1.1)-(1.2) can be expressed by the integral equation

$$\begin{aligned} v(t) \;=\; & (\psi(t)-\psi(0))^{r-1}\mathcal{R}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} \\ & E_{p,p-r+1}(\lambda(\psi(T)-\psi(s))^p)\mathcal{H}_v(s)ds \\ & + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}E_{p,p}(\lambda(\psi(t)-\psi(s))^p)\mathcal{H}_v(s)ds, \end{aligned}$$

where $\mathcal{H}_\nu$ is the solution of the functional integral equation

$$\begin{aligned}\mathcal{H}_\nu(t) &= \lambda\Bigg((\psi(t)-\psi(0))^{r-1}\mathcal{R}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r}\\ &\quad E_{p,p-r+1}\lambda(\psi(T)-\psi(s))^p\mathcal{H}_\nu(s)ds\\ &\quad +\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right)\mathcal{H}_\nu(s)ds\Bigg)\\ &\quad +f\Bigg(t,(\psi(t)-\psi(0))^{r-1}\mathcal{R}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r}\\ &\quad E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right)\mathcal{H}_\nu(s)ds\\ &\quad +\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right)\mathcal{H}_\nu(s)ds,\mathcal{H}_\nu(t)\Bigg).\end{aligned}$$

Here $\mathcal{H}_\nu(t)=\lambda\nu(t)+f(t,\nu(t),\mathcal{H}_\nu(t))$.

Let us consider the operator $\mathcal{G}: C_{1-r,\psi}(J,\mathbb{R})\longrightarrow C_{1-r,\psi}(J,\mathbb{R})$ defined as

$$\begin{aligned}\nu(t) \longrightarrow \mathcal{G}\nu(t) &= (\psi(t)-\psi(0))^{r-1}\mathcal{R}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r}\\ &\quad E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right)\mathcal{H}_\nu(s)ds\\ &\quad +\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right)\mathcal{H}_\nu(s)ds. \qquad (3.2)\end{aligned}$$

Clearly that the operator $\mathcal{G}$ is well defined. Define a bounded closed convex set

$$\mathcal{K}_\xi=\left\{\nu\in C_{1-r,\psi}(J,\mathbb{R}):\|\nu\|_{C_{1-r,\psi}}\leqslant\xi\right\},$$

of Banach space $C_{1-r,\psi}(J,\mathbb{R})$ with $\xi\geqslant\frac{\omega}{1-\sigma}$, $\sigma<1$ and

$$\omega:=\left(\frac{\mathcal{R}(\psi(T)-\psi(0))^p}{\Gamma(p-r+2)}+\frac{(\psi(T)-\psi(0))^{p+1-r}}{\Gamma(p+1)}\right)\frac{\sigma^*}{(1-\kappa^*)}.$$

Step(1) We need to show that the operator $\mathcal{G}$ is continuous. Consider a sequence $\{\nu_n\}_{n=1}^\infty$ such that $\nu_n\longrightarrow\nu$ in $C_{1-r,\psi}(J,\mathbb{R})$. In view of lemmas 2.8 and 2.11, and for $t\in J$ it follows that

$$\begin{aligned}&\left|(\psi(t)-\psi(0))^{1-r}\left[\mathcal{G}\nu_n(t)-\mathcal{G}\nu(t)\right]\right|\\ \leqslant&\left(\frac{\mathcal{R}\Gamma(r)}{\Gamma(p+1)}+\frac{\mathcal{B}(p,r)}{\Gamma(p)}\right)\lambda(\psi(T)-\psi(0))^p\|\nu_n-\nu\|_{C_{1-r,\psi}(J,\mathbb{R})}\\ &+\left(\frac{\mathcal{R}}{\Gamma(p-r+2)}+\frac{1}{\Gamma(p+1)}\right)(\psi(T)-\psi(0))^{p-r+1}\\ &\|f(\cdot,\nu_n(\cdot),\mathcal{H}_{\nu_n}(\cdot))-f(\cdot,\nu(\cdot),\mathcal{H}_\nu(\cdot))\|_{C_{1-r,\psi}(J,\mathbb{R})}.\end{aligned}$$

Since f is continuous and $\nu_n\longrightarrow\nu$ as $n\longrightarrow\infty$, we have

$$\|\mathcal{G}\nu-\mathcal{G}\nu_n\|_{C_{1-r,\psi}}\to 0 \text{ as } n\to\infty.$$

Thus, the operator $\mathcal{G}$ is continuous.

Step(2) we well show that the operator $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{K}_{\xi}$. By lemma 2.8, and for $t \in J$, $\nu \in \mathcal{K}_{\xi}$, we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-r}\mathcal{G}\nu(t)\right| \\
\leqslant\ & \frac{\mathcal{R}}{\Gamma(p-r+1)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r}\left|\mathcal{H}_{\nu}(s)\right| ds \\
&+\frac{(\psi(t)-\psi(0))^{1-r}}{\Gamma(p)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}\left|\mathcal{H}_{\nu}(s)\right| ds. \qquad (3.3)
\end{aligned}$$

By (H_2), we have

$$\begin{aligned}
\left|\mathcal{H}_{\nu}(t)\right| &= \left|\lambda\nu(t)+f(t,\nu(t),\mathcal{H}_{\nu}(t))\right| \\
&\leqslant \lambda\left|\nu(t)\right|+\sigma(t)+q(t)\left|\nu(t)\right|+p(t)\left|\mathcal{H}_{\nu}(t)\right| \\
&\leqslant \sigma^{*}+(\lambda+\mu^{*})\left|\nu(t)\right|+\kappa^{*}\left|\mathcal{H}_{\nu}(t)\right|.
\end{aligned}$$

Since $\kappa^{*}<1$, we get

$$\left|\mathcal{H}_{\nu}(t)\right| \leqslant \frac{\sigma^{*}+(\lambda+\mu^{*})\left|\nu(t)\right|}{(1-\kappa^{*})}. \qquad (3.4)$$

Put (3.4) in (3.3), we get

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-r}\mathcal{G}\nu(t)\right| \\
\leqslant\ & \frac{\mathcal{R}}{\Gamma(p-r+1)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r}\frac{\sigma^{*}+(\lambda+\mu^{*})\left|\nu(s)\right|}{(1-\kappa^{*})} ds \\
&+\frac{(\psi(t)-\psi(0))^{1-r}}{\Gamma(p)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1}\frac{\sigma^{*}+(\lambda+\mu^{*})\left|\nu(s)\right|}{(1-\kappa^{*})} ds \\
\leqslant\ & \left(\frac{\mathcal{R}(\psi(T)-\psi(0))^{p}}{\Gamma(p-r+2)}+\frac{(\psi(t)-\psi(0))^{p+1-r}}{\Gamma(p+1)}\right)\frac{\sigma^{*}}{(1-\kappa^{*})} \\
&+\left(\frac{\mathcal{R}\Gamma(r)}{\Gamma(p+1)}+\frac{\mathcal{B}(p,r)}{\Gamma(p)}\right)\frac{(\lambda+\mu^{*})(\psi(T)-\psi(0))^{p}}{(1-\kappa^{*})}\left\|\nu\right\|_{C_{1-r,\psi}(J,\mathbb{R})} \\
\leqslant\ & \omega+\sigma\xi \\
\leqslant\ & \xi,
\end{aligned}$$

which implies

$$\left\|\mathcal{G}\nu\right\|_{C_{1-r,\psi}(J,\mathbb{R})} \leqslant \xi.$$

Thus, $\mathcal{G}:\mathcal{K}_{\xi}\longrightarrow\mathcal{K}_{\xi}$, that is $\mathcal{G}\mathcal{K}_{\xi}$ is uniformly bounded.

Step(3) We need to show that the operator $\mathcal{G}$ maps bounded sets into equicontinuous set of $\mathcal{K}_{\xi}$.

For any $\nu \in \mathcal{K}_{\xi}$ and for $t_1, t_2 \in J$ such that $t_1 \leqslant t_2$. Then by using lemmas 2.8, 2.11, we have

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(0))^{1-r}\mathcal{G}\nu(t_2)-(\psi(t_1)-\psi(0))^{1-r}\mathcal{G}\nu(t_1)\right| \\
\leqslant\ & \left|\frac{E_{p,r}\left(\lambda(\psi(t_2)-\psi(0))^{p}\right)-E_{p,r}\left(\lambda(\psi(t_1)-\psi(0))^{p}\right)}{1-E_{p,1}\left(\lambda(\psi(T)-\psi(0))^{p}\right)}\right|
\end{aligned}$$

$$\times\left[\frac{(\psi(T)-\psi(0))^{p-r+1}}{\Gamma(p-r+2)}+\frac{\Gamma(r)(\psi(T)-\psi(0))^{p}\,(\lambda+\mu^{*})}{\Gamma(p+1)\,(1-\kappa^{*})}\xi\right]$$
$$+\frac{(\psi(t_2)-\psi(0))^{p+1-r}\sigma^{*}}{\Gamma(p+1)\,(1-\kappa^{*})}\left((\psi(t_2)-\psi(0))^{p+1-r}-(\psi(t_1)-\psi(0))^{p+1-r}\right)$$
$$+\frac{\Gamma(r)}{\Gamma(p+r)}\frac{(\lambda+\mu^{*})}{(1-\kappa^{*})}\xi\left((\psi(t_2)-\psi(0))^{p}-(\psi(t_1)-\psi(0))^{p}\right)$$
$$+\frac{\mathcal{B}(p,r)}{\Gamma(p)}\left((\psi(t_2)-\psi(0))^{p}-(\psi(t_1)-\psi(0))^{p}\right). \tag{3.5}$$

By Lagrange Mean value theorem, and Eq.(2.1) as $t_1 \longrightarrow t_2$, the right-hand side of the preceding inequality is not dependent on ν and goes to zero and hence

$$\left|(\psi(t_2)-\psi(0))^{1-r}\mathcal{G}\nu(t_2)-(\psi(t_1)-\psi(0))^{1-r}\mathcal{G}\nu(t_1)\right|\to 0,\ \forall\ |t_2-t_1|\to 0,\ \nu\in\mathcal{K}_{\xi}. \tag{3.6}$$

From the above steps, together with Arzela–Ascoli theorem, we infer that the operator $\mathcal{G}$ is completely continuous. In the last step, we need to prove that the set

$$\Delta=\left\{\nu\in C_{1-r,\psi}\left(J,\mathbb{R}\right):\nu=\delta\mathcal{G}\nu,\ \text{for some }\delta\in(0,1)\ \right\}$$

is bounded set. For each $t\in J$, let $\nu\in\Delta$, and $\nu=\delta\mathcal{G}\nu$ for some $\delta\in(0,1)$. Then we have

$$\nu(t)<\mathcal{G}\nu(t).$$

Hence, by virtue of step (2) and definitions of ω and σ, we obtain

$$\begin{aligned}\|\nu\|_{C_{1-r}} &< \|\mathcal{G}\nu\|_{C_{1-r,\psi}(J,\mathbb{R})}\\ &\leqslant \omega+\sigma\|\nu\|_{C_{1-r,\psi}(J,\mathbb{R})}.\end{aligned}$$

Since $\sigma<1$ it follows that

$$\|\nu\|_{C_{1-r}}\leqslant\frac{\omega}{1-\sigma}\leqslant\xi.$$

Thus, the set Δ is bounded. The Schaefer's fixed point theorem shows that $\mathcal{G}$ has a fixed point which is a solution of the problem (1.1)-(1.2). The proof is completed. □

Theorem 3.2. *Assume that* $(H_1)-(H_3)$ *hold. Then the Hilfer problem (1.1) -(1.2) has a unique solution in* $C_{1-r,\psi}\left(J,\mathbb{R}\right)$.

Proof. In view of Theorem (3.1) we have known that the operator $\mathcal{G}$ defined by 3.2 is well defined and continuous.

Next, we need only to prove that $\mathcal{G}$ is a contraction map on $C_{1-r,\psi}\left(J,\mathbb{R}\right)$. For each $\nu,\nu^{*}\in C_{1-r,\psi}\left(J,\mathbb{R}\right)$ and for all $t\in J$ with the help of lemmas 2.8 and 2.11, we have

$$\begin{aligned}&\left|(\psi(t)-\psi(0))^{1-r}\left[\mathcal{G}\nu(t)-\mathcal{G}\nu^{*}(t)\right]\right|\\ \leqslant\ &\frac{\mathcal{R}}{\Gamma(p-r+1)}\int_0^T\psi'(s)(\psi(T)-\psi(s))^{p-r}\left|\mathcal{H}_{\nu}(s)-\mathcal{H}_{\nu^{*}}(s)\right|ds\\ &+\frac{(\psi(t)-\psi(0))^{1-r}}{\Gamma(p)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{p-1}\left|\mathcal{H}_{\nu}(s)-\mathcal{H}_{\nu^{*}}(s)\right|ds,\end{aligned} \tag{3.7}$$

Since

$$\begin{aligned} |\mathcal{H}_{\nu}(s)-\mathcal{H}_{\nu^*}(s)| &\leqslant \lambda|\nu(t)-\nu^*(t)|+|f(s,\nu(s),\mathcal{H}_{\nu}(s))-f(s,\nu^*(s),\mathcal{H}_{\nu^*}(s))| \\ &\leqslant \frac{(\lambda+\vartheta)}{1-\varpi}|\nu(t)-\nu^*(t)|. \end{aligned} \tag{3.8}$$

Bringing (3.8) into (3.7), we obtain

$$\begin{aligned} &\left|(\psi(t)-\psi(0))^{1-r}\left[\mathcal{G}\nu(t)-\mathcal{G}\nu^*(t)\right]\right| \\ \leqslant\ & \frac{(\lambda+\vartheta)}{1-\varpi}\left\{\frac{\mathcal{R}\Gamma(r)}{\Gamma(p+1)}(\psi(T)-\psi(0))^{p}+\frac{\mathcal{B}(p,r)}{\Gamma(p)}(\psi(T)-\psi(0))^{1-r+p}\right\} \\ & \|\nu-\nu^*\|_{C_{1-r,\psi}(J,\mathbb{R})} \leqslant \Omega\|\nu-\nu^*\|_{C_{1-r,\psi}(J,\mathbb{R})} \end{aligned}$$

By (H_3), the operator $\mathcal{G}$ is a contraction map. According to Banach contraction principle we conclude that the ψ-Hilfer problem (1.1)-(1.2) has a unique solution in $C_{1-r,\psi}(J,\mathbb{R})$ □

4. Ulam-Hyers and Ulam-Hyers-Rassias stabilities

In this section, we analyze the HU and HUR stabilities of solution for ψ-Hilfer fractional implicit differential equation (1.1) with the periodic condition (1.2). Let $\epsilon > 0$. Consider the problem (1.1)-(1.2) and below inequality

$$\left|{}^{H}D_{0^+}^{p,\beta,\psi}x(t)-\lambda x(t)-\mathcal{H}_x(t)\right| \leqslant \epsilon, \quad t \in J. \tag{4.1}$$

The following observations are taken from[18]

Definition 4.1. Problem (1.1)-(1.2) is Ulam-Hyers stable if there exists a real number $\oslash_f > 0$ such that for each $\epsilon > 0$ there exists $x \in C_{1-r,\psi}(J,\mathbb{R})$ satisfies the inequality (4.1) corresponding to a solution $\nu \in C_{1-r,\psi}(J,\mathbb{R})$ of the problem (1.1)-(1.2) such that

$$|x(t)-\nu(t)| \leqslant \oslash_f \epsilon, \qquad t \in J.$$

Definition 4.2. Problem (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $\Phi_f \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\Phi_f(0) = 0$ such that for each solution $x \in C_{1-r,\psi}(J,\mathbb{R})$ satisfies the inequality (4.1) there exists a solution $\nu \in C_{1-r,\psi}(J,\mathbb{R})$ of the problem (1.1)-(1.2) with

$$|x(t)-\nu(t)| \leqslant \Phi_f(\epsilon), \qquad t \in J.$$

Definition 4.3. Problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to $S \in C_{1-r,\psi}(J,\mathbb{R})$ if there exists a real number $\oslash_{f,S} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{1-r,\psi}(J,\mathbb{R})$ of the inequality

$$\left|{}^{H}D_{0^+}^{p,\beta,\psi}x(t)-\lambda x(t)-\mathcal{H}_x(t)\right| \leqslant \epsilon S(t), \quad t \in J, \tag{4.2}$$

there exists a solution $\nu \in C_{1-r,\psi}(J,\mathbb{R})$ of the problem (1.1)-(1.2) with

$$|x(t)-\nu(t)| \leqslant \oslash_{f,S}\epsilon S(t), \quad t \in J.$$

Definition 4.4. Problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to $S \in C_{1-r,\psi}(J,\mathbb{R})$ if there exists a real number $\oslash_{f,S} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{1-r,\psi}(J,\mathbb{R})$ of the inequality

$$\left| {}^{H}D_{0^+}^{p,\beta,\psi} x(t) - \lambda x(t) - \mathcal{H}_x(t) \right| \leqslant S(t), \quad t \in J,$$

there exists a solution $v \in C_{1-r,\psi}(J,\mathbb{R})$ of the problem (1.1)-(1.2) with

$$|x(t) - v(t)| \leqslant \oslash_{f,S} S(t), \quad t \in J.$$

Remark 4.5. A function $x \in C_{1-r,\psi}(J,\mathbb{R})$ is a solution of the inequality (4.1) if and only if there exist a function $z \in C_{1-r,\psi}(J,\mathbb{R})$ such that

(i) $|z(t)| \leqslant \epsilon, \ t \in J;$

(ii) ${}^{H}D_{0^+}^{p,\beta,\psi} x(t) = \lambda x(t) + \mathcal{H}_x(t) + z(t), \ t \in J.$

Lemma 4.6. *Let $x \in C_{1-r,\psi}(J,\mathbb{R})$ be a function satisfies the inequality (4.1). Then x satisfies the following inequality*

$$\begin{aligned} &\left| x(t) - \mathcal{N}_x - \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_x(s) ds \right| \\ \leqslant\ & \epsilon \left[\frac{\mathcal{R}}{\Gamma(p-r+2)} + \frac{1}{\Gamma(p+1)} \right] (\psi(T)-\psi(0))^p \end{aligned}$$

where

$$\mathcal{N}_x = (\psi(t)-\psi(0))^{r-1} \mathcal{R} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) \mathcal{H}_x(s) ds.$$

Proof. In view of remark 4.5 and theorem 3.1, we get

$$\begin{aligned} x(t) = &\ (\psi(t)-\psi(0))^{r-1} \mathcal{R} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} \\ & E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) \mathcal{H}_x(s) ds \\ & + (\psi(t)-\psi(0))^{r-1} \mathcal{R} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} \\ & E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) z(s) ds \\ & + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_x(s) ds \\ & + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) z(s) ds. \end{aligned}$$

By using lemma 2.8, we have

$$\begin{aligned} &\left| x(t) - \mathcal{N}_x - \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_x(s) ds \right| \\ \leqslant\ & \epsilon \left[\frac{\mathcal{R}}{\Gamma(p-r+2)} + \frac{1}{\Gamma(p+1)} \right] (\psi(T)-\psi(0))^p \end{aligned}$$

□

Theorem 4.7. *Assume that* (H_1) *and* (H_3) *are satisfied. Then Eq.(1.1) is Ulam-Hyers stable and generalized Ulam-Hyers stable.*

Proof. Let $x \in C_{1-r,\psi}(J,\mathbb{R})$ be a function satisfies the inequality (4.1) and let $v \in C_{1-r,\psi}(J,\mathbb{R})$ be a solution of the problem

$$\begin{aligned} {}^{H}D_{0^+}^{p,\beta;\psi} v(t) - \lambda v(t) &= f(t, v(t), \mathcal{H}_v(t)) \\ \lim_{t\to 0^+} I_{0^+}^{1-r;\psi} v(t) &= \lim_{t\to 0^+} I_{0^+}^{1-r;\psi} x(t) \\ \lim_{t\to T^-} I_{0^+}^{1-r;\psi} v(t) &= \lim_{t\to T^-} I_{0^+}^{1-r;\psi} x(t). \end{aligned}$$

Using Lemma 2.13, we have

$$v(t) = \mathcal{N}_v + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_v(s)ds,$$

where

$$\mathcal{N}_v = (\psi(t)-\psi(0))^{r-1}\mathcal{R}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{p-r} E_{p,p-r+1}\left(\lambda(\psi(T)-\psi(s))^p\right) \mathcal{H}_v(s)ds,$$

and $\mathcal{H}_v(s) = \lambda v(t) + f(t, v(t), \mathcal{H}_v(t))$. By our assumptions and Lemma 2.8, we can easily conclude that $\mathcal{N}_v = \mathcal{N}_x$ and hence

$$\begin{aligned} |x(t)-v(t)| &= \left| x(t) - \mathcal{N}_x - \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_v(s)ds \right| \\ &\leqslant \left| x(t) - \mathcal{N}_x - \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_x(s)ds \right| \\ &\quad + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) |\mathcal{H}_v(s) - \mathcal{H}_x(s)|\, ds \\ &\leqslant \epsilon \left[\frac{\mathcal{R}}{\Gamma(p-r+2)} + \frac{1}{\Gamma(p+1)} \right] (\psi(T)-\psi(0))^p \\ &\quad + \frac{(\lambda+\vartheta)}{1-\varpi} \frac{1}{\Gamma(p)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} |v(s)-x(s)|\, ds \end{aligned}$$

By Lemma 2.12, we get

$$\begin{aligned} |x(t)-v(t)| &\leqslant \epsilon \left[\frac{\mathcal{R}}{\Gamma(p-r+2)} + \frac{1}{\Gamma(p+1)} \right] (\psi(T)-\psi(0))^p \\ &\quad E_p\left(\frac{(\lambda+\vartheta)}{1-\varpi} (\psi(T)-\psi(0))^p \right) \\ &= \epsilon \oslash_f \end{aligned}$$

Thus for $\oslash_f := \left[\frac{\mathcal{R}}{\Gamma(p-r+2)} + \frac{1}{\Gamma(p+1)} \right] (\psi(T)-\psi(s))^p E_p\left(\frac{(\lambda+\vartheta)}{1-\varpi} (\psi(T)-\psi(0))^p \right)$, we conclude that Eq. (1.1) is Ulam-Hyers stable. Choosing $\Phi(\epsilon) = \oslash_f \epsilon$, $\Phi(0) = 0$, we deduce that Eq. (1.1) is generalized Ulam-Hyers stable. □

Now, we need to introduce the following hypothesis:

(H_4) There exists an increasing function $S \in C_{1-r,\psi}(J, \mathbb{R})$ and there exists $\mu_{S_p} > 0$ such that for any $t \in J$

$$I_{0^+}^{p,\psi} S(t) \leqslant \mu_S S(t).$$

Remark 4.8. A function $x \in C_{1-r,\psi}(J, \mathbb{R})$ is a solution of the inequality (4.2) if and only if there exist a function $z \in C_{1-r,\psi}(J, \mathbb{R})$ (where z depends on solution x) such that

(i) $|z(t)| \leqslant \epsilon S(t)$ for all $t \in J$,

(ii) ${}^{H}D_{0^+}^{p,\beta,\psi} x(t) - \lambda x(t) = \mathcal{H}_x(t) + z(t), \quad t \in J.$

Theorem 4.9. *Assume that* (H_1), (H_3) *and* (H_4) *are satisfied. Then the Eq.(1.1) is Ulam–Hyers–Rassias stable with respect to* S *as well as generalized Ulam–Hyers–Rassias stable.*

Proof. Let $\epsilon > 0$ and $x \in C_{1-r,\psi}(J, \mathbb{R})$ be a solution of the inequality

$$\left| {}^{H}D_{0^+}^{p,\beta,\psi} x(t) - \lambda x(t) - \mathcal{H}_x(s) \right| \leqslant \epsilon S(t), \quad t \in J. \tag{4.3}$$

By the same way of lemma 4.6, (H_4) and remark 4.8, we get

$$\begin{aligned}
&\left| x(t) - \mathcal{N}_x - \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_x(s) ds \right| \\
\leqslant\ & \epsilon \left[\frac{\mathcal{R}}{\Gamma(2-r)} + 1 \right] \mu_S S(t),
\end{aligned}$$

where $\mathcal{H}_x(s) = \lambda x(t) + f(t, x(t), \mathcal{H}_x(s))$. Let $\nu \in C_{1-r,\psi}(J, \mathbb{R})$ be a unique solution of the implicit fractional differential equation

$$\begin{aligned}
{}^{H}D_{0^+}^{p,\beta;\psi} \nu(t) - \lambda \nu(t) &= f(t, \nu(t), \mathcal{H}_\nu(t)) \\
\lim_{t \to 0^+} I_{0^+}^{1-r;\psi} \nu(t) &= \lim_{t \to 0^+} I_{0^+}^{1-r;\psi} x(t) \\
\lim_{t \to T^-} I_{0^+}^{1-r;\psi} \nu(t) &= \lim_{t \to T^-} I_{0^+}^{1-r;\psi} x(t).
\end{aligned}$$

Using Lemma 2.13, we have

$$\nu(t) = \mathcal{N}_\nu + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_\nu(s) ds.$$

By Lemma 2.8, we can easily checked that $\mathcal{N}_\nu = \mathcal{N}_x$ and hence

$$\begin{aligned}
&|x(t) - \nu(t)| \\
\leqslant\ & \left| x(t) - \mathcal{N}_x - \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) \mathcal{H}_x(s) ds \right| \\
& + \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} E_{p,p}\left(\lambda(\psi(t)-\psi(s))^p\right) |\mathcal{H}_x(s) - \mathcal{H}_\nu(s)| \, ds \\
\leqslant\ & \epsilon \left[\frac{\mathcal{R}}{\Gamma(2-r)} + 1 \right] \mu_S S(t) \\
& + \frac{\lambda + \vartheta}{1 - \varpi} \frac{1}{\Gamma(p)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{p-1} |x(s) - \nu(s)| \, ds
\end{aligned} \tag{4.4}$$

Using Lemma 2.12, we get,

$$\begin{aligned} & |x(t)-v(t)| \\ \leqslant\ & \epsilon\left[\frac{\mathcal{R}}{\Gamma(2-r)}+1\right]\mu_S S(t)\left[1+\sum_{n=1}^{\infty}\left[\frac{\lambda+\vartheta}{1-\varpi}\mu_S\right]^n\right] \\ =\ & \epsilon \oslash_{f,S} S(t), \end{aligned}$$

where $\oslash_{f,S}=\left[\frac{\mathcal{R}}{\Gamma(2-r)}+1\right]\mu_S\left[1+\sum_{n=1}^{\infty}\left[\frac{(\lambda+\vartheta)}{1-\varpi}\mu_S\right]^n\right]$, that means

$$|x(t)-v(t)|\leqslant \epsilon \oslash_{f,S} S(t).$$

Thus Eq.(1.1) is Ulam-Hyers Rassias stable. Moreover, an argument similar to above in the previous steps with putting $\epsilon=1$ in Remark 4.8, we get

$$|x(t)-v(t)|\leqslant \oslash_{f,S} S(t).$$

This proves that the problem (1.1)-(1.2) is generalized Ulam-Hyers Rassias stable. □

5. An example

In this section, we give one example to illustrate our results.

Example 5.1. Conseder the following Hilfer fractional differential equation with integral condition

$$\begin{cases} {}^{H}D_{0^+}^{\frac{1}{3},\frac{1}{4};e^t} v(t) = -\frac{1}{20}v(t)+\frac{t}{20}\left(1+|v(t)|+\left|{}^{H}D_{0^+}^{\frac{1}{3},\frac{1}{4};e^t} v(t)\right|\right), \quad t\in J:=(0,1] \\ I_{0^+}^{\frac{1}{2};e^t} v(0) = I_{0^+}^{\frac{1}{2};e^t} v(1). \end{cases} \tag{5.1}$$

Here $p=\frac{1}{3}$, $\beta=\frac{1}{4}$, $r=p+\beta-p\beta=\frac{1}{2}$, $\lambda=-\frac{1}{20}$, $\psi(t)=e^t$, $J=(0,1]$ and $f(t,v(t),\mathcal{H}_v(t))=\frac{t}{20}\left(1+|v(t)|+\left|{}^{H}D_{0^+}^{\frac{1}{3},\frac{1}{4};e^t} v(t)\right|\right)$.

Clearly the function $\left(e^t-1\right)^{\frac{1}{2}} f(t,v(t),\mathcal{H}_v(t))$ is continuous on J, i.e. $f(t,v(t),\mathcal{H}_v(t))\in C_{\frac{1}{2},e^t}(J,\mathbb{R})$, and for $v\in\mathbb{R}$, $t\in J$, we have

$$|f(t,v,\mathcal{H}_v(t))-f(t,x,\mathcal{H}_x(t))|\leqslant\frac{1}{20}\left[|v-x|+|\mathcal{H}_v-\mathcal{H}_x|\right].$$

Hence the first hypothesis (H_1) is satisfied with $\vartheta=\varpi=\frac{1}{20}$. Also, the direct computation shows that $\Omega<1$. It follows from Theorem 3.2, that the problem (5.1) has a unique solution on J.

Moreover, for $v\in\mathbb{R}$ we find that

$$|f(t,v,\mathcal{H}_v(t))|\leqslant\frac{t}{20}\left(1+|v(t)|+\left|{}^{H}D_{0^+}^{\frac{1}{3},\frac{1}{4};e^t} v(t)\right|\right).$$

Thus the hypothesis (H_2) is satisfied with $\sigma(t)=\mu(t)=\kappa(t)=\frac{t}{20}$ and $\sigma^*=\mu^*=\kappa^*=\sup_{t\in[0,1]}\frac{t}{20}=\sup\{0,\frac{1}{20}\}=\frac{1}{20}<1$. Now, by simple calculations, we get $\sigma<1$. Thus all conditions in Theorem 3.1 are satisfied, then, the problem (5.1) has at least one solution on J.

Setting $S(t)=\psi(t)-\psi(0)$, by using lemma 2.6, we have

$$I_{0^+}^{\frac{1}{3},e^t} S(t) \leqslant \frac{1}{\Gamma(\frac{7}{3})}\left(\psi(t)-\psi(0)\right)^{\frac{1}{3}} S(t) \leqslant \frac{(e-1)^{\frac{1}{3}}}{\Gamma(\frac{7}{3})} S(t) = \mu_S S(t),$$

where $\mu_S = \frac{(e-1)^{\frac{1}{3}}}{\Gamma(\frac{7}{3})}$.

On the other hand, as shown in Theorem (4.9), for $\epsilon = 1$, if $x \in C_{\frac{1}{2},e^t}(J,\mathbb{R})$ satisfies

$$\left|{}^{H}D_{0^+}^{p,\beta;\psi} x(t) - \lambda x(t) - \mathcal{H}_x(s)\right| \leqslant \psi(t)-\psi(0), \quad t \in J,$$

there exists a unique solution $\nu(t) \in C_{\frac{1}{3},e^t}(J,\mathbb{R})$ such that

$$|x(t)-\nu(t)| \leqslant \oslash_{f,S}\left(\psi(t)-\psi(0)\right).$$

where $\oslash_{f,S} := \left[\frac{\mathcal{R}}{\Gamma(2-r)}+1\right]\mu_S\left[1+\sum_{n=1}^{\infty}\left[\frac{(\lambda+\vartheta)}{1-\varpi}\mu_S\right]^n\right] > 0$.

It follows from Theorem (4.9) that the problem (5.1) is generalized Ulam–Hyers–Rassias stable.

Conclusion

In this paper, we have successfully established the existence and uniqueness results of fractional implicit differential equations with period condition involving ψ-Hilfer derivative. Moreover, we have discussed the different types of stability of solutions to such equations in the weighted space $C_{1-r,\psi}(J,\mathbb{R})$. In addition, an example is presented to illustrate our results.

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References

[1] Kilbas AA, Srivastava HM, and Trujillo J.J (2006). "Theory and Applications of Fractional Differential Equations". North-Holland Math Stud, Elsevier, Amsterdam. **207**.

[2] Samko SG, Kilbas AA and Marichev OI (1987). "Fractional Integrals and Derivatives: Theory and Applications". Gordon and Breach, Yverdon.

[3] Hilfer R (1999). "Application of Fractional Calculus in Physics". World Scientific, Singapore.

[4] Podlubny I (1999). "Fractional Differential Equations". San Diego, CA: Academic Press.

[5] Ulam SM (1960). "A Collection of Mathematical Problems". Interscience Tracts in Pure and Applied Mathematics, Inter-science, New York-London.

[6] Hyers DH, Isac G and Rassias ThM (1998). "Stability of functional equations in several variables". Progr. Nonlinear Diff. Equ. Appl. **34**.

[7] Rassias TM (1978). *On the stability of the linear mapping in Banach spaces*. Proc. Amer. Math. Soc. **72**(2): 297-300.

[8] Abbas S, Benchohra M, Graef JR and Henderson J (2018). *Implicit fractional differential and integral equations: existence and stability*. Walter de Gruyter GmbH & Co KG., **26**.

[9] Jarad F and Abdeljawad T (2020). *Generalized fractional derivatives and Laplace transform*. Discrete Contin. Dyn. Syst. Ser. S **13**(3): 709-722. https://doi.org/10.3934/dcdss.2020039

[10] Ali A, Shah K, Jarad F, Gupta V and Abdeljawad T (2019). *Existence and stability analysis to a coupled system of implicit type impulsive boundary value problems of fractional-order differential equations*. Adv. Differ. Equ. **2019**(1): 101. https://doi.org/10.1186/s13662-019-2047-y

[11] Sousa JVDC, Jarad F and Abdeljawad T (2021). *Existence of mild solutions to Hilfer fractional evolution equations in Banach space*. Ann. Funct. Anal. **12**(12): 1-16. https://doi.org/10.1007/s43034-020-00095-5

[12] Abdo MS and Panchal SK (2019). *Fractional integro-differential equations involving ψ-Hilfer fractional derivative*. Adv. Appl. Math. Mech. **11**(2): 338-359. https://doi.org/10.4208/aamm.OA-2018-0143

[13] Abdo MS, Panchal SK and Shafei HH (2019). *Fractional integro-differential equations with nonlocal conditions and ψ-Hilfer fractional derivative*. Math. Model. Anal. **24**(4): 564-584. https://doi.org/10.3846/mma.2019.034

[14] Almalahi MA, Abdo MS and Panchal SK (2020). *ψ- Hilfer fractional differential equations by Picard operator*, J. Appl. Noninear. Dyn. **9**(4): 685-702. https://doi.org/10.5890/JAND.2020.12.011

[15] Oliveira EC and Sousa JVDC (2018). *Ulam–Hyers–Rassias stability for a class of fractional integro-differential equations*. Res. Math. **73**: 111. https://doi.org/10.1007/s40314-018-0639-x

[16] Shah K, Ali A and Bushnaq S (2018). *Hyers-Ulam stability analysis to implicit Cauchy problem of fractional differential equations with impulsive conditions*. Math. Meth. Appl. Sci. **41**: 8329-8343. https://doi.org/10.1002/mma.5292

[17] Sousa JVDC, Kucche KD and Oliveira EC (2019). *Stability of ψ-Hilfer impulsive fractional differential equations*. Appl. Math. Lett. **88**: 73-80. https://doi.org/10.1016/j.aml.2018.08.013

[18] Sousa JVDC and de Oliveira EC (2018). *On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator*. Fixed Point Theory Appl. **20**, Article ID 96. https://doi.org/10.1007/s11784-018-0587-5

[19] Sousa JVDC and de Oliveira EC (2018). *Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation*. Appl. Math. Lett. **81**: 50-56. https://doi.org/10.1016/j.aml.2018.01.016

[20] Wang J, Zhou Y and Feĉkan YM (2012). *Nonlinear impulsive problems for fractional differential equations and Ulam stability*. Comput. Math. Appl. **64**: 3389-3405. https://doi.org/10.1016/j.camwa.2012.02.021

[21] Khan A, Khan H, Gómez-Aguilar JF and Abdeljawad T (2019). *Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel*. Chaos Solitons Fractals **127**: 422-427. https://doi.org/10.1016/j.chaos.2019.07.026

[22] Abdo MS, Shah K, Panchal SK and Wahash HA (2020). *Existence and Ulam stability results of a coupled system for terminal value problems involving ψ-Hilfer fractional operator*. Adv. Differ. Equ. **2020**:316. https://doi.org/10.1186/s13662-020-02775-x

[23] Wahash HA, Abdo MS and Panchal SK (2020). *Existence and Ulam-Hyers stability of the implicit fractional boundary value problem with ψ-Caputo fractional derivative*. J. Appl. Math. Comput. Mech. **19**(1): 89-101. https://doi.org/10.17512/jamcm.2020.1.08

[24] Furati KM and Kassim MD (2012). *Existence and uniqueness for a problem involving Hilfer fractional derivative*. Comput. Math. Appl. **64**: 1616-1626. https://doi.org/10.1016/j.camwa.2012.01.009

[25] Sousa JVDC and de Oliveira EC (2018). *On the ψ-Hilfer fractional derivative*. Commun. Nonlinear Sci. Numer. Simul. **60**: 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005

[26] Teodoro GS, Machado JT and de Oliveira EC (2019). *A review of definitions of fractional derivatives and other operators*. J. Comput. Phys. **388**: 195-208. https://doi.org/10.1016/j.jcp.2019.03.008

[27] Sousa JVDC and de Oliveira EC (2019). *Leibniz type rule: ψ-Hilfer fractional operator*. Commun. Nonlinear Sci. Numer. Simulat. **77:** 305-311. https://doi.org/10.1016/j.cnsns.2019.05.003

[28] Sousa JVDC and de Oliveira EC (2019). *A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator*. Differ. Equ. Appl. **11**(1): 87-106.

[29] Abbas MI (2020). *On the Nonlinear Sequential ψ-Hilfer Fractional Differential Equations*. Int. J. Math. Anal. **14**: 77-90. https://doi.org/10.12988/ijma.2020.91283

[30] Almalahi MA, Abdo MS and Panchal SK (2020). *On the theory of fractional terminal value problem with*

ψ-Hilfer fractional derivative. Mathematics **5**(5): 4889–4908. https://doi.org/10.3934/math.2020312

[31] Abbas MI (2015). *Existence and Uniqueness of Mittage-Leffler-Ulam Stable Solution for Fractional Integrodifferential Equations with Nonlocal Initial Conditions*. Eur. J. Pure. Appl. Math. **8**: 478-498. https://doi.org/10.3846/mma.2019.034

[32] Abdo MS, Panchal SK and Wahash HA (2020). *Ulam–Hyers–Mittag-Leffler stability for a ψ-Hilfer problem with fractional order and infinite delay*. Results Appl. Math. **7**: 100115. https://doi.org/10.1016/j.rinam.2020.100115

[33] Wahash HA, Abdo MS and Panchal SK (2019). *Fractional integrodifferential equations with nonlocal conditions and generalized Hilfer fractional derivative*. Ufa Math. J. **11**(4): 151-170. https://doi.org/10.13108/2019-11-4-151

[34] Sousa JVDC, Kucche KD and de Oliveira EC (2019). *On the Ulam-Hyers stabilities of the solutions of ψ-Hilfer fractional differential equation with abstract Volterra operator*. Math. Meth. Appl. Sci. **42**(9): 3021-3032. https://doi.org/10.1002/mma.5562

[35] Gao Z and Yu X (2018). *Existence results for BVP of a class of Hilfer fractional differential equations*. J. Appl. Math. Comput. **56**(1-2), 217-233. https://doi.org/10.1007/s12190-016-1070-3

[36] Almalahi MA, Abdo MS and Panchal SK (2020). *Periodic boundary value problems for fractional implicit differential equations involving Hilfer fractional derivative*. Probl. Anal. Issues Anal. **9(27)**(2): 16-44. https://doi.org/10.15393/j3.art.2020.7410

[37] Wang JR, Feckan M and Zhou Y (2013). *Presentation of solutions of impulsive fractional Langevin equations and existence results*. Eur. Phys. J. Spec. Top. **222**: 1857–1874. https://doi.org/10.1140/epjst/e2013-01969-9

[38] Kucche KD, Mali AD and Sousa JVDC (2019). *Theory of Nonlinear ψ-Hilfer Fractional Differential Equations*. Comput. Appl. Math. https://doi.org/10.1007/s40314-019-0833-5