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Effect of Material Anisotropy on Buckling Load for Laminated Composite Decks Plates

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Abstract
New numerical results are generated for in-plane compressive biaxial buckling which serves to quantify the effect of material anisotropy on buckling loading. The coupling effect on buckling loads is more pronounced with the increasing degree of anisotropy. It is observed that the variation of buckling load becomes almost constant for higher values of elastic modulus ratio.
Keywords: *material anisotropy; biaxial buckling; classical laminated plate theory; finite element; fortran program; composite laminated decks plates*

Introduction

The effects of lamination scheme on the non – dimensional critical buckling loads of laminated composite plates are investigated. The material chosen has the following properties: $E_1/E_2 = 5, 10, 20, 25, 40$; $G_{12} = G_{13} = G_{23} = 0.5E_2$; $\nu_{12} = 0.25$. Several numerical methods could be used in this study, but the main ones are finite difference method (FDM), dynamic relaxation coupled with finite difference method (DR) as is shown in references (Turvey & Osman, 1990; Turvey & Osman, 1989; Turvey & Osman, 1991, Osama Mohammed Elmardi, 2014; Osama Mohammed Elmardi Suleiman, 2015; Osama Mohammed Elmardi Suleiman, 2015; Osama Mohammed Elmardi Suleiman, 2015; Osama Mohammed Elmardi Suleiman, 2016), and finite element method (FEM).

In the present work, a numerical method known as the finite element method (FEM) is used. It is a numerical procedure for obtaining solutions to many of the problems encountered in engineering analysis. It has two primary subdivisions. The first utilizes discrete elements to obtain the joint displacements and member forces of a structural framework. The second uses the continuum elements to obtain approximate solutions to heat transfer, fluid mechanics, and solid mechanics problem. The formulation using the discrete element is referred to as matrix analysis of structures and yields results identical with the classical analysis of structural frameworks. The second approach is the true finite element method. It yields approximate values of the desired parameters at specific points called nodes. A general finite element computers program, however, is capable of solving both types of problems and the name" finite element method" is often used to denote both the discrete element and the continuum element formulations.

The finite element method combines several mathematical concepts to produce a system of linear and non – linear equations. The number of equations is usually very large, anywhere from 20 to 20,000 or more and requires the computational power of the digital computer.

It is impossible to document the exact origin of the finite element method because the basic concepts have evolved over a period of 150 or more years. The method as we know it today is an outgrowth of several papers published in the 1950th that extended the matrix analysis of structures to continuum bodies. The space exploration of the 1960th provided money for basic research, which placed the method of a firm mathematical foundation and stimulated the development of multi-purpose computer programs that implemented the method. The design of airplanes, unmanned drones, missiles, space capsules, and the like, provided application areas.

The finite element method (FEM) is a powerful numerical method, which is used as a computational technique for the solution of differential equations that arise in various fields of engineering and applied sciences. The finite element method is based on the concept that one can replace any continuum by an assemblage of simply shaped elements, called finite elements with well-defined force, displacement, and material relationships. While one may not be able to derive a closed – form solution for the continuum, one can derive approximate solutions for the element assemblage that replaces it. The approximate solutions or approximation functions are often constructed using ideas from interpolation theory, and hence they are also called interpolation functions. For more details refer to References (Larry, 1984; Amir & John, 1986; Eastop & McConkey, 1993).

Mathematical Formulations

Introduction: Unlike homogeneous plates, where the coordinates are chosen solely based on the plate shape, coordinates for laminated plates should be chosen carefully. There are two main factors for the choice of the coordinate system. The first factor is the shape of the plate. Where rectangular plates will be best represented by the choice of rectangular (i.e. Cartesian) coordinates. It will be relatively easy to represent the boundaries of such plates with coordinates. The second factor is the fiber orientation or orthotropic. If the fibers are set straight within each lamina, then rectangular orthotropic would result. It is possible to set the fibers in a radial and circular fashion, which would result in circular orthotropic. Indeed, the fibers can also be set in elliptical directions, which would result in elliptical orthotropic**.**

The choice of the coordinate system is of critical importance for laminated plates. This is because plates with rectangular orthotropic could be set on rectangular, triangular, circular or other boundaries. Composite materials with rectangular orthotropic are the most popular, mainly because of their ease in design and manufacturing. The equations that follow are developed for materials with rectangular orthotropic.

Figure 1 shows the geometry of a plate with rectangular orthotropic drawn in the Cartesian coordinates X, Y, and Z or 1, 2, and 3. The parameters used in such a plate are: (1) the length in the X-direction, (a); (2) the length in the Y – direction (i.e. breadth), (b); and (3) the length in the Z – direction (i.e. thickness), (h).

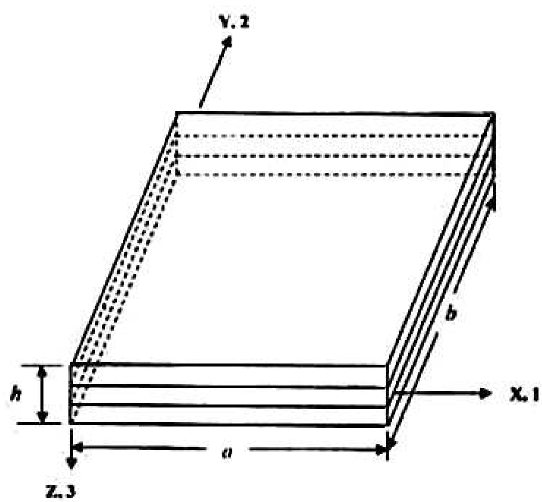


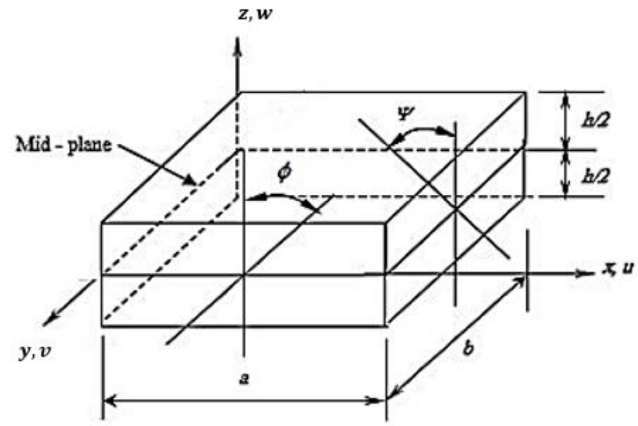
Figure 1. The geometry of a laminated composite plate

Fundamental equations of elasticity: Classical laminated plate theory (CLPT) is selected to formulate the problem. Consider a thin plate of length a, breadth b, and thickness has shown in Figure 2(a), subjected to in-plane loads Rx, Ry and Rxy as shown in Figure 2(b). The in-plane displacements $\boldsymbol{u}\ (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ and $\boldsymbol{v}\ (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ can be expressed in terms of the out of plane displacement $\boldsymbol{w}\ (\boldsymbol{x},\boldsymbol{y})$ as shown below:

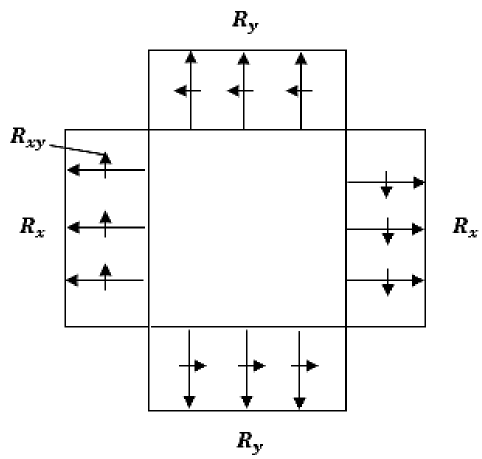
The displacements are:

$$\left.\begin{aligned} u(x,y,z) &= u_o(x,y) - z\frac{\partial w}{\partial x} \\ v(x,y,z) &= v_o(x,y) - z\frac{\partial w}{\partial y} \\ w(x,y,z) &= w_o(x,y) \end{aligned}\right\} \qquad (1)$$

Where u_o, v_o and w_o are mid – plane displacements in the direction of the x, y and z axes respectively; z is the perpendicular distance from mid – plane to the layer plane.



(a)



(b)

Figure 2. A plate showing dimensions and deformations

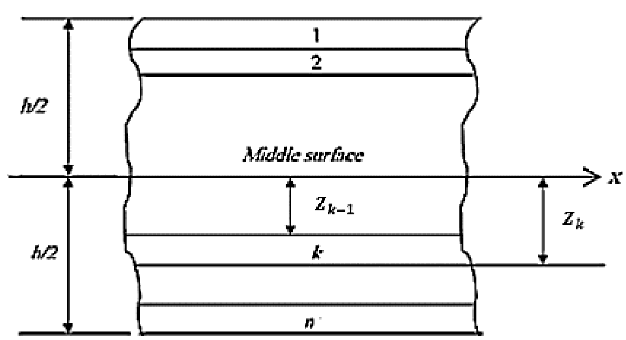


Figure 3. Geometry of an n-layered laminate

The plate is shown in Figure 2(a) is constructed of an arbitrary number of orthotropic layers bonded together as in Figure 3.

The strains are:

$$\left.\begin{aligned}\epsilon_x &= \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \epsilon_y &= \frac{\partial v_o}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma &= \frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\end{aligned}\right\} \quad (2)$$

The virtual strains:

$$\left.\begin{aligned}\delta\epsilon_x &= \frac{\partial}{\partial x}\delta u_o - z\frac{\partial^2}{\partial x^2}\delta w + \frac{\partial w}{\partial x}\frac{\partial}{\partial x}\delta w \\ \delta\epsilon_y &= \frac{\partial}{\partial y}\delta v_o - z\frac{\partial^2}{\partial y^2}\delta w + \frac{\partial w}{\partial y}\frac{\partial}{\partial y}\delta w \\ \delta\gamma &= \frac{\partial}{\partial x}\delta v_o + \frac{\partial}{\partial y}\delta u_o - 2z\frac{\partial^2}{\partial x \partial y}\delta w \\ &\quad + \frac{\partial w}{\partial x}\frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial x}\delta w\frac{\partial w}{\partial y}\end{aligned}\right\} \quad (3)$$

The virtual strain energy:

$$\delta U = \int_V \delta\epsilon^T \sigma dV \quad (4)$$

But,

$$\sigma = C\epsilon$$

Where,

$$C = C_{ij}(i, j = 1, 2, 6)$$

$$\therefore\ \delta U = \int_V \delta\epsilon^T\ C\ \delta\epsilon\ dV \quad (5)$$

If we neglect the in-plane displacements u_o and v_o and considering only the linear terms in the strain – displacement equations, we write:

$$\delta\epsilon = -z\begin{vmatrix}\dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y}\end{vmatrix}\delta w \quad (6)$$

The Numerical Method

The finite element is used in this analysis as a numerical method to predict the buckling loads and shape modes of buckling of laminated rectangular plates (Osama Mohammed Elmardi Suleiman, 2016 and Osama Mohammed Elmardi Suleiman, 2016). In this method of analysis, four – a noded type of elements is chosen. These elements are the four – noded bilinear rectangular elements of a plate. Each element has three degrees of freedom at each node. The degrees of freedom are the lateral displacement (w), and the rotations (ϕ) and (ψ) about the (X) and (Y) axes respectively.

Now express w in terms of the shape functions N and noded displacements a^e, equation (6) can be written as:

$$\delta\epsilon = -zB\delta a^e \quad (7)$$

Where,

$$B^T = \left[\frac{\partial^2 N_i}{\partial x^2} \quad \frac{\partial^2 N_i}{\partial y^2} \quad 2\frac{\partial^2 N_i}{\partial x \partial y}\right]$$

and

$$N_i a_i^e = [w_i] \quad i = 1, n$$

The stress-strain relation is:

$$\sigma = C\,\epsilon$$

Where C are the material properties which could be written as follows:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}$$

$$\delta U = \int_V (B\delta a^e)^T (Cz^2) B a^e dV$$

Where V denotes volume.

$$\delta U = \delta a^{eT} \int_V B^T D B a^e dx\, dy = \delta a^{eT} K^e a^e \qquad (8)$$

Where $D_{ij} = \sum_{k=1}^{n} \int_{Z_{k-1}}^{Z_k} C_{ij} Z^2\, dZ$ is the bending stiffness, and K^e is the element stiffness matrix which could be written as follows:

$$K^e = \int B^T D B\; dxdy \qquad (9)$$

The virtual work done by external forces can be expressed as follows: Refer to Figure 4.
Denoting the nonlinear part of strain by $\delta\epsilon'$

$$\delta W = \iint \delta\epsilon'^T \sigma' dV = \int \delta\epsilon'^T N\; dxdy \qquad (10)$$

Where

$$N^T = \left[N_x\; N_y\; N_{xy}\right] = \left[\sigma_x\; \sigma_y\; \tau\right] dZ$$

$$\delta\epsilon' = \begin{bmatrix} \delta\epsilon_x \\ \delta\epsilon_y \\ \delta\gamma \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 \\ 0 & \frac{\partial}{\partial y}\delta w \\ \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} \qquad (11)$$

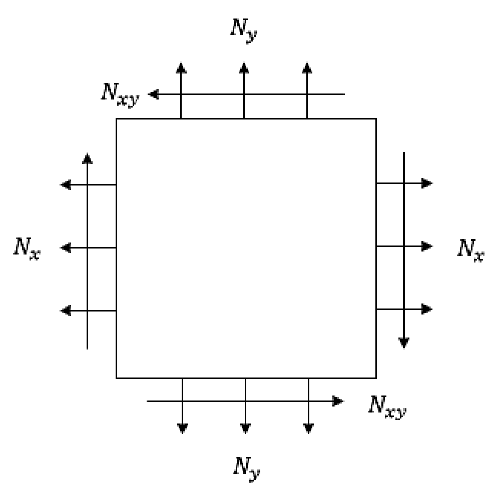


Figure 4. External forces acting on an element

Hence,

$$\delta W = \iint \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}^T \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 & \frac{\partial}{\partial y}\delta w \\ 0 & \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} dx\, dy \qquad (12)$$

This can be written as:

$$\delta W = \iint \begin{bmatrix} \frac{\partial}{\partial x}\delta w \\ \frac{\partial}{\partial y}\delta w \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} dx\, dy \qquad (13)$$

Now $w = N_i a_i^e$

$$\delta W = \delta a^{eT} \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} a^e\, dx\, dy \qquad (14)$$

Substitute $P_x = -N_x, P_y = -N_y, P_{xy} = -N_{xy}$

$$\delta W = -\delta a^{eT} \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} a^e\, dx\, dy \qquad (15)$$

Therefore, equation (15) could be written in the following form:

$$\delta W = -\delta a^{eT} K^D a^e \qquad (16)$$

Where,

$$K^D = \iint \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}^T \begin{bmatrix} P_x & P_{xy} \\ P_{xy} & P_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} dx\, dy$$

K^D is the differential stiffness matrix known also as geometric stiffness matrix, initial stress matrix, and initial load matrix.

The total energy:

$$\delta U + \delta W = 0 \qquad (17)$$

Since δa^e is an arbitrary displacement which is not zero, then

$$K^e a^e - K^D a^e = 0 \qquad (18)$$

Now let us compute the elements stiffness and the differential matrices.

$$K^e = \iint B^T D B\, dx\, dy$$

$$K^e = \iint \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ 2\frac{\partial^2 N_i}{\partial x \partial y} \end{bmatrix}^T \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ 2\frac{\partial^2 N_i}{\partial x \partial y} \end{bmatrix} dx\, dy$$

The elements of stiffness matrix can be expressed as follows:

$$K_{ij}^{e} = \iint \left[D_{11}\frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x^2} + D_{12}\left(\frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2}\right) + 2D_{16}\left(\frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x \partial y}\right) + D_{22}\frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2} + 2D_{26}\left(\frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial y^2} + \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x \partial y}\right) + 4D_{66}\frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x \partial y}\right] dx\, dy \quad (19)$$

The elements differential stiffness matrix can be expressed as follows;

$$K_{ij}^{D} = \iint \left[P_x \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + P_{xy}\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\right) + P_y \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right] dxdy \qquad (20)$$

The shape local co – ordinate for a 4 – noded element is shown below in Figure 5.

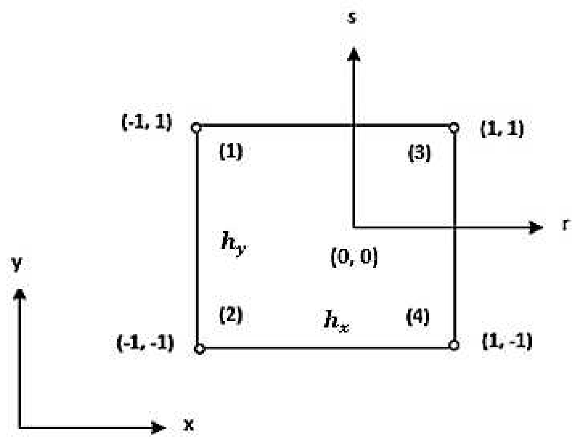


Figure 5. A four noded element with local and global co – ordinates

The shape functions for the 4 – noded element expressed in global co – ordinates (x, y) are as follows:

$$w = N_1 w_1 + N_2 \phi_1 + N_3 \psi_1 + N_4 w_2 + N_5 \phi_2 + N_6 \psi_2$$
$$+ N_7 w_3 + N_8 \phi_3 + N_9 \psi_3 + N_{10} w_4 + N_{11} \phi_4 + N_{12} \psi_4$$

Where,

$$\phi = \frac{\partial w}{\partial x}, \qquad \psi = \frac{\partial w}{\partial y}$$

The shape functions in local co – ordinates are as follows:

$$N_i = a_{i1} + a_{i2} r + a_{i3} s + a_{i4} r^2 + a_{i5} rs + a_{i6} s^2 + a_{i7} r^3 + a_{i8} r^2 s + a_{i9} rs^2$$
$$+ a_{i10} s^3 + a_{i11} r^3 s + a_{i12} rs^3$$
$$N_j = a_{j1} + a_{j2} r + a_{j3} s + a_{j4} r^2 + a_{j5} rs + a_{j6} s^2 + a_{j7} r^3 + a_{j8} r^2 s + a_{j9} rs^2$$
$$+ a_{j10} s^3 + a_{j11} r^3 s + a_{j12} rs^3$$

The integrals of the shape functions in local co – ordinates are as follows:

$$q_1 = \iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 16\left[a_{i4}a_{j4} + 3a_{i7}a_{j7} + \frac{1}{3}a_{i8}a_{j8} + a_{i11}a_{j11}\right]$$
$$q_2 = \iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 16\left[a_{i6}a_{j6} + \frac{1}{3}a_{i9}a_{j9} + 3a_{i10}a_{j10} + a_{i12}a_{j12}\right]$$
$$q_3 = \iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial s^2}\, dr\, ds = 16\left[a_{i4}a_{j6} + a_{i7}a_{j9} + a_{i8}a_{j10} + a_{i11}a_{j12}\right]$$
$$q_4 = \iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r^2}\, dr\, ds = 16\left[a_{i6}a_{j4} + a_{i9}a_{j7} + a_{i10}a_{j8} + a_{i12}a_{j11}\right]$$
$$q_5 = \iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial r \partial s}\, dr\, ds = 8[a_{i4}a_{j5} + a_{i4}a_{j11} + 2a_{i7}a_{j8} + a_{i4}a_{j12}$$

$$+\frac{2}{3}a_{i8}a_{j9}\Big]$$

$$q_6 = \iint \frac{\partial^2 N_i}{\partial r \partial s}\frac{\partial^2 N_j}{\partial r^2}\ dr\ ds = 8\Big[a_{i5}a_{j4} + 2a_{i8}a_{j7} + a_{i11}a_{j4} + \frac{2}{3}a_{i9}a_{j8}$$

$$+a_{i12}a_{j4}\Big]$$

$$q_7 = \iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r \partial s}\ dr\ ds = 8\Big[a_{i6}a_{j5} + a_{i6}a_{j11} + \frac{2}{3}a_{i9}a_{j8}\Big]$$

$$q_8 = \iint \frac{\partial^2 N_i}{\partial r \partial s}\frac{\partial^2 N_j}{\partial s^2}\ dr\ ds = 8\Big[a_{i5}a_{j6} + \frac{2}{3}a_{i8}a_{j9} + a_{i11}a_{j6}\Big]$$

$$q_9 = \iint \frac{\partial^2 N_i}{\partial r \partial s}\frac{\partial^2 N_j}{\partial r \partial s}\ dr\ ds = 4\Big[a_{i5}a_{j5} + a_{i5}a_{j11} + \frac{4}{3}a_{i8}a_{j8} + a_{i5}a_{j12}$$

$$+\frac{4}{3}a_{i9}a_{j9} + a_{i11}a_{j12} + a_{i12}a_{j11} + \frac{9}{5}a_{i12}a_{j12}\Big]$$

$$q_{10} = \iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\ dr\ ds = 4\Big[a_{i2}a_{j2} + \frac{1}{3}(3a_{i2}a_{j7} + 4a_{i4}a_{j4} + 3a_{i7}a_{j2}$$

$$+a_{i7}a_{j9} + a_{i5}a_{j5} + a_{i9}a_{j2} + a_{i5}a_{j11} + a_{i7}a_{j9} + \frac{4}{3}a_{i8}a_{j8} + a_{i9}a_{j7}$$

$$a_{i11}a_{j5}) + \frac{1}{5}(a_{i5}a_{j12} + a_{i9}a_{j9} + a_{i12}a_{j5} + 9a_{i7}a_{j7} + 3a_{i11}a_{j11} + a_{i11}a_{j12}$$

$$+a_{i12}a_{j11}) + \frac{1}{7}a_{i12}a_{j12}\Big]$$

$$q_{11} = \iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\ dr\ ds = 4\Big[a_{i3}a_{j3} + \frac{1}{3}(a_{i3}a_{j8} + a_{i5}a_{j5} + a_{i8}a_{j3}$$

$$+3a_{i3}a_{j10} + 4a_{i6}a_{j6} + 3a_{i10}a_{j3} + a_{i5}a_{j12} + a_{i8}a_{j10} + \frac{4}{3}a_{i9}a_{j9} + a_{i10}a_{j8}$$

$$+a_{i12}a_{j5}) + \frac{1}{5}(a_{i5}a_{j11} + a_{i8}a_{j8} + a_{i11}a_{j5} + 9a_{i10}a_{j10} + a_{i11}a_{j12} + a_{i12}a_{j11}$$

$$+3a_{i2}a_{j12}) + \frac{1}{7}a_{i11}a_{j11}\Big]$$

$$q_{12} = \iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial s}\ dr\ ds = 4\Big[a_{i2}a_{j3} + \frac{1}{3}(a_{i2}a_{j8} + 2a_{i4}a_{j5} + 3a_{i7}a_{j8}$$

$$+3a_{i2}a_{j10} + 2a_{i5}a_{j6} + a_{i9}a_{j3} + 2a_{i4}a_{j12} + 3a_{i7}a_{j10} + \frac{4}{3}a_{i8}a_{j9} + \frac{1}{3}a_{i9}a_{j8}$$

$$+2a_{i11}a_{j6})\Big]$$

$$q_{13} = \iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial r}\ dr\ ds = 4\Big[a_{i3}a_{j2} + \frac{1}{3}(3a_{i3}a_{j7} + 2a_{i5}a_{j4} + a_{i8}a_{j2}$$

$$+a_{i3}a_{j9} + 2a_{i6}a_{j5} + 3a_{i10}a_{j2} + 2a_{i6}a_{j11} + \frac{1}{3}a_{i8}a_{j9} + \frac{4}{3}a_{i9}a_{j8} + 3a_{i10}a_{j7}$$

$$+2a_{i12}a_{j4}) + \frac{1}{5}(2a_{i6}a_{j12} + 3a_{i10}a_{j9} + 3a_{i8}a_{j7} + 2a_{i11}a_{j4})\Big]$$

The values of the integrals are converted from local co – ordinate (r, s) to global co – ordinates.

The integrals of the shape functions in global co – ordinates are as follows:

$$r_1 = \iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial N_j}{\partial x^2}\ dx\ dy = \left(\frac{4h_y}{h_x^3}\right)q_1 = \frac{4n^3 b}{ma^3}q_1$$

$$r_2 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2}\ dx\ dy = \left(\frac{4h_x}{h_y^3}\right)q_2 = \frac{4am^3}{nb^3}q_2$$

$$r_3 = \iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2}\ dx\ dy = \left(\frac{4}{h_y h_x}\right)q_3 = \frac{4mn}{ab}q_3$$

$$r_4 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2}\ dx\ dy = \left(\frac{4}{h_y h_x}\right)q_4 = \frac{4mn}{ab}q_4$$

$$r_5 = \iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x \partial y}\ dx\ dy = \left(\frac{4}{h_x^2}\right)q_5 = \frac{4n^2}{a^2}q_5$$

$$r_6 = \iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x^2}\, dx\, dy = \left(\frac{4}{h_x^2}\right) q_6 = \frac{4n^2}{a^2} q_6$$
$$r_7 = \iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x \partial y}\, dx\, dy = \left(\frac{4}{h_y^2}\right) q_7 = \frac{4m^2}{a^2} q_7$$
$$r_8 = \iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial y^2}\, dx\, dy = \left(\frac{4}{h_y^2}\right) q_8 = \frac{4m^2}{b^2} q_8$$
$$r_9 = \iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x \partial y}\, dx\, dy = \left(\frac{4}{h_y h_x}\right) q_9 = \frac{4mn}{ab} q_9$$
$$r_{10} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\, dx\, dy = \left(\frac{h_y}{h_x}\right) q_{10} = \frac{bn}{am} q_{10}$$
$$r_{11} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\, dx\, dy = \left(\frac{h_x}{h_y}\right) q_{11} = \frac{am}{bn} q_{11}$$
$$r_{12} = \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\, dx\, dy = q_{12}$$
$$r_{13} = \iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\, dx\, dy = q_{13}$$

In the previous equations $h_x = \frac{a}{n}$ and $h_y = \frac{b}{m}$ where a and b are the lengths of the plate along the x – and y – axis respectively. n and m are the number of elements in the x – and y – directions respectively.

The elements of the stiffness matrix and the differential matrix can be written as follows:

$$K_{ij} = D_{11}r_1 + D_{12}r_4 + 2D_{16}r_3 + D_{12}r_3 + D_{22}r_2 + 2D_{66}r_8 + 2D_{16}r_5 + 2D_{26}r_7 + 4D_{66}r_9$$
$$K_{ij}^D = P_x r_{10} + P_{xy}(r_{12} + r_{13}) + P_y r_{11}$$

or in the non – dimensional form:

$$K_{ij} = \frac{4n^3}{m}\left(\frac{b}{a}\right)\bar{D}_{11}q_1 + 4mn\left(\frac{a}{b}\right)\bar{D}_{12}q_4 + 4n^2\bar{D}_{16}q_6 + 4mn\left(\frac{a}{b}\right)\bar{D}_{12}q_3$$
$$+\frac{4m^3}{n}\left(\frac{a}{b}\right)\bar{D}_{22}q_2 + 4m^2\left(\frac{a}{b}\right)^2\bar{D}_{26}q_8 + 4n^2\bar{D}_{16}q_5 + 4m^2\left(\frac{a}{b}\right)^2\bar{D}_{26}q_7$$
$$+4mn\left(\frac{a}{b}\right)\bar{D}_{66}q_9$$
$$K_{ij}^D = \bar{P}_x\frac{n}{m}\left(\frac{b}{a}\right)q_{10} + \bar{P}_{xy}(q_{12} + q_{13}) + \bar{P}_y\frac{m}{n}\left(\frac{a}{b}\right)q_{11}$$

where

$$\bar{D}_{ij} = \left(\frac{1}{E_1 h^3}\right)D_{ij}\,, \quad \bar{P}_i = \left(\frac{a}{E_1 h^3}\right)P_i$$

Also

$$\bar{w} = \left(\frac{1}{h}\right)w, \quad \bar{\phi} = \left(\frac{h}{a}\right)\phi, \quad \bar{\psi} = \left(\frac{h}{a}\right)\psi, \quad \bar{b} = b/a$$

The transformed stiffnesses are as follows:

$$C_{11} = C'_{11}c^4 + 2c^2s^2(C'_{11} + 2C'_{66}) + C'_{22}s^4$$
$$C_{12} = c^2s^2(C'_{11} + C'_{22} + 4C'_{66}) + C'_{12}(c^4 + s^4)$$
$$C_{16} = cs[C'_{11}c^4 + C'_{22}s^2 - (C'_{12} + 2C'_{66})(c^2 - s^2)]$$
$$C_{22} = C'_{11}s^4 + 2c^2s^2(C'_{12} + 2C'_{66}) + C'_{22}c^4$$
$$C_{26} = cs[C'_{11}s^2 - C'_{22}c^2 - (C'_{12} + 2C'_{66})(c^2 - s^2)]$$
$$C_{66} = (C'_{11} + C'_{22} - 2C'_{12})c^2s^2 + C'_{66}(c^2 - s^2)^2$$

Where

$$C'_{11} = \frac{E_1}{1 - v_{12}v_{21}}$$

$C'_{12} = \dfrac{v_{21}\,E_1}{1 - v_{12}v_{21}} = \dfrac{v_{12}\,E_1}{1 - v_{12}v_{21}}$

$C'_{22} = \dfrac{E_2}{1 - v_{12}v_{21}}$

$C'_{44} = G_{23}\,, \qquad C'_{55} = G_{13} \quad$ and $\; C'_{66} = G_{12}$

E_1 and E_2 are the elastic moduli in the direction of the fiber and the transverse directions respectively, v is the Poisson's ratio. G_{12}, G_{13}, and G_{23} are the shear moduli in the $x - y$ plane, $y - z$ plane, and $x - z$ plane respectively, and the subscripts 1 and 2 refer to the direction of fiber and the transverse direction respectively.

Effect of Material Anisotropy

The buckling loads as a function of the modulus ratio of symmetric cross-ply plates (0/ 90/ 90/ 0) are illustrated in Table 1 and Figure 6. As confirmed by other investigators, the buckling load decreases with increase in modulus ratio. Therefore, the coupling effect on buckling loads is more pronounced with the increasing degree of anisotropy. It is observed that the variation of buckling load becomes almost constant for higher values of elastic modulus ratio.

Table 1
The first three non – dimensional buckling loads $\overline{\mathrm{P}} = \mathrm{Pa}^2/\mathrm{E}_1\mathrm{h}^3$ of symmetric cross – ply (0/ 90/ 90/ 0) square laminated plates for different modulus ratios with $\mathrm{a/h} = 20$

E_1/E_2	Mode Number	Boundary Conditions SS	CC	CS
	1	0.6972	2.1994	1.8225
5	2	1.2552	2.5842	2.0097
	3	2.4284	4.1609	2.7116
	1	0.5505	1.8548	1.3928
10	2	0.8557	1.8951	1.8292
	3	1.6532	2.9814	1.9089
	1	0.5019	1.6663	1.2505
15	2	0.7232	1.7248	1.6428
	3	1.3966	2.6049	1.7694
	1	0.4775	1.5515	1.1791
20	2	0.6569	1.6524	1.5096
	3	1.2683	2.4228	1.7394
	1	0.4629	1.4828	1.1365
25	2	0.6172	1.6055	1.4299
	3	1.1916	2.3171	1.7214
	1	0.4531	1.4366	1.1078
30	2	0.5907	1.5723	1.3766
	3	1.1402	2.2481	1.7094
	1	0.4462	1.4044	1.0877
35	2	0.5723	1.5479	1.3391
	3	1.1043	2.2006	1.7009
	1	0.4409	1.3795	1.0723
40	2	0.5580	1.5286	1.3105
	3	1.0763	2.1648	1.6946

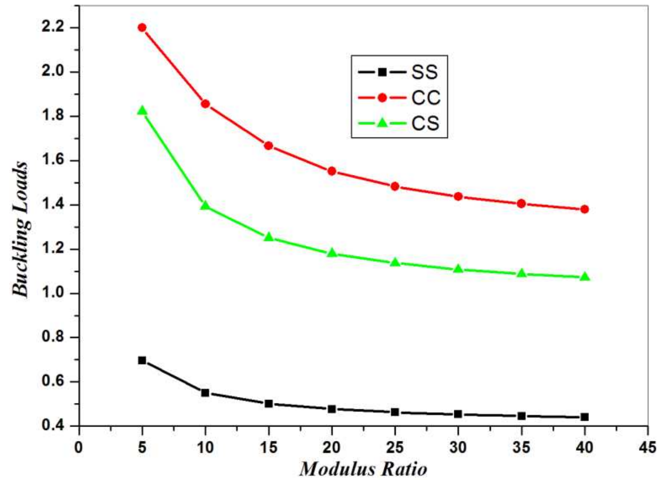


Figure 6. Effect of material anisotropy

Conclusions

The finite element model has been formulated to compute the buckling loads of laminated plates with rectangular cross-section and to study the effects of lamination scheme on the non – dimensional critical buckling loads of laminated composite plates. New results have been presented. The coupling effect on buckling loads is more pronounced with the increasing degree of anisotropy. It is observed that the variation of buckling load becomes almost constant for higher values of elastic modulus ratio.

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