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Firefly Algorithm (FA) for solving extended fuzzy portfolio selection problem

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Abstract

Portfolio selection problem is generally a nonlinear programming which has been solved by a variety of heuristic and non-heuristic techniques. This paper presents a novel heuristic method for solving an extended fuzzy portfolio selection model. The significance of this paper is twofold. First, it extends fuzzy mean-variance-skewness model to fuzzy mean-variance-skewness-kurtosis model. Second, a powerful heuristic called Firefly Algorithm (FA) is proposed for solving model. No study has ever proposed and solved this expanded model. Finally, several numerical examples are provided to illustrate the modeling idea and performance and effectiveness of the proposed algorithm is compared against the exact approach (LINGO software) in terms of fitness value and required computational time. Results show that the proposed FA is very promising and achieves quality results for fuzzy portfolio selection in a reasonable time.

1. Introduction

Portfolio selection deals with the problem of allocating one's capital to a large number of securities to meet investor's satisfaction. The earliest work in this field is due to Markowitz [17] who introduced the well-known mean-variance model considering trade-off between return and risk. The main idea of this model was to characterize the securities of individual returns as random variables with normal probability distribution. Lots of efforts have been performed by researchers in order to expand and solve Markowitz's model. These attempts, regarding the higher moments such as skewness and kurtosis, have tried to make his model more practical [1,5,9,11,20,26].

According to Konno and Suzuki [9], maximizing the skewness in a portfolio would result in better return and Samuelson [20] has indicated that most of the investors would prefer a portfolio with larger skewness where mean and variance are the same and the higher moments such as skewness can be neglected only where there are reasons to trust that the returns are symmetrically distributed (e.g. normal) or the higher moments are irrelevant to the investors' decisions. Furthermore, many researchers have showed the importance of kurtosis in financial markets which is a measure of whether the data are peaked or flat relative to a normal distribution. Tang and Shum [22] and Gondzio and Grothey [7] have indicated that returns of individual securities in most cases do not have normal distributions and are characterized by significant kurtosis.

In the field of models solving, some heuristic methods based on Genetic Algorithm [21], Tabu Search [19], Particle Swarm Optimization [6], Simulated Annealing [4], Ant Colony Optimization [15] and Memetic Algorithm [16], Artificial bee colony algorithm [3,10,13], bacteria foraging optimization [18] have been reported in the literatures for the portfolio selection problem. In addition to, [2] proposed three heuristics based on GA, TS, and SA to solve the problem and compared the results. Almost more the above literatures assume that the security returns are random variables. However, when there is not the enough historical data or the new assets are listed in the market, probability distribution is not reasonable to explain the variables while the fuzzy theory appropriately handles the case. Li et al in [12] were the first who proposed an extended portfolio selection model in fuzzy environment. Their model includes the skewness and lacks the kurtosis moment. In addition to, they employed GA for solving their model.

After review the literature, the main contribution of this paper is twofold. First we extend Li et al. [12] model into mean-variance-skewness-kurtosis model in fuzzy environment. Second we will propose a heuristic named as Firefly Algorithm (FA) in order to solve our model.

To achieve to this purpose the remainder of this paper is organized as follows. Section 2 defines the moments for fuzzy variables. Section 3 presents the fuzzy mean-variance-skewness-kurtosis model and its variations. A Firefly Algorithm (FA) for solving the models is developed in Section 4. Section 5 provides numerical examples to illustrate the effectiveness of proposed algorithm and the paper ends up with conclusions and summarizes the research in Section 6.

1. Preliminaries

The possibility theory is a branch of mathematics that studies the behavior of fuzzy phenomena which has been introduced by Zadeh [24] via membership function. To measure a fuzzy event, Zadeh [25] has also proposed the concept of possibility measure. Although the possibility measure is widely used in portfolio selection problems, it has some limitations. One of its important limitations is that the measure does not have self-duality property. Using the measure which is not self-dial may result in the same possibility value for two fuzzy events with different occurring chances. In addition, if the possibility value of a portfolio return being greater than a target value is less than one, the possibility value of the opposite event, i.e. the portfolio return being less than or equal to the target value, has the maximum value of one. Similarly, if the possibility value of a portfolio return being less than or equal to a target value is less than one, the possibility value of the opposite event, i.e. the portfolio return being greater than the target value, has the maximum value of one, as well. These results are quite awkward and confuse the decision maker [8].

In order to define a self-dual measure, [25] have introduced the concept of credibility measure which is more appropriate than the possibility measure in portfolio selection problems. Let ξ be a fuzzy variable with membership function μ, and u and r are real numbers. The credibility of a fuzzy event characterized by $\xi \geq r$, is then defined as in Eq. (1).

$$\mathrm{Cr}\{\xi \geq r\} = \frac{1}{2}\left(\sup_{u \geq r} \mu(u) + 1 - \sup_{u < r} \mu(u)\right) \tag{1}$$

Now consider a triangular fuzzy variable ξ which is fully determined by the triplet (a,b,c) of crisp numbers with $a < b < c$ and the membership function given by:

$$\mu(r)=\begin{cases}\frac{r-a}{b-a} & a\le r\le b\\ \frac{r-c}{b-c} & b\le r\le c\\ 0 & otherwise\end{cases} \tag{2}$$

The credibility of $\xi \ge r$ is then defined as follows.

$$Cr\{\xi\ge r\}=\begin{cases}1 & r\le a\\ \frac{a-2b+r}{2(a-b)} & a\le r\le b\\ \frac{r-c}{2(b-c)} & b\le r\le c\\ 0 & r\ge c\end{cases} \tag{3}$$

Liu and Liu [14] have also provided a more general definition of the expected value of a fuzzy variable ξ based on the credibility measure given in Eq. (4).

$$E[\xi]=\int_{0}^{+\infty} Cr\{\xi\ge r\}dr-\int_{-\infty}^{0} Cr\{\xi\le r\} \tag{4}$$

where at least one of the two integrals is finite.

According to Liu and Liu [25], if the fuzzy variable ξ has a finite expected value, its variance is then defined by:

$$V[\xi]=E\left[\left(\xi-E[\xi]\right)^2\right] \tag{5}$$

Now, let ξ be a fuzzy variable with finite expected value. The skewness of ξ is defined as follows [21]:

$$S[\xi]=E\left[\left(\xi-E[\xi]\right)^3\right] \tag{6}$$

Moreover, if the fuzzy variable ξ has a finite expected value, its kurtosis can be then defined as:

$$K[\xi]=E\left[\left(\xi-E[\xi]\right)^4\right] \tag{7}$$

Theorem 1: Let ξ be a fuzzy variable with finite expected value. For any real numbers *a* and *b*, we have:

$$K[a\xi+b]=a^4K[\xi] \tag{8}$$

Proof: According to [21], we have:

$$E[a\xi+b]=aE[\xi]+b, \tag{9}$$

Therefore, we can write:

$$K[a\xi+b]=E\left[\left(a\xi+b-\left(aE[\xi]+b\right)\right)^4\right]=E\left[a^4\left(\xi-E[\xi]\right)^4\right]=a^4E\left[\left(\xi-E[\xi]\right)^4\right]=a^4K[\xi]$$

2. The mean-variance-skewness-kurtosis models of portfolio selection

Let ξ be a fuzzy variable representing the return of the ith security and x_i be the proportion of total capital invested in security i. In general, ξ_i is given as $(p'_i+d_i-p_i)/p_i$ where p_i is the closing price of the ith security at present, p'_i is the estimated closing price in the next year, and d_i is the estimated dividends during the coming year.

The following notations are used in developing the models:

E: The operator of expected return

V: The operator of risk

S: The operator of skewness

K: The operator of kurtosis

ψ : Minimal expected return which the investors can accept

δ :Maximal risk which the investors can bear

η : Minimal skewness which is desired by the investors

φ : Maximal kurtosis which the investors can endure

Now suppose that minimal expected return and kurtosis and maximal risk are given, the investors interested in the use of skewness prefer a portfolio with large skewness. Therefore, we have the following mean-variance-skewness-kurtosis model:

$$\begin{cases} \max & S\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \\ \textit{subject to}: & V\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \le \delta \\ & E\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \ge \psi \\ & K\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \le \phi \\ & x_1 + \dots\dots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, .., n \end{cases} \tag{10}$$

The second model is as follows:

$$\begin{cases} \max & E\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \\ \textit{subject to}: & V\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \le \delta \\ & S\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \ge \eta \\ & K\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \le \phi \\ & x_1 + \dots\dots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, .., n \end{cases} \tag{11}$$

The purpose of the model presented in (11) is to select a portfolio with maximum return where the minimal variance and kurtosis as well as the maximal skewness are given.

The third model is as follows:

$$\begin{cases} \min & V\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \\ \textit{subject to}: & S\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \ge \eta \\ & E\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \ge \psi \\ & K\left[\xi_1 x_1 + \dots\dots + \xi_n x_n\right] \le \phi \\ & x_1 + \dots\dots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, .., n \end{cases} \tag{12}$$

The purpose of this model is to choose a portfolio with minimum risk where the maximal skewness and expected return as well as the minimal kurtosis are given.

Now if the maximal variance and the minimal expected value and skewness are given, then the investors would prefer a portfolio with lower kurtosis. Therefore, the forth model is as follows:

$$\begin{cases} \min & K\left[\xi_1 x_1 + ... + \xi_n x_n\right] \\ subject\ to: & S\left[\xi_1 x_1 + ... + \xi_n x_n\right] \ge \eta \\ & E\left[\xi_1 x_1 + ... + \xi_n x_n\right] \ge \psi \\ & V\left[\xi_1 x_1 + ... + \xi_n x_n\right] \le \delta \\ & x_1 + ... + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, .., n \end{cases} \tag{13}$$

Theorem 2: Assume that $\xi_i = \left(a_i, b_i, c_i\right)$ be independent triangular fuzzy variables for $i = 1, 2, ..., n$. Therefore, the above models can convert to deterministic programming. For instance, the model (13) in this case is as follows:

$$\begin{cases} \min \dfrac{253\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)^5 + 395\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)^4\left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)}{10240\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)} \\ + \dfrac{17\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)\left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)^4 + 290\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)^3\left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)^2}{10240\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)} \\ + \dfrac{70\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)^2\left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)^3 - \left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)^5}{10240\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)} \\ subject\ to: \sum_{i=1}^{n}(c_i - a_i)x_i . \sum_{i=1}^{n}(c_i + a_i - 2b_i)x_i \ge 32\eta \\ \dfrac{33\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)^3 + 21\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)^2\left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)}{384\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)} \\ + \dfrac{11\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)\left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)^2 - \left(\sum_{i=1}^{n}(b_i - a_i)x_i\right)^3}{384\left(\sum_{i=1}^{n}(c_i - b_i)x_i\right)} \le \delta \\ \sum_{i=1}^{n}\dfrac{(a_i + 2b_i + c_i)}{4}x_i \ge \psi \\ x_1 + x_2 + ... + x_n = 1, l_i \le x_i \le u_i, \quad i = 1, 2, ..., n \end{cases} \tag{14}$$

Proof: This deterministic programming for mean-variance-skewness model has been proved in [12]. Herein, only the kurtosis equation is investigated.

Suppose that $E\left[\xi_i\right] = e$, $\alpha = b - a$, $\beta = c - b$. Based on the mathematical credibility theory we know that $Cr\left\{\left(\xi - e\right)^4 \ge r\right\} = Cr\left\{\xi - e \ge \sqrt[4]{r}\right\} \vee Cr\left\{\xi - e \le \sqrt[4]{r}\right\}$. Thus, there are three different states regarding α and β as follows.

1) Assume that $\alpha > \beta$, then $e < b$. So we have:

$$Cr\left\{(\xi-e)^4 \geq r\right\}=\begin{cases}1-\dfrac{\sqrt[4]{r}+e-a}{2\alpha} & 0 \leq r \leq (b-e)^4 \\ -\dfrac{\sqrt[4]{r}+e-c}{2\beta} & (b-e)^4 \leq r \leq \left(\dfrac{c-a}{4}\right)^4 \\ \dfrac{-\sqrt[4]{r}+e-a}{2\alpha} & \left(\dfrac{c-a}{4}\right)^4 \leq r \leq (e-a)^4 \\ 0 & r \geq (e-a)^4\end{cases}$$

As a result,

$$K[\xi]=E\left[\left(\xi-E[\xi]\right)^4\right]=\int_0^{+\infty} Cr\left\{(\xi-e)^4 \geq r\right\} dr=\int_0^{(b-e)^4}\left(1-\frac{\sqrt[4]{r}+e-a}{2(b-a)}\right) dr$$
$$+\int_{(b-e)^4}^{\left(\frac{c-a}{4}\right)^4}\left(\frac{-\sqrt[4]{r}+e-c}{2(c-b)}\right) dr+\int_{\left(\frac{c-a}{4}\right)^4}^{(e-a)^4}\left(\frac{-\sqrt[4]{r}+e-a}{2(b-a)}\right) dr$$

Then we have:

$$K[\xi]=\frac{253\left(\sum_{i=1}^n (b_i-a_i)\right)^5+395\left(\sum_{i=1}^n (c_i-b_i)\right)\left(\sum_{i=1}^n (b_i-a_i)\right)^4}{10240\left(\sum_{i=1}^n (b_i-a_i)\right)}+\frac{17\left(\sum_{i=1}^n (c_i-b_i)\right)^4\left(\sum_{i=1}^n (b_i-a_i)\right)+290\left(\sum_{i=1}^n (c_i-b_i)\right)^2\left(\sum_{i=1}^n (b_i-a_i)\right)^3}{10240\left(\sum_{i=1}^n (b_i-a_i)\right)} \quad (15)$$
$$+\frac{70\left(\sum_{i=1}^n (c_i-b_i)\right)^3\left(\sum_{i=1}^n (b_i-a_i)\right)^2-\left(\sum_{i=1}^n (c_i-b_i)\right)^5}{10240\left(\sum_{i=1}^n (b_i-a_i)\right)}$$

2) Assume that $\beta>\alpha$, then $e<b$. Similar to state (1), it can be proved that the kurtosis value is equal with:

$$K[\xi_i]=\frac{253\left(\sum_{i=1}^n (c_i-b_i)\right)^5+395\left(\sum_{i=1}^n (c_i-b_i)\right)^4\left(\sum_{i=1}^n (b_i-a_i)\right)}{10240\left(\sum_{i=1}^n (c_i-b_i)\right)}+\frac{17\left(\sum_{i=1}^n (c_i-b_i)\right)\left(\sum_{i=1}^n (b_i-a_i)\right)^4+290\left(\sum_{i=1}^n (c_i-b_i)\right)^3\left(\sum_{i=1}^n (b_i-a_i)\right)^2}{10240\left(\sum_{i=1}^n (c_i-b_i)\right)} \quad (16)$$
$$+\frac{70\left(\sum_{i=1}^n (c_i-b_i)\right)^2\left(\sum_{i=1}^n (b_i-a_i)\right)^3-\left(\sum_{i=1}^n (b_i-a_i)\right)^5}{10240\left(\sum_{i=1}^n (c_i-b_i)\right)}$$

3) Now suppose that the security returns are symmetric, i.e. $\alpha=\beta$. Then we have:

$$Cr\left\{(\xi-e)^4 \geq r\right\}=\begin{cases}\dfrac{(b-a)-\sqrt[4]{r}}{2(b-a)} & 0 \leq r \leq \dfrac{(b-a)^4}{16} \\ 0 & r \geq \dfrac{(b-a)^4}{16}\end{cases}$$

With simple mathematical operations, the kurtosis value is obtained as:

$$K[\xi_i] = \sum_{i=1}^{n} \frac{(b_i - a_i)^4}{64} \tag{17}$$

Remark: Herein, state (2) is applied in portfolio selection since the investors choose the risky assets in such a way that the right width of the fuzzy number is greater than the left width, that is $\beta > \alpha$.

3. Firefly Algorithm (FA)

Heuristic algorithms are one of the most powerful algorithms for optimization problems. Firefly Algorithm (FA) was developed by Yang [23], based on the idealization of the flashing characteristics of fireflies. According to Yang [23], there are three major components in the FA optimization; 1) All fireflies are unisex which means that they are attracted to other fireflies regardless of their sex; 2) The degree of the attractiveness of a firefly is proportion to its brightness, thus for any two flashing fireflies, the less brighter one will move towards the brighter one and the more brightness means the less distance between two fireflies. If there is no brighter one than a particular firefly, it will move randomly; 3) The brightness of a firefly is determined by the value of the objective function.

In the firefly algorithm, there are three important issues:

Attractiveness: In the firefly algorithm, the main form of attractiveness function *β(r)* can be any monotonically decreasing functions such as the following generalized form:

$$\beta(r) = \beta_0 e^{-\gamma r^m}, \quad m \geq 1 \tag{18}$$

where r is the distance between two fireflies, β_0 is the attractiveness at $r = 0$ and γ is a fixed light absorption coefficient. Furthermore every member of the swarm is characterized by its light intensity I_i which can be directly expressed as a inverse of a cost function $f(x_i)$.

Distance: The distance between any two fireflies i and j at x_i and x_j , respectively, is the Cartesian distance

$$r_{ij} = \|x_i - x_j\| = \sqrt{\sum_{k=1}^{d} (x_{i,k} - x_{j,k})^2} \tag{19}$$

where $x_{i,k}$ is the kth component of the spatial coordinate x_i of ith firefly.

Movement: The movement of a firefly i is attracted to another more attractive (brighter) firefly j is determined by

$$x_i = x_i + \beta_0 e^{-\gamma r_{ij}^2} (x_j - x_i) + \alpha \left(rand - \frac{1}{2} \right) \tag{20}$$

where the second term is due to the attraction while the third term is randomization with α being the randomization parameter and "rand" is a random number generator uniformly distributed in [0, 1].

Based on these three rules, the basic steps of the firefly algorithm (FA) can be summarized as the pseudo code shown in Fig. 1.

Objective function $f(x)$, $x=(x_1,\ldots,x_d)$
Generate initial population of fireflies x_i (I,2,…,n)
Light intensity I_i at x_i is determined by $f(x_i)$

Define light absorption coefficient γ

while *(t <MaxGeneration)*

for *i = 1 : n* all *n* fireflies

for *j = 1 : i* all *n* fireflies

if ($I_j > I_i$), Move firefly *i* towards *j* in d-dimension; **end if**

Attractiveness varies with distance *r* via exp[−r]

Evaluate new solutions and update light intensity

end for *j*

end for *i*

Rank the fireflies and find the current best

end while

Postprocess results and visualization

Fig 1: Pseudo code of the firefly algorithm (FA).

3.1. Constraints handling

During the algorithm running**,** the position of firefly may be out of feasible solution. Therefore, we employ an approach for constraints handling. First we try to convert constraint problem to unconstraint problem. There are many specific methods for constraints handling has been proposed with Evolutionary Algorithms, of which the most popular are the penalty function methods for their simplicity and their easy application. Here, a penalty method is considered for constrained optimization with the proposed algorithm.

Let the following constrained problem:

$$\begin{aligned} &\min\ f(x) \\ &\textit{subject to}: \\ &\quad g_i(x) \le 0 \quad i = 1,2,\dots,N \\ &\quad l_i \le x_i \le u_i \quad i = 1,2,\dots,n \end{aligned} \tag{21}$$

As a consequence of penalty function can be following as [23]:

$$F(x,\mu_i,\nu_j) = f(x) + \sum_{j=1}^{N} \nu_j g_j^{\,2}(x) \tag{22}$$

Where, $\nu_j \ge 0$ which should be large enough, depending on the solution quality needed. In general, for most applications, ν_j can be taken as 10^{10} to 10^{15}. Note that an equality constraint can be converted to two inequality constraint.

4. Numerical experiments

In this section, some numerical examples are provided to evaluate performance of the proposed FA. The algorithm is coded using Matlab software and tested on a Pentium Dual 2.6 GHz with 256 GB memory. The proposed algorithm is compared with model results in LINGO which can generate global optima for small problems, but because of LINGO's limitation, this comparison couldn't be performed in large scale data. Herein, the mean-variance-skewness-kurtosis model is applied to the data adopted from Huang [8] which is given in Table 1. The parameters of the proposed FA used to find the optimal solution for the portfolio selection problem are also given in Table 2. In addition to, we assume that in all examples upper bounds on assets are $(0.5,0.3,0.25,0.45,0.5,0.25,0.35,0.25,0.15,0.45)$.

Table 1: Fuzzy returns of 10 securities

Security i	Return	Security i	Return
1	(-0.4,2.7,3.4)	6	(-0.1,2.5,3.6)
2	(-0.1,1.9,2.6)	7	(-0.3,2.4,3.5)
3	(-0.2,3.0,4.0)	8	(-0.1,3.3,4.5)
4	(-0.5,2.0,2.9)	9	(-0.7,1.1,2.7)
5	(-0.6,2.2,3.3)	10	(-0.2,2.1,3.8)

Table 2: Parameters of firefly algorithm approach

Parameters	Value
The number of fireflies	20
Iterations	1500
Alpha	0.5
Beta	0.3
Gama	1

Example 1: Assume that an investor wants to select his portfolio from 10 securities given in Table 1. Let the maximum kurtosis that the investor can accept is 2, the bearable maximum risk is 1.2 and the skewness of his portfolio is at least -1. Based on the model presented in Eq. (11), we have the following model:

$$\begin{cases} \max & E\left[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}\right] \\ \textit{subject to}: & V\left[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}\right] \le 1.2 \\ & S\left[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}\right] \ge -1 \\ & K\left[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}\right] \le 2 \\ & x_1 + \ldots.. + x_{10} = 1 \\ & u \ge x_i \ge 0, \quad i = 1, 2, .., 10 \end{cases}$$

According to the results of the firefly algorithm, the investor should allocate his money as shown in Table 3. The corresponding maximum expected value in this example is 2.33.

Table 3: Asset allocation of the mean-variance-skewness-kurtosis model in Example 1

Security i	1	2	3	4	5	6	7	8	9	10	Expected value	Time(min)
LINGO	0	0	0.25	0	0.5	0	0.25	0	0	0	2.35	18
FA	0.0002	0.0001	0.21	0	0.5	0.01	0.24	0.0297	0.01	0	2.33	1

Example 2: Assume that an investor wishes to choose his portfolio from 10 securities given in Table 1. Let the minimum expected return that the investor can accept is 1.8, the bearable maximum kurtosis is 2, and the skewness of his portfolio is at least -1. Based on the model presented in Eq. (12), we have the following model:

$$\begin{cases} \min & V[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \\ subject\ to: & S[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \geq -1 \\ & E[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \geq 1.8 \\ & K[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \leq 2 \\ & x_1 + \ldots.. + x_{10} = 1 \\ & u \geq x_i \geq 0, \quad i = 1, 2, .., 10 \end{cases}$$

Results of the firefly algorithm show that the investor should allocate his money as given in Table 4.

Table 4: Asset allocation of the mean-variance-skewness-kurtosis model in Example 2

Security *i*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Variance value**	**Time(min)**
LINGO	0.5	0	0	0.2	0	0	0.15	0	0.15	0	0.1620	24
FA	0.49	0	0	0.19	0	0.03	0.14	0	0.14	0.01	0.1629	1

The corresponding minimum variance in this example is 0.1629.

Example 3: Suppose that an investor wants to select his portfolio from 10 securities given in Table 1. Let the minimum expected return that the investor can accept is 1.5, the bearable maximum risk is 1.2 and the skewness of his portfolio is at least -1. Based on the model presented in Eq. (13), we have the following model:

$$\begin{cases} \min & K[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \\ subject\ to: & S[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \geq -1 \\ & E[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \geq 1.5 \\ & V[\xi_1 x_1 + \ldots.. + \xi_{10} x_{10}] \leq 1.2 \\ & x_1 + \ldots.. + x_{10} = 1 \\ & u \geq x_i \geq 0, \quad i = 1, 2, .., 10 \end{cases}$$

Applying the proposed algorithm shows that the investor should allocate his money according to Table 5.

Table 5: Asset allocation of the mean-variance-skewness-kurtosis model in Example 3

Security *i*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Kurtosis value**	**Time(min)**
LINGO	0.5	0	0	0.45	0	0	0	0	0	0.05	0.098	27
FA	0.5	0	0	0.44	0	0.004	0.0004	0.01	0	0.0456	0.099	1

In this example, the corresponding minimum kurtosis is 0.099.

4.1. Discussion

To investigate trade-off between the four moments, it is supposed that two of them are constant in any case. Then, we observe how the other moments change. First, assume that the variance and skewness are constant. In this case, the investors' higher preference for expected return in portfolio leads to a larger kurtosis. As shown in Example 3, when the minimum expected return that the investor can accept is 1.5, the value of kurtosis is 0.099. If the investor increases the level of expected return to 1.7, 1.8, 1.9 and 2, then the value of kurtosis increases to 0.112, 0.133, 0.162 and 0.190 respectively.

Now consider the value of skewness and kurtosis to be constant. If the maximum bearable risk for the investors is 1.5, the expected return is 2.33 as shown in Example 1. If the investor sets the maximal risk at 1.8, then the expected return will be 2.22. This implies that if the investor wants to get higher expected return, he/she should bear a higher risk. Also, in order to investigate the influence of kurtosis considers the example 1. Suppose that variance and skewness are constant. Now we change the different levels of kurtosis that investors can bear. The results show that if the levels of kurtosis decrease of 2 to 0.8, the expected value will not change. However, decrease the level of kurtosis of 0.8 to 0 reduces the expected value. Such results can be achieved considering other cases.

5. Summary and Conclusions

As an extension of the fuzzy portfolio selection model, a fuzzy mean-variance-skewness-kurtosis model was presented in this research. Since the variables are triangular fuzzy variables, the models were converted to deterministic programming. In order to solve these models a firefly algorithm (FA) was developed to find the final optimal portfolio solutions. The proposed algorithm was tested against LINGO software in some numerical examples. The results clearly showed that that the proposed algorithm is robust and effective. In addition to, the computational testing indicated that the kurtosis has a direct impact on portfolio performance.

There still remain some further directions for future research. Other algorithms such as genetic algorithm (GA), simulated annealing (SA), particle swarm optimization (PSO), and tabu search (TS) can be employed instead of firefly algorithm (FA) and compared with the results obtained in current paper. In addition to, other realistic constraints, e.g. minimum transaction lots, sector capitalization, and cardinality constraint with various measures of risk such as semi-variance, value at-risk, entropy, and so on can be added to develop more complex portfolio optimization models.

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