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Further results on the existence of solutions for generalized fractional quadratic functional integral equations

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Abstract

This paper discusses some existence results for at least one continuous solution for generalized fractional quadratic functional integral equations. Some results on nonlinear functional analysis including Schauder fixed point theorem are applied to establish the existence result for proposed equations. We improve and extend the literature by incorporated of some well known and commonly cited results as special cases in this topic. Further, we prove the existence of maximal and minimal solutions for these equations.

Keywords: Fractional differential equations, ψ–Caputo fractional derivative, Existence, Maximal and minimal solutions, Fixed point theorem.
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1. Introduction

In the past few years, many author have utilized the fractional calculus as a path of describing natural phenomena in diverse fields such as mathematics, applied science, and engineering. Recently, this topic has aroused deep intense interest, and increasing studies of some researchers, both in mathematics and in applications. As a result of this interest, many studies have been presented on this topic. In this regard, we refer to the monographs of leading researchers [1, 2, 3, 4, 5, 6, 7], and the papers [8, 9, 10, 11, 12]. There are various definitions of fractional derivatives and fractional integrals, the most famous of which are Riemann-Liouville [4], Caputo [13], Hilfer [2], Hadamard [14], Katugampola [15] and Prabhakar [16]. On the other hand, new, more general, local, and non-local fractional operators have emerged incorporates several operators dealing with a different kernel of integrations, such as Caputo-Hadamard [17], Caputo-Katugampola [18], Caputo-Fabrizio [19], Atangana-Baleanu-Caputo [20], ψ-Caputo [21], ψ-Hilfer [22] and some other operators [23, 24, 25]. These operators combine a wide class of fractional derivatives as the aforementioned ones. It is notable that integral equations (IEs) have many useful applications in describing various events and problems of the real world. Additionally, the theory of IEs is quickly developing utilizing the instruments of fixed point

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theory and nonlinear analysis. Specifically, quadratic integral equations (QIEs) have numerous helpful applications in the real world. For instance, QIEs are regularly pertinent in the theory of radiative exchange, the theory of neutron transport, the dynamic theory of gases, traffic theory, and queuing theory. Numerous creators studied the existence of solutions for sundry classes of nonlinear QIEs (see e.g. [26, 27, 28, 29, 30, 31, 32, 33, 34] and [35, 36, 37, 38, 39, 40]. Notwithstanding, in a large portion of the above literature, the main results were investigated with the assistance of the method associated with the measure of noncompactness and the classical fractional operators. Rather than utilizing the method of measure of noncompactness, we use Schauder fixed point theorem. Moreover, our results will be more generalized of the above literature due to the kernel of integration depends on another function ψ, thus our obtained results will cover a large number of results for different functions of ψ.

In this regard, we discuss the existence of at least one continuous solution of the nonlinear fractional quadratic functional integral equation

$$\omega(t) = h(t) + g(t, \omega(\varphi(t)) \int_0^t \frac{[\psi(t) - \psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho, \omega(\phi(\rho)) d\rho, \ t \in \Delta = [0, 1], \tag{1.1}$$

The equation (1.1) we can express it as

$$\omega(t) = h(t) + g(t, \omega(\varphi(t)) \mathcal{I}_{0^+}^{\alpha,\psi} f(t, \omega(\phi(t)), \ t \in \Delta = [0, 1], \tag{1.2}$$

where $\alpha > 0$, $\psi : \Delta \to \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in \Delta$, $\mathcal{I}_{0^+}^{\alpha,\psi}$ is generalized fractional operator in Reimann-Liouville sense, and h, g, φ, f, ϕ are given functions satisfying some suppositions that will be specified in the next section. Moreover, the existence of maximal and minima solutions of (1.1) will be proved. For some recent results involving ψ-fractional derivatives, one can refer to [41, 42, 43, 44, 45].

In the next analysis, we shall use the following as an abbreviation:

Fractional Quadratic Functional Integral Equation (FQFIE); Fractional Quadratic Integral Equation (FQIE); Fractional Functional Integral Equation (FFIE); Fractional Functional Differential Equation (FFDE); Fractional Functional Differential Equations (FDEs); Differential Equations (DEs); Reimann-Liouville (RL).

Remark 1.1.

1. If $\psi(t) = t$, then FQFIE (1.1) reduces to the following equation

$$\omega(t) = h(t) + g(t, \omega(\varphi(t)) \int_0^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho, \omega(\phi(\rho)) d\rho, \ t \in \Delta, \ \alpha > 0$$

   which have been studied by EL-Sayed and Hashem in [46].
2. If $\psi(t) = \varphi(t) = \phi(t) = t$ then the existence of continuous solutions of

$$\omega(t) = h(t) + g(t, \omega(t) \int_0^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho, \omega(\rho) d\rho, \ t \in \Delta, \ \alpha > 0$$

   was proved in [34, 47].

3. If $\psi(t) = \phi(t) = t$ and $g(t, \omega(\varphi(t)) \equiv 1$, then the existence of continuous solutions of
$$\omega(t) = h(t) + \int_0^t \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} f(\rho, \omega(\rho) d\rho,\ t \in \Delta,\ \alpha > 0$$
was proved in [40].
4. If $\psi(t) = \phi(t) = t$, $\alpha = 1$, $g(t, \omega(\varphi(t)) \equiv 1$, and $h(t) = \omega_0$, then the existence of continuous solutions of
$$\omega(t) = \omega_0 + \int_0^t f(\rho, \omega(\rho)) d\rho,\ t \in \Delta,$$
was proved in [48].

To our knowledge, no contributes to study of the fractional quadratic functional integral equations exist in the literature, especially for those involving the ψ-fractional derivatives.

This paper is structured as follows: In Section 2, we recall some fundamental definitions and fact preliminary. Section 3 contains the existence result obtained by using Schauder fixed point theorem. Applications are given in Section 4. We discuss the maximal and minimal solutions of (1.1) in Section 5.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers whereas $\Delta = [0, 1] \subset \mathbb{R}$. $C(\Delta)$ be the Banach space of all real continuous functions defined on Δ with the supremum norm $\|\cdot\|_\infty$, and let $L^1(\Delta)$ be the space of Lebesgue integrable functions on Δ. The existence results will be rely on the following definitions, lemmas and theorems.

Definition 2.1. [4] The generalized RL fractional integral and derivative of order $\alpha > 0$ for an integrable function $z : [a, b] \to \mathbb{R}$ with respect to another function ψ are defined as follows
$$\mathcal{I}_{a^+}^{\alpha,\psi} z(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\rho)(\psi(t) - \psi(\rho))^{\alpha-1} z(\rho) d\rho,\ t > a, \tag{2.1}$$
and
$$^{RL}\mathcal{D}_{a^+}^{\alpha,\psi} z(t) = \mathcal{D}^{n,\psi}\ \mathcal{I}_{a^+}^{n-\alpha,\psi} z(t),\ t > a, \tag{2.2}$$
respectively, where $\psi : [a, b] \to \mathbb{R}$ is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$, $\mathcal{D}^{n,\psi} = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n$, $n = [\alpha] + 1$, and $\Gamma(\cdot)$ the Euler gamma function is defined by
$$\Gamma(\delta) = \int_0^\infty t^{\delta-1} e^{-t} dt,\ \ \delta \in \mathbb{C},\ \mathcal{R}(\delta) > 0.$$

Definition 2.2. [21] The generalized Caputo fractional derivative of order $\alpha > 0$ with respect to another function ψ can be represented by the expression
$$^{C}\mathcal{D}_{a^+}^{\alpha,\psi} z(t) = \mathcal{I}_{a^+}^{n-\alpha;\psi}\ \mathcal{D}^{n,\psi} z(t), \tag{2.3}$$

where $\mathcal{Z}, \psi \in C^n[a,b]$. Given $\mathcal{Z} \in C^{n-1}[a,b]$, we have

$$ {}^{C}\mathcal{D}^{\alpha,\psi}_{a^+}\mathcal{Z}(t) = {}^{RL}\mathcal{D}^{\alpha,\psi}_{a^+}\left[\mathcal{Z}(t) - \sum_{j=0}^{n-1} \frac{\mathcal{Z}^{[j]}_{\psi}(a)}{j!}(\psi(t)-\psi(a))^j\right], $$

where $\mathcal{Z}^{[j]}_{\psi}(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^j \mathcal{Z}(t)$, and $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$. Further, if $\alpha = n \in \mathbb{N}$, then ${}^{C}\mathcal{D}^{\alpha,\psi}_{a^+}\mathcal{Z}(t) = \mathcal{Z}^{[n]}_{\psi}(t)$.

In particular, if $0 < \alpha < 1$, then ${}^{C}\mathcal{D}^{\alpha,\psi}_{a^+}\mathcal{Z}(t) = {}^{RL}\mathcal{D}^{\alpha,\psi}_{a^+}[\mathcal{Z}(t) - \mathcal{Z}(a)]$.

Remark 2.3. In Defintions 2.1 and 2.2, if $\psi(t) = t$, then equations (2.1), (2.2) and (2.3) reduce to the following equations

$$ \mathcal{I}^{\alpha}_{a^+}\mathcal{Z}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\rho))^{\alpha-1}\mathcal{Z}(\rho)d\rho, \ t > a, $$

$$ {}^{RL}\mathcal{D}^{\alpha}_{a^+}\mathcal{Z}(t) = \mathcal{D}^n \ \mathcal{I}^{n-\alpha}_{a^+}\mathcal{Z}(t), \ t > a, $$

and

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+}\mathcal{Z}(t) = \ \mathcal{I}^{n-\alpha}_{a^+} \ \mathcal{D}^n\mathcal{Z}(t), $$

respectivaly, where $\mathcal{D}^n = \left(\frac{d}{dt}\right)^n$ and $n = [\alpha]+1$. For more details see [4].

Lemma 2.4. *[21] Let $\alpha > 0$, $\psi, \mathcal{Z} \in C^{n-1}[a,b]$ and $\mathcal{Z}^{(n)}$ exists a.e. on any bounded interval of $[a,b]$. Then*

$$ \mathcal{I}^{\alpha,\psi}_{a^+} \ {}^{C}\mathcal{D}^{\alpha,\psi}_{a^+}\mathcal{Z}(t) = \mathcal{Z}(t) - \sum_{j=0}^{n-1} \frac{\mathcal{Z}^{[j]}_{\psi}(a)}{j!}(\psi(t)-\psi(a))^j. $$

In particular, if $0 < \alpha < 1$, we have $\mathcal{I}^{\alpha,\psi}_{a^+} \ {}^{C}\mathcal{D}^{\alpha,\psi}_{a^+}\mathcal{Z}(t) = \mathcal{Z}(t) - \mathcal{Z}(a)$.

Lemma 2.5. *[4] Let $\alpha > 0$ and $\beta > 0$. Then, we have*

$$ \mathcal{I}^{\alpha,\psi}_{a^+}\mathcal{I}^{\beta,\psi}_{a^+}\mathcal{Z}(t) = \mathcal{I}^{\alpha+\beta,\psi}_{a^+}\mathcal{Z}(t), \ t \in [a,b], \ \text{and} $$

$$ \mathcal{I}^{\alpha;\psi}_{a^+}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}. $$

Lemma 2.6. *[41, 21] Let $\alpha > 0$, $\mathcal{Z} \in C[a,b]$ and $\psi \in C^1[a,b]$. Then $\mathcal{I}^{\alpha;\psi}_{a^+}\mathcal{Z} \in C[a,b]$ and*

$$ \mathcal{I}^{\alpha;\psi}_{a^+}\mathcal{Z}(a) = \lim_{t\to a^+} \mathcal{I}^{\alpha;\psi}_{a^+}\mathcal{Z}(t) = 0. $$

Theorem 2.7. *(Schauder Fixed Point Theorem [49]) Let ϖ a Banach space and P be a nonempty, convex, compact subset of ϖ. If $\mathcal{N} : P \to P$ is a continuous mapping, then $\mathcal{N}$ has at least one fixed point in P.*

3. Existence of Continuous Solutions

Now, FQFIE (1.1) will be investigated under the following hypotheses:

(i) $h : \Delta \to \mathbb{R}$ is continuous and bound with $\theta_1 = \sup_{t\in\Delta} |h(t)|$.

(ii) $g : \Delta \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded with $\theta_2 = \sup_{(t,\omega)\in\Delta\times\mathbb{R}} |g(t,\omega)|$.

(iii) There exist $\ell_1, \ell_2 > 0$ such that

$$|g(t,\omega) - g(\rho,\varpi)| \leqslant \ell_1 |t-\rho| + \ell_2 |\omega - \varpi|,$$

for all $t, \rho \in \Delta$ and $\omega, \varpi \in \mathbb{R}$.

(iv) $f : \Delta \times \mathbb{R} \to \mathbb{R}$ is satisfies Carathéodory conditions, i.e.

$f(\cdot,\omega)$ is measurable for each fixed $\omega \in \mathbb{R}$ and $f(t,\cdot)$ continuous for each fixed $t \in \Delta$.

(v) There exist constant b and function $\eta \in L^1(\Delta)$ such that $|f(t,\omega)| \leqslant \eta(t) + b|\omega|$, for each $t \in \Delta$, $\omega \in \mathbb{R}$, and $\theta_3 = \sup_{t\in\Delta} \mathcal{I}_{0^+}^{\beta,\psi}\eta(t)$, for any $\beta \leqslant \alpha$.

(vi) $\varphi, \phi : \Delta \to \Delta$ are continuous.

(vii) The inequality $\theta_2 b\,[\psi(1)-\psi(0)]^{\alpha} < \Gamma(\alpha+1)$ is satisfied.

Theorem 3.1. *Assume that (i)-(vii) be satisfied. Then FQFIE (1.1) has at least one solution* $\omega(t)$ *in the space* $C(\Delta)$.

Proof. Fix a number $\lambda > 0$, we consider the ball

$$\mathbb{P}_\lambda = \{\omega \in C(\Delta) : |\omega(t)| \leqslant \lambda \text{ for } t \in \Delta\} \subset C(\Delta),$$

with

$$\lambda \geqslant \frac{\theta_1 + \frac{\theta_2\theta_3}{\Gamma(\alpha-\beta+1)}\,[\psi(1)-\psi(0)]^{\alpha-\beta}}{1 - \frac{\theta_2 b}{\Gamma(\alpha+1)}\,[\psi(1)-\psi(0)]^{\alpha}}. \tag{3.1}$$

Let the operator $\mathcal{N} : C(\Delta) \to C(\Delta)$ defined on $\mathbb{P}_\lambda$ by

$$(\mathcal{N}\omega)(t) = h(t) + g(t,\omega(\varphi(t)) \int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho,\omega(\phi(\rho))d\rho,\ t \in \Delta.$$

Then, in light of our hypotheses, for $t \in \Delta$ and $\omega \in \mathbb{P}_\lambda$, we have

$$\begin{aligned}
|(\mathcal{N}\omega)(t)| &\leqslant |h(t)| + |g(t,\omega(\varphi(t)))| \int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho)\, |f(\rho,\omega(\phi(\rho)))|\, d\rho \\
&\leqslant \theta_1 + \theta_2 \int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho)\, [\eta(\rho) + b\,|\omega(\phi(\rho))|]\, d\rho \\
&= \theta_1 + \theta_2 \mathcal{I}_{0^+}^{\alpha-\beta,\psi} \left(\mathcal{I}_{0^+}^{\beta,\psi}\eta(t)\right) \\
&\quad + \theta_2 b \int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho)\, |\omega(\phi(\rho))|\, d\rho \\
&\leqslant \theta_1 + \theta_2\theta_3 \int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \psi'(\rho) d\rho \\
&\quad + \theta_2 b\lambda \int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) d\rho \\
&\leqslant \theta_1 + \frac{\theta_2\theta_3}{\Gamma(\alpha-\beta+1)} [\psi(1)-\psi(0)]^{\alpha-\beta} + \frac{\theta_2 b\lambda}{\Gamma(\alpha+1)} [\psi(1)-\psi(0)]^{\alpha}.
\end{aligned}$$

By assumption (vii) and inequality (3.1) we conclude that $\mathcal{N} : \mathbb{P}_\lambda \to \mathbb{P}_\lambda$.

Now, we prove that $\{\mathcal{N}\omega : \omega \in \mathbb{P}_\lambda\}$ is equi-continuous. For $t_1, t_2 \in \Delta$ $(t_1 < t_2)$, we have

$$\begin{aligned}
&(\mathcal{N}\omega)(t_2) - (\mathcal{N}\omega)(t_1) \\
= \;& h(t_2) - h(t_1) + g(t_2,\omega(\varphi(t_2)))\mathcal{I}_{0^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))) \\
& - g(t_1,\omega(\varphi(t_1)))\, \mathcal{I}_{0^+}^{\alpha,\psi} f(t_1,\omega(\phi(t_1))) \\
& + g(t_1,\omega(\varphi(t_1)))\mathcal{I}_{0^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))) \\
& - g(t_1,\omega(\varphi(t_1)))\, \mathcal{I}_{0^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))) \\
= \;& h(t_2) - h(t_1) + \big(g(t_2,\omega(\varphi(t_2))) \\
& - g(t_1,\omega(\varphi(t_1)))\big)\mathcal{I}_{0^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))) \\
& + (g(t_1,\omega(\varphi(t_1)))\, \big(\mathcal{I}_{0^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))) - \mathcal{I}_{0^+}^{\alpha,\psi} f(t_1,\omega(\phi(t_1))\big), \qquad (3.2)
\end{aligned}$$

but

$$\begin{aligned}
&\mathcal{I}_{0^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))) - \mathcal{I}_{0^+}^{\alpha,\psi} f(t_1,\omega(\phi(t_1))) \\
= \;& \int_0^{t_2} \frac{[\psi(t_2)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho,\omega(\phi(\rho)) d\rho + \int_{t_1}^{t_2} \frac{[\psi(t_2)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \\
& \times \psi'(\rho) f(\rho,\omega(\phi(\rho)) d\rho - \int_0^{t_1} \frac{[\psi(t_1)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho,\omega(\phi(\rho)) d\rho \\
\leqslant \;& \int_0^{t_1} \frac{[\psi(t_1)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho,\omega(\phi(\rho)) d\rho + \int_{t_1}^{t_2} \frac{[\psi(t_2)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \\
& \times \psi'(\rho) f(\rho,\omega(\phi(\rho)) d\rho - \int_0^{t_1} \frac{[\psi(t_1)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho,\omega(\phi(\rho)) d\rho \\
= \;& \int_{t_1}^{t_2} \frac{[\psi(t_2)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho,\omega(\phi(\rho)) d\rho = \mathcal{I}_{t_1^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))).
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\mathcal{I}_{0^+}^{\alpha,\psi} f(t_2,\omega(\phi(t_2))) - \mathcal{I}_{0^+}^{\alpha,\psi} f(t_1,\omega(\phi(t_1)))\right| \\
\leqslant\ & \mathcal{I}_{t_1^+}^{\alpha,\psi} |f(t_2,\omega(\phi(t_2)))| \leqslant \mathcal{I}_{t_1^+}^{\alpha,\psi}\eta(t_2) + b\mathcal{I}_{t_1^+}^{\alpha,\psi} |\omega(\phi(t_2)|\\
\leqslant\ & \mathcal{I}_{t_1^+}^{\alpha-\beta,\psi}\ \mathcal{I}_{t_1}^{\beta,\psi}\eta(t_2) + b\mathcal{I}_{t_1^+}^{\alpha,\psi} |\omega(\phi(t_2)|\\
\leqslant\ & \theta_3\frac{[\psi(t_2)-\psi(t_1)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + b\lambda\frac{[\psi(t_2)-\psi(t_1)]^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Hence, the inequality (3.2) becomes

$$\begin{aligned}
&|(\mathcal{N}\omega)(t_2)-(\mathcal{N}\omega)(t_1)|\\
\leqslant\ & |h(t_2)-h(t_1)| + \big(\ell_1|t_2-t_1| + \ell_2|\omega(\varphi(t_2)-\omega(\varphi(t_1)|\big)\mathcal{I}_{0^+}^{\alpha,\psi}|f(t_2,\omega(\phi(t_2)))|\\
&+(g(t_1,\omega(\varphi(t_1)))\Big(\theta_3\frac{[\psi(t_2)-\psi(t_1)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + b\lambda\frac{[\psi(t_2)-\psi(t_1)]^{\alpha}}{\Gamma(\alpha+1)}\Big)\\
\leqslant\ & |h(t_2)-h(t_1)| + \big(\ell_1|t_2-t_1| + \ell_2|\omega(t_2)-\omega(t_1)|\big)\mathcal{I}_{0^+}^{\alpha,\psi}\big(\eta(t_2)+b|\omega(\phi(t_2)|\big)\\
&+\theta_2\theta_3\frac{[\psi(t_2)-\psi(t_1)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \theta_2 b\lambda\frac{[\psi(t_2)-\psi(t_1)]^{\alpha}}{\Gamma(\alpha+1)}\\
\leqslant\ & |h(t_2)-h(t_1)| + \theta_3\frac{[\psi(t_2)-\psi(0)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\big(\ell_1|t_2-t_1| + \ell_2|\omega(t_2)-\omega(t_1)|\big)\\
&+b\lambda\frac{[\psi(t_2)-\psi(0)]^{\alpha}}{\Gamma(\alpha+1)}\big(\ell_1|t_2-t_1| + \ell_2|\omega(t_2)-\omega(t_1)|\big)\\
&+\theta_2\theta_3\frac{[\psi(t_2)-\psi(t_1)]^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \theta_2 b\lambda\frac{[\psi(t_2)-\psi(t_1)]^{\alpha}}{\Gamma(\alpha+1)}\\
\to\ & 0 \ \text{ as } t_2\to t_1.
\end{aligned}$$

This shows that $\{\mathcal{N}\omega : \omega\in\mathbb{P}_\lambda\}$ is equi-continuous on Δ. Therefore by the Arzela-Ascoli theorem [48] $\overline{\mathcal{N}\mathbb{P}_\lambda}$ is compact . Clearly, $\mathbb{P}_\lambda$ is nonempty, bounded, convex, and closed subset of $C(\Delta)$. Hypotheses (ii) and (iv) lead to $\mathcal{N}:\mathbb{P}_\lambda\to C(\Delta)$ is a continuous operator in ω.

Since all assumptions of Theorem 2.7 hold, then $\mathcal{N}$ has a fixed point in $\mathbb{P}_\lambda$. □

3.1. Special Cases

In this part, we give some special cases that are also investigated according to our previous results.

Corollary 3.2. *Under the hypotheses of Theorem 3.1 with $\varphi(t)=\phi(t)=t$, then the FQIE*

$$\omega(t) = h(t) + g(t,\omega(t))\int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)}\psi'(\rho)f(\rho,\omega(\rho))d\rho,$$

has at least one solution $\omega\in C(\Delta)$.

Corollary 3.3. *Under the hypotheses of Theorem 3.1 with* $g(t,\omega(\varphi(t)))\equiv 1$, *then the FFIE*

$$\omega(t)=h(t)+\int_0^t \frac{[\psi(t)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)}\psi'(\rho)f(\rho,\omega(\phi(\rho)))d\rho,$$

has at least one solution $\omega\in C(\Delta)$.

Remark 3.4. If $b=0$ and $\psi(t)=\phi(t)=t$ in Corollary 3.3, we get the same result proven in [40].

Now putting $\alpha,\beta\to 1$, we get the following corollary:

Corollary 3.5. *Under the hypotheses of Theorem 3.1 with* $g(t,\omega(\varphi(t)))\equiv 1$, $h(t)=\omega_0$ *and* $\alpha=\beta=1$, *then the following IE*

$$\omega(t)=\omega_0+\int_0^t \psi'(\rho)f(\rho,\omega(\phi(\rho)))d\rho$$

has at least one solution $\omega\in C(\Delta)$ *which is equivalent to the problem*

$$\omega'_\psi(t)=f(t,\omega(\phi(t)),\ \omega(0)=\omega_0,$$

where $\omega'_\psi(t)=\frac{\omega'(t)}{\psi'(t)}$.

Remark 3.6. In Corollary 3.5, if $\psi(t)=\phi(t)=t$, and $b=0$ in the condition (v), then the following IE

$$\omega(t)=\omega_0+\int_0^t f(\rho,\omega(\rho)))d\rho$$

has at least one solution $\omega\in C(\Delta)$ is equivalent to the problem

$$\omega'(t)=f(t,\omega(\phi(t)),\ \omega(0)=\omega_0,$$

which was proved in [48].

4. Applications

In this part, we apply the results of Section 3 to some FDEs. In view of these applications, we consider nonlinear Caputo FFDE, Reimann-Liouville FFDE and classical DE of the forms:

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\alpha,\psi}\omega(t)=f(t,\omega(\phi(t)),\ t\in\Delta=[0,1],\ \alpha\in(0,1),\\ \quad\mathcal{I}_{0^+}^{1-\alpha,\psi}\omega(0)=\omega_0, \end{cases} \tag{4.1}$$

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha,\psi}\omega(t)=f(t,\omega(\phi(t)),\ t\in\Delta=[0,1],\ \alpha\in(0,1),\\ \quad\omega(0)=\omega_0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \omega'(t)=f(t,\omega(\phi(t)),\ t\in\Delta=[0,1],\\ \quad\omega(0)=\omega_0 \end{cases} \tag{4.3}$$

where ${}^{R}\mathcal{D}_{0^+}^{\alpha,\psi}$ and ${}^{C}\mathcal{D}_{0^+}^{\alpha,\psi}$ are the fractional derivatives in the sense of RL and Caputo of order α, respectively.

Theorem 4.1. *Under the hypotheses of Theorem 3.1 with*

$$h(t) \equiv \frac{\omega_0}{\Gamma(\alpha)} [\psi(t) - \psi(0)]^{\alpha-1} \text{ and } g(t, \omega(\varphi(t))) \equiv 1,$$

then the RL problem (4.1) has at least one solution $\omega \in C(\Delta)$.

Proof. Operating with $\mathcal{I}_{0^+}^{\alpha,\psi}$ on (4.1), we obtain ([4, 21])

$$\omega(t) = c_1 [\psi(t) - \psi(0)]^{\alpha-1} + \mathcal{I}_{0^+}^{\alpha,\psi} f(t, \omega(\phi(t))),$$

for $c_1 \in \mathbb{R}$. By means of the condition $\mathcal{I}_{0^+}^{1-\alpha,\psi}\omega(0) = \omega_0$, we get $c_1 = \frac{\omega_0}{\Gamma(\alpha)}$. Hence

$$\omega(t) = \frac{\omega_0}{\Gamma(\alpha)} [\psi(t) - \psi(0)]^{\alpha-1} + \int_0^t \frac{[\psi(t) - \psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho, \omega(\phi(\rho))) d\rho, \quad t \in \Delta. \tag{4.4}$$

Operating with ${}^{RL}\mathcal{D}_{0^+}^{\alpha,\psi}$ on FFIE (4.4) we obtain the RL problem (4.1). Moreover, the condition $\mathcal{I}_{0^+}^{1-\alpha,\psi}\omega(0) = \omega_0$ is satisfied. Consequently, the equivalence between the problem (4.1) and the FFIE (4.4) is proved and then the results follow from Theorem 3.1. □

Theorem 4.2. *Under the hypotheses of Theorem 3.1 with* $h(t) \equiv \omega_0$ *and* $g(t, \omega(\varphi(t))) \equiv 1$, *then the Caputo problem (4.2) has at least one solution* $\omega \in C(\Delta)$.

Proof. Operating with $\mathcal{I}_{0^+}^{\alpha,\psi}$ on (4.2) we obtain we obtain ([21])

$$\omega(t) = c_0 + \mathcal{I}_{0^+}^{\alpha,\psi} f(t, \omega(\phi(t))),$$

for $c_0 \in \mathbb{R}$. By means of the condition $\omega(0) = \omega_0$, we get $c_1 = \omega_0$. Hence

$$\omega(t) = \omega_0 + \int_0^t \frac{[\psi(t) - \psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)} \psi'(\rho) f(\rho, \omega(\phi(\rho))) d\rho, \quad t \in \Delta \tag{4.5}$$

which by Theorem 3.1 has the desired solution.

Operating with ${}^{C}\mathcal{D}_{0^+}^{\alpha,\psi}$ on (4.5) we obtain the problem (4.2). Therefore the equivalence between the problem (4.2) and the FFIE (4.5) is proved and then the results follow from Theorem 3.1.

In particular, if $\alpha = 1$ and $\psi(t) = t$, then the problem (4.2) reduces to the problem (4.3). So, the IE

$$\omega(t) = \omega_0 + \int_0^t f(\rho, \omega(\phi(\rho))) d\rho, \quad t \in \Delta$$

is equivalent to the problem (4.3). □

5. Maximal and Minimal Solutions

Definition 5.1. [50] The function $\mathfrak{Z}(t)$ is called a maximal solution of (1.1) if every solution of (1.1) on Δ satisfies

$$\omega(t) \leqslant \mathfrak{Z}(t), \quad t \in \Delta.$$

Whereas, the function $\mathfrak{N}(t)$ is called a minimal solution of (1.1) if every solution of (1.1) on Δ satisfies

$$\omega(t) \geqslant \mathfrak{N}(t), \quad t \in \Delta.$$

To prove the existence of maximum and minimum solutions for (1.1), we need the following lemma:

Lemma 5.2. *Assume that $f(t,\omega)$ is nondecreasing function in ω. Let $g(t,\omega)$, $f(t,\omega)$ satisfy the hypotheses in Theorem 3.1 and let $\omega(t)$, $\varpi(t)$ be continuous functions on Δ satisfying*

$$\omega(t) \leqslant h(t) + g(t,\omega(\varphi(t)))\mathcal{I}_{0^+}^{\alpha,\psi} f(t,\omega(\phi(t))), \tag{5.1}$$

$$\varpi(t) \geqslant h(t) + g(t,\varpi(\varphi(t)))\mathcal{I}_{0^+}^{\alpha,\psi} f(t,\varpi(\phi(t))), \tag{5.2}$$

where either (5.1) or (5.2) is strict. Then

$$\omega(t) < \varpi(t). \tag{5.3}$$

Proof. Assume that the conclusion (5.3) be false. Then there exists t_1 such that

$$\omega(t_1) = \varpi(t_1), \quad t_1 > 0, \tag{5.4}$$

and

$$\omega(t) < \varpi(t), \quad 0 < t < t_1.$$

Since f, g are monotone functions in ω, it follows that

$$\begin{aligned}
\omega(t_1) &\leqslant h(t_1) + g(t_1,\omega(\varphi(t_1)))\mathcal{I}_{0^+}^{\alpha,\psi} f(t_1,\omega(\phi(t_1))) \\
&= h(t_1) + g(t_1,\omega(\varphi(t_1)))\int_0^{t_1} \frac{[\psi(t_1)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)}\psi'(\rho) f(\rho,\omega(\phi(\rho)))d\rho \\
&< h(t_1) + g(t_1,\varpi(\varphi(t_1)))\int_0^{t_1} \frac{[\psi(t_1)-\psi(\rho)]^{\alpha-1}}{\Gamma(\alpha)}\psi'(\rho) f(\rho,\varpi(\phi(\rho)))d\rho \\
&< \varpi(t_1).
\end{aligned}$$

This contradicts with our assumption (5.4). Hence

$$\omega(t) < \varpi(t).$$

□

Theorem 5.3. *Suppose the hypotheses of Theorem 3.1 be fulfilled. In addition, if $f(t,\omega)$ is nondecreasing functions in ω, then there exist maximal and minimal solutions of (1.1).*

Proof. Initially, we will prove the existence of maximal solution of (1.1). Let $\varepsilon > 0$ be given. Now consider the following FQFIE

$$\omega_\varepsilon(t) = h(t) + g_\varepsilon(t, \omega_\varepsilon(\varphi(t))\mathcal{I}_{0^+}^{\alpha,\psi} f_\varepsilon(t, \omega_\varepsilon(\phi(t)), \tag{5.5}$$

where

$$f_\varepsilon(t, \omega_\varepsilon(\phi(t)) = f(t, \omega_\varepsilon(\phi(t)) + \varepsilon,$$

and

$$g_\varepsilon(t, \omega_\varepsilon(\varphi(t)) = g(t, \omega_\varepsilon(\varphi(t)) + \varepsilon.$$

Obviously the functions $f_\varepsilon(t, \omega_\varepsilon)$ and $g_\varepsilon(t, \omega_\varepsilon)$ satisfy hypotheses (ii), (iv) and

$$|g_\varepsilon(t, \omega_\varepsilon)| \leqslant \eta + \varepsilon := \eta',$$

$$|f_\varepsilon(t, \omega_\varepsilon)| \leqslant \eta(t) + \varepsilon + b\,|\omega| := \eta'(t) + b\,|\omega|\,.$$

Consequently, the equation (5.5) has a continuous solution $\omega(t)$ due to Theorem 3.1.

Next, let ε_1 and ε_2 be such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$. Then

$$\begin{aligned} \omega_{\varepsilon_1}(t) &= h(t) + g_{\varepsilon_1}(t, \omega_{\varepsilon_1}(\varphi(t))\mathcal{I}_{0^+}^{\alpha,\psi} f_{\varepsilon_1}(t, \omega_{\varepsilon_1}(\phi(t)) \\ &= h(t) + (g(t, \omega_{\varepsilon_1}(\varphi(t)) + \varepsilon_1)\,\mathcal{I}_{0^+}^{\alpha,\psi}\,(f(t, \omega_{\varepsilon_1}(\phi(t)) + \varepsilon_1) \\ &> h(t) + (g(t, \omega_{\varepsilon_1}(\varphi(t)) + \varepsilon_2)\,\mathcal{I}_{0^+}^{\alpha,\psi}\,(f(t, \omega_{\varepsilon_1}(\phi(t)) + \varepsilon_2)\,. \end{aligned} \tag{5.6}$$

Also, we have

$$\omega_{\varepsilon_2}(t) = h(t) + (g(t, \omega_{\varepsilon_2}(\varphi(t)) + \varepsilon_2)\,\mathcal{I}_{0^+}^{\alpha,\psi}\,(f(t, \omega_{\varepsilon_2}(\phi(t)) + \varepsilon_2)\,. \tag{5.7}$$

Applying Lemma 5.2, it follows from (5.6) and (5.7) that

$$\omega_{\varepsilon_2}(t) < \omega_{\varepsilon_1}(t), \quad t \in \Delta.$$

Let the operator $\mathcal{N}_\varepsilon : C(\Delta) \to C(\Delta)$ defined by

$$\mathcal{N}_\varepsilon \omega_\varepsilon(t) = h(t) + g_\varepsilon(t, \omega_\varepsilon(\varphi(t))\mathcal{I}_{0^+}^{\alpha,\psi} f_\varepsilon(t, \omega_\varepsilon(\phi(t)).$$

By same reported arguments in proof of Theorem 3.1, the operator $\mathcal{N}_\varepsilon$ is uniformly bounded and equicontinuous. Hence, the Arzela-Ascoli Theorem shows that there exists a decreasing sequence ε_n such that $\varepsilon_n \to 0$ as $n \to \infty$ and the uniform limit $\lim_{n\to\infty} \omega_{\varepsilon_n}(t) = \mathfrak{Z}(t)$ exists on Δ. Since the functions f_{ε_n} and g_{ε_n} are continuous in the second argument, it follows that

$$\mathfrak{Z}(t) = \lim_{n\to\infty} \omega_{\varepsilon_n}(t) = h(t) + g(t, \mathfrak{Z}(\varphi(t))\mathcal{I}_{0^+}^{\alpha,\psi} f(t, \mathfrak{Z}(\phi(t)).$$

This proves that $\mathfrak{Z}(t)$ is a solution of (1.1).

Finally, we will show that $\mathfrak{Z}(t)$ is maximal solution of (1.1). Indeed, let $\omega(t)$ be any solution of (1.1). Then we have

$$\begin{aligned} \omega_\varepsilon(t) &= h(t) + g_\varepsilon(t, \omega_\varepsilon(\varphi(t))\mathcal{I}_{0^+}^{\alpha,\psi} f_\varepsilon(t, \omega_\varepsilon(\phi(t)) \\ &> h(t) + g(t, \omega_\varepsilon(\varphi(t))\mathcal{I}_{0^+}^{\alpha,\psi} f(t, \omega_\varepsilon(\phi(t)), \end{aligned}$$

and

$$\omega(t) = h(t) + g(t, \omega(\varphi(t))\mathcal{I}_{0^+}^{\alpha,\psi} f(t, \omega(\phi(t)),$$

Applying Lemma 5.2, we get

$$\omega_\varepsilon(t) > \omega(t), \quad \text{for all } t \in \Delta.$$

from the uniqueness of the maximal solution (see [50]), it is obvious that the uniform limit

$$\lim_{\varepsilon \to 0} \omega_\varepsilon(t) = \mathfrak{Z}(t)$$

exists on Δ.

Likewise, we can demonstrate that there exists a minimal solution of (1.1). □

6. conclusion

In this article, we have considered the generalized fractional quadratic functional integral equations involving ψ-fractional operators. Some existence results for at least one continuous solution for the proposed equations have investigated. Fractional differential equations with respect to another function ψ were given as an application. We have applied the Schauder fixed point technique and nonlinear functional analysis to discuss these results. Further, we have proven the existence of maximal and minimal solutions for these equations. Many results of equations that contain classical fractional operators are obtained as special cases of (1.1). The reported results in this paper are recent and significantly contribute to the existing literature on the subject. For future work, the problems studied in this paper can be extended to cover other kinds of fractional quadratic functional integral equations involving ψ-Hilfer fractional operator.

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