Еліптичним кривим притаманний певний недолік, пов'язаний з тим, що в точках перетину з осями координат еліпси мають дотичні перпендикулярні до цих осей. Проте в деяких практичних застосуваннях еліпсів подібна ситуація є небажаною. Запобігти цьому можна моделюванням вказаних кривих у косокутних координатах, які, в свою чергу, віднесені до деякої вихідної ортогональної координатної системи. Під суперелiпсами Ламе розуміються криві, в рівняннях яких застосовуються показники степенів, відмінні від двох, що є притаманним для звичайних еліпсів. Варіюванням цими показниками степенів можна отримати широке коло різноманітних кривих. У цій роботі запропоновано метод геометричного моделювання суперелiпсів у косокутних координатних системах. Вихідними даними для моделювання є координати двох точок з відомими в них кутами нахилу дотичних. За вісі косокутної системи координат приймаються прямі, проведені наступним чином. Через першу точку будується пряма паралельно дотичній в другій точці, а в другій точці – пряма паралельно дотичній в першій точці. Показано, що завдяки цим заходам можна забезпечити потрібні значення кутів нахилу дотичних в точках перетину суперелiпса з осьовими лініями. Доведено, що дугу суперелiпса можна проводити через третю задану точку з потрібним в ній кутом нахилу дотичної, але це потребує визначення числовим методом показників степенів у рівнянні суперелiпса. Подібна ситуація має місце, наприклад, при розробці проектів профілів лопаток осьових турбін. На підставі запропонованого методу моделювання дуг суперелiпсів розроблено комп'ютерний код, який можна застосовувати при описі контурів виробів технологічно складних галузей промисловості

Ключові слова: суперелiпс Ламе, геометричне моделювання, косокутна система координат, кут нахилу дотичної, розподіл кривини

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CONSTRUCTING A METHOD FOR THE GEOMETRICAL MODELING OF THE LAME SUPERELLIPSES IN THE OBLIQUE COORDINATE SYSTEMS

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1. Introduction

Elliptic curves are the generally known curves of the second order, which are characterized by the axial symmetry relative to the *Ox* and *Oy* axes, as well as the central symmetry relative to the coordinate origin [1, 2]. The techniques for constructing these curves using graphical methods are considered in descriptive geometry [3], engineering [4], and computer graphics [5]. The examples of the practical application of the elliptic curves in shipbuilding are given in reference book [6], in the theory of mechanisms and machines – in [7], in the construction of profiles of the axial turbine blades – in [8]. Given their reflecting capability, the ellipses are widely used in architecture and building, particularly when erecting domes of palaces and cathedrals, as well as amphitheaters (for example, the «Hall of Secrets» of the Alhambra in Granada and St. Peter's Cathedral in London). It is known from astronomy that the planets in the solar system revolve around the Sun in orbits in the form of ellipses. In our time, thousands of artificial satellites move around the Earth through elliptic orbits.

The elliptical curves are understood to be the closed flat lines, which can be obtained as the cross-section of a cylinder or a rotating cone by the plane inclined to their axis at a certain angle. It can also be the mapping of a circle onto a plane not parallel to the plane of the circle location. A circle is a separate case of an ellipse. With the affine transformation of a circle, you can get an elliptical curve or just an ellipse.

The elliptical curves possess some specific benefits due, for example, to the monotony of a curvature change, an angle of inclination of the tangent, derivatives, etc. However, there is a certain shortcoming in the ellipses that is related to the angles of inclination of the tangents at the intersection points of the curve with the coordinate axes. These angles accept either zero values or are equal to 90°. However, for ellipses that are built in Cartesian coordinates, there is no way to ensure arbitrary values of the angles of inclination of the tangent at intersections with the coordinate axes.

One possible way to ensure the arbitrary values of the angles of inclination of the tangents at the intersection points of the curve with the coordinate axes is to model them in oblique coordinate systems, as well as to construct new elliptic curves by transforming the already known curve.

These issues are of theoretical and practical significance. They are relevant for those industries where the articles of complex geometric shapes are made (for example, in shipbuilding, when describing waterlines, frames, battoks lines; in the gas turbine engineering, when modeling air intakes, profiles of turbine blades). Thus, in the gas turbine industry, it is important to ensure the estimated angles of a flow inlet to and out of the blade apparatus. At the same time, it is necessary to meet the conditions for a smooth transition between the angles of inclination of the tangents from the starting point to the endpoint of the modeled curve.

It should be noted that recent years have seen significant qualitative changes in designing complex highly technological products in various industries. There is a widespread shift from traditional graphic information processing to paperless technologies based on the digital descriptions of projected and manufactured objects. Computerized technologies make it possible to create numerical models of different objects. A designer can view the physically non-existent object on the computer, get the desired geometric characteristics, make certain changes, prepare the production and, finally, produce one or another product in modern machining centers.

Geometric information about products must determine them in full, meet the requirements arising from the functional, structural, strength, ergonomic, aesthetic, operating, technological, and other conditions. The most important component of the information used in the manufacture of products in the technologically sophisticated, knowledge-intensive industries is the geometric model of an object, which contains a description of its shape, as well as the description of the connecting elements in the model.

2. Literature review and problem statement

In the Cartesian coordinate system, the ellipse is described by the following equation [9, 10]:

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1, \tag{1}$$

where a and b are the ellipse semi-axes whose equality transforms the ellipse into a circle.

The concept of the superellipse was first introduced by the French mathematician Gabriel Lamé in 1818, who generalized the equation of the ellipse and, instead of the exponent equal to two in expression (1), applied an arbitrary indicator n, having recorded this equation in the following form:

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1, \tag{2}$$

where the exponent n can be any rational number, a and b are the positive numbers, which are termed the semi-axes or the half-diameters of a curve.

Equation (2) defines a closed curve, limited by a rectangle with sides $-a\leq x\leq a$ and $-b\leq y\leq b$. It is the generalized equation of the ellipse, which, depending on the exponent value and the magnitudes of the semi-axes, makes it possible to obtain a circle, an ellipse, a square, and a rectangle. At $n=1$, the curve degenerates into a straight line; at $n=2$, we obtain a regular ellipse, at $n=2/3$ – the astroid (provided $a=b$).

Some of the features of the Lamé superellipse can be found in [10], in particular, the expressions for calculating the arc length and the curve square. In the cited work, the superellipses are examined in the Cartesian coordinates not in the form of arcs, located in the region of positive coordinates, but in the so-called full form.

Now it is difficult to determine who for the first time offered to apply equation (2) with different exponent values, that is, to write it in a more general form:

$$\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^n=1. \tag{3}$$

Applying different exponent values provides even more possibilities to constructing a set of various curves.

Summing up, one can note that all the above equations are recorded in almost identical mathematical notation and differ only in the values of the exponents.

The most striking examples of the practical application of the Lamé superellipse include the Aztec Stadium, built before the Olympic Games in Mexico City, and a square in Stockholm.

Paper [11] gives examples of constructing superellipses with equal exponent values. Those examples demonstrate a change in the geometry of curves as the exponents gradually decrease. It is indicated that one of the squares in Stockholm has the shape of a superellipse.

The examples of using superellipses in computer graphics are shown in work [12]. The specified curves are considered with equal exponents in the Cartesian coordinates.

Study [13] demonstrates that nature has many examples of plants whose transverse cross-section coincides in the shape with the Lamé superellipses, which are termed the Lamé superellipses. The cited study considers superellipses with equal exponent values. Further results of the previous studies are given in [14], which considers the superellipse equations in the polar coordinate system and at different exponent values. That advanced the circle of a variety of curves whose examples found in the plant world.

Work [15] built on the search related to the description of the features in the plant world. It also uses the Lamé superellipse with equal exponent values; that ellipse is formed in the Cartesian coordinates.

Study [16] applies the Lamé superellipses to describe the symmetrical forms of bamboo leaves, which is crucial to describe the morphogenesis and development of plants. Work [17] considers the issue of modeling annular tree cuts, which could make it possible to better estimate the productivity of forests and carbon accumulation in the terrestrial forest ecosystems.

The asymptomatic behavior of the maximum curvature of the Lamé superellipses was investigated in the paper [18]. This is the only known source, which argues on the curvature of the Lamé superellipses, although it also applies curves with equal exponent values. In this case, an optimum value of exponent was derived, which provides for the «most exquisite» shape. The main result of the cited paper is the expression for finding the asymptote of a point with maximum curvature.

It is proposed in work [19] to use the Lamé superellipses with equal exponent values to reproduce and categorize the mine-like forms in the sonarbic images. The superellipses were used to level the irregularities occurring in actual mine-like forms.

Study [20] suggests employing the Lamé superellipses to describe antennas aimed to receive electromagnetic vibrations. More or less the same topic is addressed in work [21], which reported the development of an omnidirectional ultra-wideband antenna in the superellipse shape.

Many contact rolling or sliding mechanisms, such as roller bearings, gearbox bearings, gears, execute a contact between two semi-infinite bodies, with the concentrated stress occurring at the edges of the contact. Paper [22] described a new type of profile based on the superellipse equation (the ellipse is generalized to an order of n). Applying this profile makes it quite simple to set the parameters according to the alleged scope of application. The superellipse is easily adjusted to all types of contact by changing the order of a superellipse profile. The advantages of the superellipse profile are the uniform distribution of pressure and the absence of an edge effect while it remains easy to make.

It follows from our analysis that the most commonly considered issues are those related to the construction of the Lamé superellipses as closed curves, in the Cartesian coordinate systems. The curves are built on the condition that the exponents in the superellipse equations are the magnitudes that are set with the source data. This makes it possible to obtain a variety of lines, from a straight line to a rectangle, with clearly defined right angles. The tasks of building the superellipses with the assigned angles of inclination of the tangents at points of intersection with the axes of coordinates have not been considered. When designing articles of a complex geometric shape (cam mechanisms, turbine blades, and the like), the designers face the need to draw curves through the two or three set points at the known angles of inclination of the tangents. Given the appealing properties of elliptical curves, it is necessary to devise tools that would satisfy the practitioners in the development of the geometric models of projected articles.

3. The aim and objectives of the study

The aim of this study is to construct a method to model the Lamé superellipses, which would ensure that they pass through two or three points at the assigned angles of inclination of the tangents, applicable for the articles of complex geometric shape.

To accomplish the aim, the methods to build the following must be constructed:

– the arc of an ellipse in the oblique systems of coordinates;

– the arc of a Lamé superellipse in the oblique systems of coordinates;

– the arc of a Lamé superellipse through three set points with the known angles of inclination of the tangents at them.

4. A method for constructing the arc of an ellipse in the oblique coordinate system

Consider an ellipse arc, described by equation (1), located in the region of the coordinates' positive values. We shall construct this arc in an oblique coordinate system with the angles of inclination of the axial lines of this system α and β to the axis of the abscissa in a certain rectangular coordinate system.

Take two arbitrary points T_1 and T_2 in the Cartesian coordinate system, draw the straight lines through these points, inclined to the abscissa axis at angles α and β. Assume that these straight lines are the coordinate axes in the oblique coordinate system.

Denote the axes in the oblique coordinate system and its center via $\bar{x}$, $\bar{y}$ and $\bar{O}$. In Fig. 1, in the xOy coordinate system, two points T_1 and T_2 are assigned, for which the angles of inclination of the tangents α and β are known. It is required to draw the arc of the curve so that it passes through the specified points and accepts, at the endpoints, the assigned angles of inclination of the tangents.

Based on the known initial data, we build a parallelogram. The side $\bar{O}T_1$ of this parallelogram is taken as the $\bar{x}$, axis; the side $\bar{O}T_2$ – the $\bar{y}$ axis. Thus, an oblique coordinate system $\bar{x}\bar{O}\bar{y}$. is formed. In this coordinate system, we model the arc of the ellipse so that it passes through points T_1 and T_2 and is the tangent to the sides TT_1 and TT_2 of the built parallelogram.

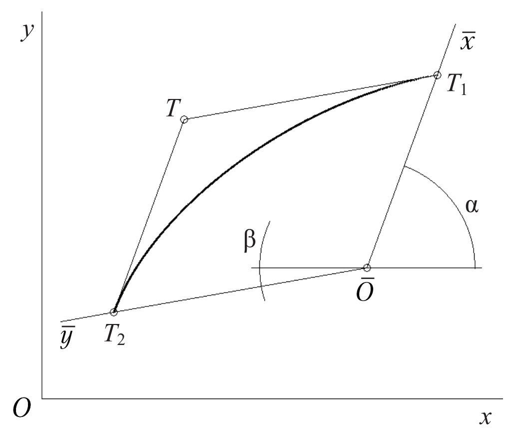

Fig. 1. The arc of an ellipse in the oblique coordinates

Since the arc of the ellipse is to be built in an oblique coordinate system, the equation (1) is re-recorded in the form, which reflects the fact of its application in this coordinate system:

$$\left(\frac{\bar{x}}{a}\right)^2+\left(\frac{\bar{y}}{b}\right)^2=1. \quad (4)$$

The equations of the axial lines are constructed based on the known coordinates of the endpoints of the modeled curve and the angles of inclination of the tangents at them:

$$y-y_{T_1}=\left(x-x_{T_1}\right)k_2;\quad y-y_{T_2}=\left(x-x_{T_2}\right)k_1, \quad (5)$$

where

$$k_1=\text{tg}\alpha;\quad k_2=-\text{tg}\beta.$$

By solving the derived system of equations, we find the coordinates of point $\bar{O}$, which is the origin of the oblique coordinate system:

$$x_{\bar{O}}=\frac{y_{T_2}-y_{T_1}+k_2x_{T_1}-k_1x_{T_2}}{k_2-k_1}; \quad (6)$$

$$y_{\bar{O}}=y_{T_1}+k_2\left(x_{\bar{O}}-x_{T_1}\right). \quad (7)$$

In the oblique coordinate system $\bar{x}\bar{O}\bar{y}$, the segments $\bar{O}T_1$ and $\bar{O}T_2$ define the values for the major a and the minor b semi-axes of the ellipse:

$$a=\sqrt{\left(x_{T_1}-x_{\bar{O}}\right)^2+\left(y_{T_1}-y_{\bar{O}}\right)^2};\; b=\sqrt{\left(x_{T_2}-x_{\bar{O}}\right)^2+\left(y_{T_2}-y_{\bar{O}}\right)^2}.$$

It is possible to show that the relation between the Cartesian coordinates x, y of a certain point in the ellipse with coordinates $\bar{x}$, $\bar{y}$ and the oblique coordinate system takes the following form:

$$x=x_{\bar{O}}+\bar{x}\cos\alpha-\bar{y}\cos\beta; \quad (8)$$

$$C=C_{\bar{O}}+\bar{x}\sin\alpha-\bar{y}\sin\beta. \quad (9)$$

Since, from equation (4),

$$\bar{y}=b\sqrt{1-\left(\frac{\bar{x}}{a}\right)^2},$$

then, by substituting $\bar{y}$ in expressions (8) and (9), we obtain:

$$x=x_{\bar{O}}+\bar{x}\cos\alpha-b\sqrt{1-\left(\frac{\bar{x}}{a}\right)^2}\cos\beta; \quad (10)$$

$$y=y_{\bar{O}}+\bar{x}\sin\alpha-b\sqrt{1-\left(\frac{\bar{x}}{a}\right)^2}\sin\beta. \quad (11)$$

Expressions (10) and (11) make it possible to determine the x and y coordinates at any point in the curve and thereby describe the section of an ellipse in the rectangular xOy coordinate system.

5. A method for building the arc of a Lamé superellipse in the oblique coordinate system

The superellipses are built in the rectangular coordinate systems based on equations (2) and (3). In this case, the tangents, drawn at the point of intersection of the curves with the coordinate axes, form the right angles with these axes. However, for some practical applications, it is required that the specified angles should differ from the right angles. This result can be achieved only when constructing the Lamé superellipses in the oblique coordinate systems whose axes are oriented to each other at an angle required for building the desired curve.

Based on the source data given above, a parallelogram can be built. The side $\bar{O}T_1$ of this parallelogram is taken to be the $\bar{x}$, axis, the side $\bar{O}T_2$ – the $\bar{y}$ axis. Thus, the oblique coordinate system $\bar{x}\bar{O}\bar{y}$ will be formed. The coordinates of point $\bar{O}$ – the origin of the oblique coordinate system – are determined from expressions (6), (7). In this coordinate system, we model the arc of a Lamé superellipse so that it passes through points T_1 and T_2 and is the tangent to the sides TT_1 and TT_2 of the parallelogram.

The superellipse equation is taken in the following form:

$$\left(\frac{\bar{x}}{a}\right)^m+\left(\frac{\bar{y}}{b}\right)^n=1. \quad (12)$$

Since the coordinates of the modeled curve are computed in the oblique coordinates, the relation between the oblique and Cartesian coordinates of the points in the curve are determined from expressions (10)–(11).

The construction of the arc of a Lamé superellipse implies a sequential change in the $\bar{x}$ coordinate from zero to a magnitude equal to the length of the semi-axis a. For the current value of the coordinate $\bar{x}$, one calculates the oblique coordinate $\bar{y}$, which is determined from the following expression:

$$\bar{y}=b\sqrt[n]{1-\left(\frac{\bar{x}}{a}\right)^m}. \quad (13)$$

By substituting $\bar{y}$ in expressions (10) and (11), we ultimately obtain:

$$x=x_{\bar{O}}+\bar{x}\cos\alpha-b\sqrt[n]{1-\left(\frac{\bar{x}}{a}\right)^m}\cos\beta; \quad (14)$$

$$y=y_{\bar{O}}+\bar{x}\sin\alpha-b\sqrt[n]{1-\left(\frac{\bar{x}}{a}\right)^m}\sin\beta. \quad (15)$$

Expressions (14) and (15) define the x and y coordinates at any point in the curve and thus describe the arc of a Lamé superellipse in the rectangular coordinate system.

Fig. 2 shows an example of the three arcs of the Lamé superellipses, constructed at the exponent values $m=2$ and $n=2$ (curve 1), $m=\pi$ and $n=e$ (curve 2, where e is the base of the natural logarithms), $m=5$ and $n=4$ (curve 3). Curve 1 is the arc of a conventional ellipse, built in the oblique coordinates; in fact, it is the repeated curve shown in Fig. 1.

It follows from Fig. 2 that all three curves at their endpoints are the tangents to the corresponding sides of the parallelogram $\bar{O}T_1TT_2$. One can also note that the exponent values m an n significantly affect the shape of the curves. With an increase in these exponents, the curves are increasingly approaching the sides TT_1 and TT_2 of the parallelogram $\bar{O}T_1TT_2$. Consequently, changing the exponent values m and n can produce a wide range of different curves.

The graphical information shown in Fig. 3 demonstrates the influence of angles α and β on the arcs of the Lamé superellipses. When modeling curves, the angle α gradually decreased by 10°, and the angle β – by 5°. Under these circumstances, the centers of the oblique coordinates are shifted up to the left, and the point T moves down to the right. The points T_1 and T_2 did not change their positions. The points $\bar{O}$ and T are indicated near the vertices of the original parallelogram. The arcs of the Lamé superellipses were built at the exponent values $m=\pi$ and $n=e$.

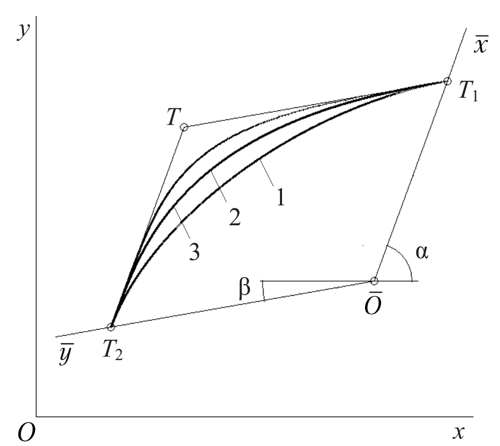

Fig. 2. The superellipses arcs in the oblique coordinates

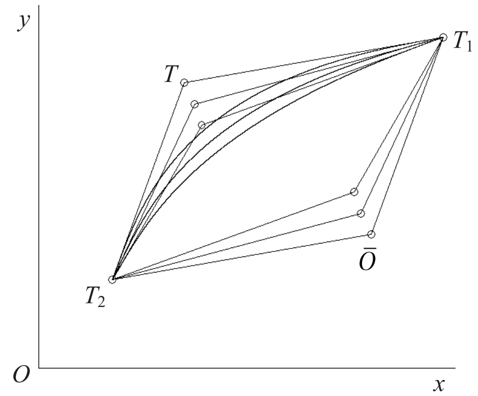

Fig. 3. The influence of angles α and β on the arcs of the Lamé superellipses

Fig. 4 shows the graphic information demonstrating the influence of the coordinates of the original points T_1 and T_2 on the arcs of the Lamé superellipses. The point T_1 gradually moved to the right and bottom, and the point T_2 – to the left and down. The angles α and β remained unchanged, equal to their original values. The vertices of the parallelogram are indicated in its representation corresponding to the initial data.

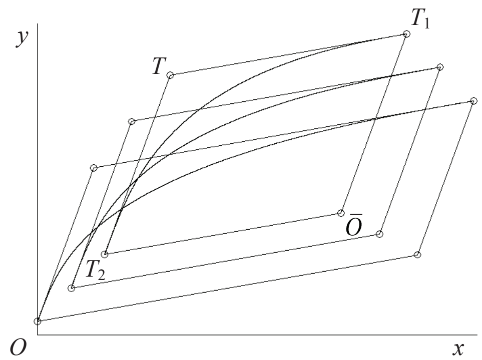

Fig. 4. The influence of the points T_1 and T_2 position on the arcs of the Lamé superellipses

The joint influence of the α and β angles and the coordinates of the original points T_1 and T_2 on the arcs of the Lamé superellipses is shown in Fig. 5. The initial data were the angles α and β, which were applied in the construction of the curves shown in Fig. 3; the coordinates of the T_1 and T_2 points, based on which the curves were built, are shown in Fig. 4.

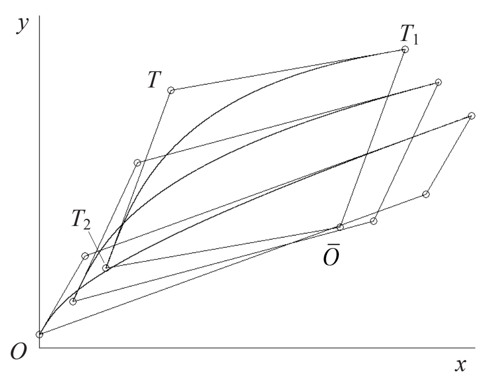

Fig. 5. The influence of angles α and β and the points T_1 and T_2 position on the arcs of the Lamé superellipses

Thus, the graphical information shown in Fig. 3–5 clearly confirms the possibility of modeling the arcs of the Lamé superellipse in a wide range of variation of initial data.

6. A method for building the arc of a Lamé superellipse through three set points at the known angles of inclination of the tangents at them

In the above-examined examples, the arcs of the Lamé superellipses were built under the condition that the exponents *m* and *n* in the Lamé superellipse equation were known values. At the same time, there was no task to draw a curve through a set point at the assigned angle of inclination of the tangent at it. Such a task is very common in a variety of practical applications of the Lamé superellipse arcs. The set task could be solved by developing a specific algorithm to find such *m* and *m* exponents that would ensure that the curve passes through the set point and the angle of inclination of the tangent.

First of all, we shall determine the dependences, which relate the oblique coordinates $\bar{x}$, $\bar{y}$ of an arbitrary point to its Cartesian coordinates. These dependences can be derived by solving the system of equations (8), (9) relative to $\bar{x}$ and $\bar{y}$. Following the transforms, we obtain:

$$\bar{x}=\frac{\left(x-x_{\bar{O}}\right)\sin\beta+\left(y-y_{\bar{O}}\right)\cos\beta}{\sin\left(\alpha+\beta\right)};$$

$$\overline{C}=\frac{\left(y-y_{\bar{O}}\right)\cos\alpha+\left(x-x_{\bar{O}}\right)\sin\alpha}{\sin\left(\alpha+\beta\right)}.$$

The $\bar{x}$ and $\bar{y}$ coordinates can be represented in the following way:

$$\bar{x}=a_1x+a_2y+a_3; \tag{16}$$

$$\bar{y}=a_4x+a_5y+a_6, \tag{17}$$

where

$$a_1=\frac{\sin\beta}{\sin\left(\alpha+\beta\right)};\quad a_2=\frac{\cos\beta}{\sin\left(\alpha+\beta\right)};$$

$$a_3=-\frac{x_{\bar{O}}\sin\beta+y_{\bar{O}}\cos\beta}{\sin\left(\alpha+\beta\right)};\quad a_4=\frac{\sin\alpha}{\sin\left(\alpha+\beta\right)};$$

$$a_5=\frac{\cos\alpha_c}{\sin\left(\alpha+\beta\right)};\quad a_6=\frac{x_{\bar{O}}\sin\alpha-y_{\bar{O}}\cos\alpha}{\sin\left(\alpha+\beta\right)}.$$

Substituting expressions (16), (17) in the ellipse equation (14) makes it possible to establish a reciprocal relationship between the Cartesian coordinates *x* and *y*:

$$\left(\frac{a_1x+a_2y+a_3}{a}\right)^m+\left(\frac{a_4x+a_5y+a_6}{b}\right)^n=1$$

or

$$b^n\left(a_1x+a_2y+a_3\right)^m+$$
$$+a^m\left(a_4x+a_5y+a_6\right)^n-a^mb^n=0. \tag{18}$$

The m and n exponents for equations (12) or (18) will be determined under condition that the curve passes through the set point A at the angle of inclination of the tangent δ at it.

Since the coordinates x and y are implicitly related via a dependence in the form

$$f(x,y)=0,$$

then its derivative is determined as follows:

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}, \tag{19}$$

where

$$\frac{\partial f}{\partial x}=b^n m(a_1x+a_2y+a_3)^{m-1}a_1+a^m n(a_4x+a_5y+a_6)^{n-1}a_4;$$

$$\frac{\partial f}{\partial y}=b^n m(a_1x+a_2y+a_3)^{m-1}a_2+a^m n(a_4x+a_5y+a_6)^{n-1}a_5.$$

By equating the derivative at point A to the tangent of angle δ, after the transforms, we obtain the following equation:

$$b^n m(a_1\bar{x}+a_2\bar{y}+a_3)^{m-1}(a_1+\mathrm{tg}\delta a_2)+$$
$$+a^m n(a_4\bar{x}+a_5\bar{y}+a_6)^{m-1}(a_4+\mathrm{tg}\delta a_5)=0.$$

The expressions recorded in parentheses are constant values. Apply the notation:

$$A=a_1\bar{x}+a_2\bar{y}+a_3;\quad B=a_1+\mathrm{tg}\delta a_2;$$

$$C=a_4\bar{x}+a_5\bar{y}+a_6;\quad D=a_4+\mathrm{tg}\delta a_5.$$

Hence

$$b^n mA^{m-1}B+a^m nC^{n-1}D=0. \tag{20}$$

In this expression, the unknown values are the m and n exponents. However, one of these exponents can be expressed through another by using equation (13) written for point A:

$$n=\frac{\ln\left[1-\left(\frac{\bar{x}}{a}\right)^m\right]}{\ln\left(\frac{\bar{y}}{b}\right)}. \tag{21}$$

To solve equations (20), (21) in combination, a highly efficient algorithm, proposed in work [23], was applied. This algorithm combines the reliability of bisection with the asymptotic velocity of the secant method.

The graphic data shown in Fig. 6 indicate that the built arcs of the Lamé superellipses clearly pass through the set points. They also accept the assigned angles of inclination of the tangents. At the same time, it should be noted that since a Lamé superellipse possesses a so-called «rigid» character, then choosing the point that the curve must pass through must be approached reasonably. Firstly, the point should be inside the parallelogram formed by the axes and the straight lines parallel to them. Secondly, the angle of inclination of the tangent at the set point should roughly correspond to the character of the curve path. It is clear that there is a certain margin when assigning an angle but it is undesirable to set an arbitrary angle. A program would be terminated once the angle is improperly set.

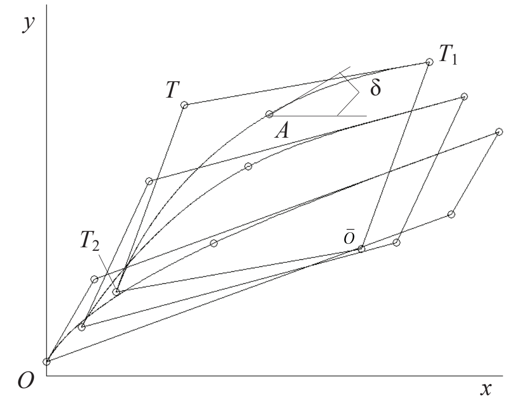

Fig. 6. The construction of arcs of the superellipses passing through the original points at the assigned angles of inclination of the tangents at them

It should be noted that in practical applications the user will be able to predict both the position of an intermediate point and the probable value for an angle of inclination of the tangent.

One of the most important characteristics of a flat curve is the curvature and the related radius of the curvature.

Let us devise tools to define the curvature of the Lamé superellipse arc. Generally, it is known that the curvature k of any curve is determined from the following formula:

$$k=\frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3\right]^{3/2}}. \tag{22}$$

The first derivative $\mathrm{d}y/\mathrm{d}x$ is found from expression (19). The second derivative of the implicit function is determined as follows:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2-2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3},$$

where

$$\frac{\partial^2 f}{\partial x^2}=b^n m(m-1)(a_1x+a_2y+a_3)^{m-2}a_1^2+$$
$$+a^m n(n-1)(a_4x+a_5y+a_6)^{n-2}a_4^2;$$

$$\frac{\partial^2 f}{\partial x\partial y}=b^n m(m-1)(a_1x+a_2y+a_3)^{m-2}a_1a_2+$$
$$+a^m n(n-1)(a_4x+a_5y+a_6)^{n-2}a_4a_5;$$

$$\frac{\partial^2 f}{\partial y^2}=b^n m(m-1)(a_1x+a_2y+a_3)^{m-2}a_2^2+$$
$$+a^m n(n-1)(a_4x+a_5y+a_6)^{n-2}a_5^2.$$

Fig. 7 shows an example of the three curves constructed in the oblique coordinates, two of which are the arcs of the Lamé superellipses (curves 1 and 2), while one is the arc of a regular ellipse (curve 3). The Lamé superellipse arcs were built at the following exponent values: curve 1 – m=5 and n=4, curve 2 – $m=\pi$ and $n=e$.

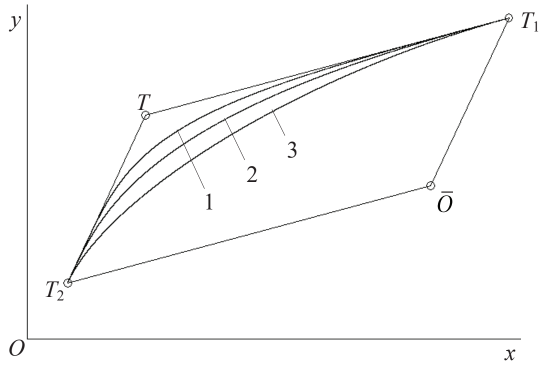

Fig. 7. The arcs of elliptic curves with different exponent values

For these curves, based on expression (22), we determined the curvature whose shape is shown in Fig. 8. The numbering of curves in Fig. 7 and 8 are identical. The curves were built depending on the relative length of the arc of the modeled curve.

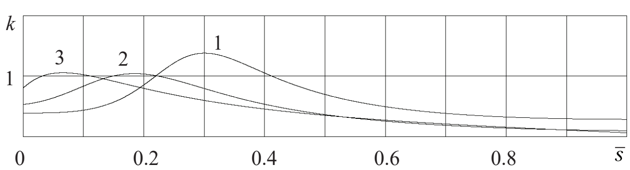

Fig. 8. Charts of curvature distribution

When considering the charts of the curvature distribution, the following conclusions can be drawn:

1) all curves are smooth with a single extremum located over the original sections of the curves;

2) after the extremum, the curves take a monotonous, decreasing character with the curvature value slightly greater than zero (curves 2 and 3) and somewhat larger than zero (curve 1).

7. Results of modeling the Lamé superellipses in the oblique coordinate systems

Based on the proposed tools for the geometrical modeling of the Lamé superellipses, a computer code was developed in the programming environment Fortran PowerStation. Employing this code helps perform the calculations related to determining the coordinates of the points in the modeled line. It uses the subprograms for deriving the first and second derivatives, the curvature of a curve, for transforming the oblique coordinates into Cartesian ones; a subprogram to solve transcendental equations was borrowed from work [23]. The developed code, in addition to numerical results, which are the coordinates of points in the modeled lines, makes it possible to visualize the lines on a computer monitor screen. The graphic data are the visual confirmation of the operability of the proposed method of geometric modeling of the Lamé superellipses in the oblique coordinates at two and three preset coordinates of the points and the angles of inclination of the tangents. When considering the built arcs of the ellipses in the Cartesian coordinates, one can clearly see that these arcs pass through the original points and accept the assigned angles of inclination of the tangents at them. An error in the divergence between the original points' coordinates and those computed does not exceed 10^{-6}, which is sufficient for practical application.

Fig. 9 shows an example of the profile of an axial turbine's nozzle blade with the flow inlet and outlet angles equal to 90° and 15°, respectively. The pressure and suction sides' profiles were modeled using the arcs of the Lamé superellipse. Both arcs were built based on three points and the assigned angles of inclination of the tangents at them. The two points of each arc were in the places that join the leading and trailing edges of the profile. At the at the suction side, the third point was in the throat of an interblade channel, on the trough – at the point that determines the assigned maximum thickness of the profile. The slope of the tangent in the throat of the channel was determined by the bending angle of the profile. The angle of inclination of the tangent at the third point on the trough was equal to the angle that was accepted by the tangent on the suction in the place where it touches the circle of the maximum thickness of the profile.

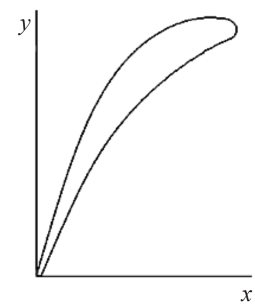

Fig. 9. A turbine blade profile

The proposed method for modeling the arcs of superellipses in the oblique coordinate can be applied for the analytical representation of ship curves, first of all, it concerns waterlines.

8. Discussing the method of modeling the Lamé superellipses in the oblique coordinate systems

This paper has shown that it is necessary, in order to ensure the desired angles of inclination of the tangents at the points of intersecting the ellipses, and, especially, the superellipses, with the axes of the coordinates, to construct these curves in the oblique coordinates. The axes of oblique coordinates are chosen in such a way that one of the axes passes through the first point and is parallel to the tangent at the second point. Accordingly, the second axis must pass through the second point and be parallel to the tangent at the second point. It is clear that the tangents must not be parallel to each other. This follows from the consideration of formulae (6) and (7); in this case, an indefinite result would be obtained, associated with the division by zero.

The positive results obtained in the geometrical modeling of Lamé superellipses in the oblique coordinates are predetermined by the correctness of mathematical calculations, which are based on the provisions from the analytical and differential geometry and numerical methods. Algorithmizing the methods that match the tasks of this research has allowed us to develop operational computer code. All the tasks set for this study have been practically implemented, which is confirmed by the represented graphic results. Thus,

Fig. 1 demonstrates the possibility of constructing a regular ellipse in the oblique coordinates and ensuring the assigned, different from 90°, angles of inclination of the tangents at the endpoints. Fig. 2–5 confirm the possibility of modeling the Lamé superellipses in the oblique coordinates when changing a position of the original points and the angles of inclination of the tangents at them. In our problem, the superellipse exponents are the values that are set by the initial data. This is the consequence of the fact that there are no additional conditions to find the exponents.

A condition for calculating the exponents by solving the equations (20) and (21) numerically is the presence of the third set point at the known angle of inclination of the tangent at it. The solution results are graphically represented in Fig. 6, which demonstrates the impact of both the coordinates of the initial points and the angles of inclination of the tangents at them. The solution to this problem is important for the practical application of the Lamé superellipse arcs.

When constructing the Lamé superellipses at two assigned points and different values for the angles of inclination of the tangents at them, there are no problems although it is necessary to specify the exponent values. Drawing a superellipse arc through three set points at the known angles of inclination of the tangents at them requires determining the exponent values by a numerical method and necessitates a reasonable approach to choosing a position of the «medium» point and the angle of inclination of the tangent at it, which is predetermined by the «rigid» character of a Lamé superellipse. However, this is not critical for experts in the subject area of the elliptic superellipses application.

It is advisable to advance the research into the geometric modeling of Lamé superellipses towards extending the circle of practical tasks for which it is necessary to build the contours of parts of a complex geometric shape. Interesting results could be obtained when constructing the Lamé superellipses in the polar coordinates.

9. Conclusions

1. The construction of the arc of a regular ellipse, which has the same exponent values equal to two, in the oblique coordinates makes it possible to obtain these arcs at the assigned angles of inclination of the tangents at the endpoints. The axes of the oblique coordinate are determined by the position of two points at the known angles of inclination of the tangents at them or by directly setting the inclination angles of the axes relative to a certain original Cartesian coordinate system.

2. The practical calculations have shown that the proposed method for modeling the superellipse arcs in the oblique coordinates makes it possible to build the arcs of these curves in a wide variation range of the angles' initial data. The method implies the presence of the coordinates of two points, the angles of inclination of the tangents at them, as well as the exponents. Based on these data, one determines the position of the axes in the oblique coordinate system in which the required curve is constructed. Changing the exponents in superellipse equations can yield a diverse circle of curves, which has been confirmed by the above graphic results.

3. The devised method for constructing the arc of a superellipse, if there are three set points and the angles of inclination of the tangents at them, has been implemented in the form of computer code. That enabled finding, by using a numerical method, the exponent values for the equation of the modeled Lamé superellipse. It has been determined that the error of the curve passing through an intermediate point does not exceed 10^{-6}. An example of the profile of an axial turbine's nozzle blade has been given to show that the method for modeling the Lamé superellipses can be practically applied.

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IMPROVEMENT OF THE BRANCH AND BOUND ALGORITHM FOR SOLVING THE KNAPSACK LINEAR INTEGER PROBLEM

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У статті представлений новий підхід до переформулювання, що дозволяє зменшити складність алгоритму розгалуження і меж для вирішення лінійної цілочисельної задачі про рюкзак. Алгоритм розгалуження і обмеження в цілому спирається на звичайну стратегію, яка полягає в першому ослабленні цілочисельного завдання в моделі лінійного програмування (ЛП). Якщо оптимальне рішення лінійного програмування є цілочисельним, то є оптимальне рішення цілочисельного завдання. Якщо оптимальне рішення лінійного програмування не є цілочисельним, то обирається змінна з дробовим значенням для створення двох підзадач, так що частина допустимої області відкидається без усунення будь-якого з можливих цілочисельних рішень. Процес повторюється для всіх змінних з дробовими значеннями, поки не буде знайдено цілочисельне рішення. У цьому підході змінна сума і додаткові обмеження генеруються і додаються до вихідної задачі перед її рішенням. Для цього швидко визначається об'єктивна межа задачі про рюкзак. Потім межа використовується для генерації набору меж змінної суми і чотирьох додаткових обмежень. Виходячи за межі змінної суми, вихідні підзадачі будуються і вирішуються. Оптимальне рішення потім виходить як краще рішення з усіх підзадач з точки зору об'єктивного значення. Пропонована процедура призводить до підзадач, які мають меншу складність і легше вирішуються, ніж вихідна задача, з точки зору кількості гілок і пов'язаних ітерацій або підзадач.

Задача про рюкзак – це особлива форма загальної лінійної цілочисельної задачі. Є багато видів задач про рюкзак. Вони включають в себе задачі «нуль-один», «множинного вибору», «обмежену», «необмежену», «квадратичну», «багатоцільову», «багатовимірну», «колапсу нуль-один» та задачу про об'єднання рюкзаків. Задачі про рюкзаки «нуль-один» – ті, в яких змінні приймають тільки 0 і 1. Причина в тому, що предмет може бути обрано або не обрано. Іншими словами, немає можливості отримати дробові суми або предмети. Це найпростіший клас завдань про рюкзаки, і він єдиний, який може бути вирішений в поліномі за допомогою алгоритмів внутрішніх точок і в псевдополіноміальному часі за допомогою методів динамічного програмування. Задачі з множинним вибором рюкзаків – це узагальнення звичайної задачі про рюкзаки, коли набір предметів розбивається на класи. Нульовий варіант вибору предмета замінюється вибором рівно одного предмета з кожного класу предметів

Ключові слова: цілочисельна задача про рюкзаки, переформулювання, алгоритм гілок і меж, унімодулярний, обчислювальна складність

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1. Introduction

In general the linear integer programming problem has very important real life applications. The general linear integer problem comes in the form of capital budgeting, transportation, traveling salesman, facility location, scheduling, knapsack etc. This model even though it is very easy to model mathematically, has proved to be very difficult to solve. See [1–5] for more on linear integer models.

The paper presents a new reformulation approach to reduce the complexity of a branch and bound algorithm for solving the knapsack linear integer problem. The branch and bound algorithm [6, 7] in general relies on the usual strategy of first relaxing the integer problem into a linear