Barekeng Jurnal Ilmu Matematika dan Terapan | Desember 2015 | Volume 9 Nomor 2 | Hal. 103–108

THE ENTIRE FACE IRREGULARITY STRENGTH OF A BOOK WITH POLYGONAL PAGES

Meilin I. Tilukay[1], Venn Y. I. Ilwaru[2]

[1,2]Jurusan Matematika FMIPA Universitas Pattimura
Jl. Ir. M. Putuhena, Kampus Unpatti, Poka-Ambon, Indonesia
e-mail: [1]meilin.tilukay@fmipa.unpatti.ac.id

Abstract

A face irregular entire labeling is introduced by Baca *et al.* recently, as a modification of the well-known vertex irregular and edge irregular total labeling of graphs and the idea of the entire colouring of plane graph. A face irregular entire k-labeling $\lambda: V \cup E \cup F \to \{1, 2, \cdots, k\}$ of a 2-connected plane graph $G = (V, E, F)$ is a labeling of vertices, edges, and faces of G such that for any two different faces f and g, their weights $w_\lambda(f)$ and $w_\lambda(f)$ are distinct. The minimum k for which a plane graph G has a face irregular entire k-labeling is called the entire face irregularity strength of G, denoted by $efs(G)$.

This paper deals with the entire face irregularity strength of a book with m n-polygonal pages, where embedded in a plane as a closed book with n −sided external face.

Keywords and phrases: Book, entire face irregularity strength, face irregular entire $\boldsymbol{k}$-labeling, plane graph, polygonal page.

NILAI KETAKTERATURAN SELURUH MUKA GRAF BUKU SEGI BANYAK

Abstrak

Pelabelan tak teratur seluruh muka diperkenalkan oleh Baca *et al.* baru-baru ini, sebagai suatu modifikasi atas pelabelan total tak teratur titik dan tak teratur sisi suatu graf serta ide tentang pewarnaan lengkap pada graf bidang. Pelabelan k- tak teratur seluruh muka $\lambda: V \cup E \cup F \to \{1, 2, \cdots, k\}$ dari suatu graf bidang 2-*connected* $G = (V, E, F)$ adalah suatu pelabelan seluruh titik, sisi, dan muka internal dari G sedemikian sehingga untuk sebarang dua muka f and g berbeda, bobot muka $w_\lambda(f)$ and $w_\lambda(f)$ juga berbeda. Bilangan bulat terkecil k sedemikian sehingga suatu graf bidang G memiliki suatu pelabelan k-tak teratur seluruh muka disebut nilai ketakteraturan seluruh muka dari G, dinotasikan oleh $efs(G)$.

Kami menentukan nilai eksak dari nilai ketakteraturan seluruh muka graf buku segi-n, dimana pada bidang datar dapat digambarkan seperti suatu buku tertutup.

***Kata Kunci*:** Graf bidang, graf buku segi-$\boldsymbol{n}$, nilai ketakteraturan seluruh muka, pelabelan lengkap $\boldsymbol{k}$-tak teratur muka.

1. Introduction

Let G be a finite, simple, undirected graph with vertex set $V(G)$and edge set $E(G)$. A total labeling of G is a mapping that sends $V \cup E$ to a set of numbers (usually positive or nonnegative integers). According to the condition defined in a total labeling, there are many types of total labeling have been investigated.

Baca, Jendrol, Miller, and Ryan in [1] introduced a vertex irregular and edge irregular total labeling of graphs. For any total labeling $f: V \cup E \to \{1, 2, \dots, k\}$, the *weight of a vertex* v and the *weight of an edge* $e = xy$ are defined by $w(v) = f(v) + \sum_{uv \in E} f(uv)$ and $w(xy) = f(x) + f(y) + f(xy)$, respectively. If all the vertex weights are distinct, then f is called a *vertex irregular total k-labeling*, and if all the edge weights are distinct, then f is called an *edge irregular total k-labeling*. The minimum value of k for which there exist a vertex (an edge) irregular total labeling $f: V \cup E \to \{1, 2, \dots, k\}$ is called *the total vertex (edge) irregularity* *strength* of G and is denoted by $tvs(G)$ ($tes(G)$), respectively. There are several bounds and exact values of tvs and tes were determined for different types of graphs given in [1] and listed in [2].

Furthermore, Ivanco and Jendrol in [3] posed a conjecture that for arbitrary graph G different from K_5 and maximum degree $\Delta(G)$,

$$tes(G) = max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}.$$

Combining previous conditions on irregular total labeling, Marzuki *et al.* [4] defined a totally irregular total labeling. A total k-labeling $f: V \cup E \to \{1, 2, \dots, k\}$ of G is called a *totally irregular total k-labeling* if for any pair of vertices x and y, their weights $w(x)$ and $w(y)$ are distinct and for any pair of edges x_1x_2 and y_1y_2, their weights $w(x_1x_2)$ and $w(y_1y_2)$ are distinct. The minimum k for which a graph G has totally irregular total labeling, is called *total irregularity strength* of G, denoted by $ts(G)$. They have proved that for every graph G,

$$ts(G) \geq \max\{tes(G), tvs(G)\} \quad (6)$$

Several upper bounds and exact values of ts were determined for different types of graphs given in [4], [5], [6], and [7].

Motivated by this graphs invariants, Baca *et al.* in [8] studied irregular labeling of a plane graph by labeling vertices, edges, and faces then considering the weights of faces. They defined a face irregular entire labeling.

A *2-connected plane graph* $G = (V, E, F)$ is a particular drawing of planar graph on the Euclidean plane where every face is bound by a cycle. . Let $G = (V, E, F)$ be a plane graph.

A labeling $\lambda : V \cup E \cup F \to \{1, 2, \cdots, k\}$ is called a *face irregular entire k-labeling* of the plane graph G if for any two distinct faces f and g of G, their weights $w_\lambda(f)$ and $w_\lambda(f)$ are distinct. The minimum k for which a plane graph G has a face irregular entire k-labeling is called *the entire face irregularity strength* of G, denoted by $efs(G)$. The *weight* of a *face f* under the labeling λ is the sum of labels carried by that face and the edges and vertices of its boundary. They also provided the boundaries of $efs(G)$.

Teorema A. Let $G = (V, E, F)$ be a 2-connected plane graph G with n_i i-sided faces, $i \geq 3$. Let $a = \min\{i|n_i \neq 0\}$ and $b = \max\{i|n_i \neq 0\}$. Then

$$\left\lceil\frac{2a + n_3 + n_4 + \cdots + n_b}{2b+1}\right\rceil \leq efs(G) \leq \max\{n_i|3 \leq i \leq b\}.$$

For $n_b = 1$, they gave the lower bound as follow

Teorema B. Let $G = (V, E, F)$ be a 2-connected plane graph G with n_i i-sided faces, $i \geq 3$. Let $a = \min\{i|n_i \neq 0\}$, $b = \max\{i|n_i \neq 0\}$, $n_b = 1$ and $c = \max\{i|n_i \neq 0, i < b\}$. Then

$$efs(G) \geq \left\lceil\frac{2a + |F| - 1}{2c+1}\right\rceil.$$

Moreover, by considering the maximum degree of a 2-connected plane graph G, they obtained the following theorem.

Theorem C. Let $G = (V, E, F)$ be a 2-connected plane graph G with maximum degree Δ. Let x be a vertex of degree Δ and let the smallest (and biggest) face incident with x be an a-sided (and a b-sided) face, respectively. Then

$$efs(G) \geq \left\lceil\frac{2a + \Delta - 1}{2b}\right\rceil.$$

They proved that Theorem B is tight for Ladder graph L_n, $n \geq 3$, and its variation and Theorem C is tight for wheel graph W_n, $n \geq 3$. In this paper, we determine the exact value of efs of a book with m n-polygonal pages which is greater than the lower bound given in Theorem A - C.

2. Main Results

Considering Theorem C, $efs(W_n)$, and a condition where every face of a plane graph shares common vertices or edges, our first result provide a lower bound of the entire face irregularity strength of a graph with this condition. This can be considered as generalization of Theorem A, B, and C.

Lemma 2.1. Let $G = (V, E, F)$ be a 2-connected plane graph with n_i i-sided faces, $i \geq 3$. Let $a = \min\{i|\, n_i \neq 0\}$, $b = \max\{i|\, n_i \neq 0\}$, $c = max\{i|\, n_i \neq 0, i < b\}$, and d be the number of common labels of vertices and edges which have bounded every face of G. Then

$$efs(G) \geq \begin{cases} \left\lceil \dfrac{2a + |F| - d - 1}{2c - d + 1} \right\rceil, & \text{for } n_b = 1, \\ \left\lceil \dfrac{2a + |F| - d}{2b - d + 1} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda : V \cup E \cup F \to \{1, 2, \cdots, k\}$ be a face irregular entire k-labeling of 2-connected plane graph $G = (V, E, F)$ with $efs(G) = k$. Our first proof is for $n_b \neq 1$. By Theorem A, the minimum face-weight is at least $2a + 1$ and the maximum face-weight is at least $2a + |F|$. Since G is 2-connected, each face of G is a cycle. It implies that every face might be bounded by common vertices and edges.

Let d be the number of common labels of vertices and edges which have bounded every face of G and D be the sum of all common labels. Then the face-weights $w_\lambda(f_1), w_\lambda(f_1), \cdots, w_\lambda(f_{|F|})$ are all distinct and each of them contains D, implies the variation of face-weights is depend on $2a - d + 2 \leq i \leq 2b - d + 1$ labels. Without adding D, the maximum sum of a face label and all vertices and edges-labels surrounding it is at least $2a + |F| - d$. This is the sum of at most $2b - d + 1$ labels. Thus, we have $\mathrm{efs(G)} \geq \left\lceil \frac{2a+|F|-d}{2b-d+1} \right\rceil$.

For $n_b = 1$, it is a direct consequence from Theorem B with the same reason as in the result above. ■

This lower bound is tight for ladder graphs and its variation and wheels given in [8].

A book with m n-polygonal pages $B_m^n, m \geq 1, n \geq 3$, is a plane graph obtained from m-copies of cycle C_n that share a common edge. There are many ways drawing B_m^n for which the external face of B_m^n can be an n-sided face or a $(2n - 2)$-sided face.

By considering that topologically, B_m^n can be drawn on a plane as a closed book such that B_m^n has an n-sided external face, an n-sided internal face, and $m - 1$ number of $(2n - 2)$-sided internal faces, the entire face irregularity strength of B_m^n is provided in the next theorem.

Theorem 2.2. For B_m^n, $m \geq 1, n \geq 3$, be a book with m n-polygonal pages whose an n-sided external face, an n-sided internal face, and $m - 1$ $(2n - 2)$-sided internal faces, we have

$$efs(B_m^n) = \begin{cases} 2, & \text{for } m \in \{1, 2\}; \\ \left\lceil \dfrac{4n + m - 7}{4n - 5} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. Let $B_m^n, m \geq 1, n \geq 3$, be a 2-connected plane graph. For $m \in \{1, 2\}$, by Lemma 2.1, we have $efs(B_m^n) \geq 2$. Labeling the n-sided external face by label 2 and all the rests by label 1, then all face-weights are distinct. Thus, $efs(B_m^n) = 2$.

Now for $m > 2$, let $z = efs(B_m^n)$. Since every internal face of B_m^n shares 2 common vertices, $a = n$, $b = 2n - 2$, and $n_b > 1$, by Lemma 2.1, we have $z \geq \left\lceil \frac{2a+|F|-2}{2b-1} \right\rceil = \left\lceil \frac{2n+m-1}{4n-5} \right\rceil$. Consider that $z = \left\lceil \frac{2n+m-1}{4n-5} \right\rceil$ is not valid, since for $m \leq 2n - 4$, the maximum label is 1.

Moreover, since B_m^n has at least 2 face-weights which are contributed by the same number of labels, there must be 2 faces of the same weight. Then the divisor must be at least $4n - 4$. Thus we have $z \geq \left\lceil \frac{4n+m-7}{4n-5} \right\rceil$.

Next, to show that z is an upper bound for entire face irregularity strength of B_m^n, let $B_m^n, m \geq 1, n \geq 3$, be the 2-connected plane graph with an n-sided internal face f_{int}^n, $m-1$ $(2n-2)$-sided internal faces and an external n-sided face f_{ext}^n.

Let $m_1 = \left\lceil \frac{m}{2} \right\rceil$ and $m_2 = m - m_1$. Our goal is to have m_1 distinct even face-weights and m_2 distinct odd face-weights such that m $(2n-2)$-sided face-weights are distinct and form an arithmetic progression.

Let $z = \left\lceil \frac{4n+m-7}{4n-5} \right\rceil$. It can be seen that B_m^n has m different paths of length $(n-1)$. Next, we divide m_1 paths into $S = \left\lceil \frac{m_1}{4n-5} \right\rceil$ parts, where part s-th consists of $(4n-5)$ paths, for $1 \leq s \leq S-1$, and part S-th consists of $r_1 = m_1 - (S-1)(4n-5)$ paths. Also, we divide m_2 paths into $T = \left\lceil \frac{m_2+1}{4n-5} \right\rceil$ parts, where the first part consists of $(4n-6)$ paths, part t-th consists of $(4n-5)$ paths, for $2 \leq t \leq T-1$, and part T-th consists of $r_2 = m_2 - (T-1)(4n-5)$ paths.

Let

$V(B_m^n) = \{x,\ y,\ u(s)_i^{2j},\ u(S)_k^{2j},\ v(t)_i^{2j} \neq v(1)_1^{2j},\ v(T)_l^{2j} \mid 1 \leq s \leq S-1, 1 \leq t \leq T-1,\ 1 \leq i \leq 4n-5,\ 1 \leq j \leq 2n-2,\ 1 \leq k \leq r_1,\ 1 \leq l \leq r_2\}$;

$E(B_m^n) = \{xy\} \cup$

$\{u(s)_i^1 = x\,u(s)_i^2,\ u(s)_i^{2j-1} = u(s)_i^{2j-2}\,u(s)_i^{2j},\ u(s)_i^{2n-3} = u(s)_i^{2n-4}y \mid 1 \leq s \leq S-1,\ 1 \leq i \leq 4n-5,\ 2 \leq j \leq n-2\} \cup$

$\{u(S)_i^1 = x\,u(S)_i^2,\ u(S)_i^{2j-1} = u(S)_i^{2j-2}\,u(S)_i^{2j},\ u(S)_i^{2n-3} = u(S)_i^{2n-4}y \mid 1 \leq i \leq r_1,\ 2 \leq j \leq n-2\} \cup$

$\{v(t)_i^1 = xv(t)_i^2,\ v(t)_i^{2j-1} = v(t)_i^{2j-2}v(t)_i^{2j}, v(t)_i^{2n-3} = v(t)_i^{2n-4}y \mid 1 \leq t \leq T, 1 \leq i \leq 4n-5, 2 \leq j \leq n-2\} \cup$

$\{v(T)_i^1 = xv(T)_i^2,\ v(T)_i^{2j-1} = v(T)_i^{2j-2}v(T)_i^{2j}, v(T)_i^{2n-3} = v(T)_i^{2n-4}y \mid 1 \leq i \leq r_2, 2 \leq j \leq n-2\}$;

$F(B_m^n) = \{f_{ext}^n, f_{int}^n,\ u(s)_i^{2n-2},\ u(S)_k^{2n-2},\ v(t)_i^{2n-2} \neq v(1)_1^{2n-2},\ v(T)_j^{2n-2} \mid 1 \leq s \leq S-1, 1 \leq t \leq T-1, 1 \leq i \leq 4n-5, 1 \leq k \leq r_1,\ 1 \leq l \leq r_2\}$;

Where f_{ext}^n is bounded by cycle $xv(1)_2^2 v(1)_2^4 \cdots v(1)_2^{2n-4} yx$;

f_{int}^n is bounded by cycle $xu(1)_1^2 u(1)_1^4 \cdots u(1)_1^{2n-4} yx$;

$u(s)_i^{2n-2}$ is bounded by cycle $xu(s)_i^2 u(s)_i^4 \cdots u(s)_i^{2n-4} yu(s)_{i+1}^{2n-4} u(s)_{i+1}^{2n-6} \cdots u(s)_{i+1}^2 x$, for $1 \leq s \leq S, i \neq r_1$;

$u(S)_{r_1}^{2n-2}$ is bounded by cycle $xu(S)_{r_1}^2 u(S)_{r_1}^4 \cdots u(S)_{r_1}^{2n-4} yv(T)_{r_2}^{2n-4} v(T)_{r_2}^{2n-6} \cdots v(T)_{r_2}^2 x$; and

$v(t)_i^{2n-2}$ is bounded by cycle $xv(t)_i^2 v(t)_i^4 \cdots v(t)_i^{2n-4} yv(t)_{i+1}^{2n-4} v(t)_{i+1}^{2n-6} \cdots v(t)_{i+1}^2 x$, for $1 \leq t \leq T, i \neq r_2$;

Our notations above imply that, without losing generality, for $v(t)_i^j$, we let $2 \leq i \leq 4n-5$ for $t = 1$. It means that there is no vertex or edge or face $v(1)_1^j$.

Now, we divide our labeling of B_m^n into 2 cases as follows:

Case 1. For odd m with $2 \leq r_2 \leq 2n-1$ or even m;

Define an entire k-labeling $\lambda : V \cup E \cup F \to \{1, 2, \cdots, k\}$ of B_m^n as follows.

$\lambda(x) = \lambda(y) = \lambda(xy) = \lambda(f_{ext}^n) = 1$;

$\lambda(f_{int}^n) = 2$;

$$\lambda\big(u(s)_i^j\big)=\begin{cases}2s-1 & \text{for } 1\le s\le S, 1\le i\le \min\{r_1,\ 2n-2\} \text{ and } 1\le j\le 2n-i-1\\ 2s & \text{for } 1\le s\le S, 1\le i\le \min\{r_1,\ 2n-2\} \text{ and } 2n-i\le j\le 2n-2\\ 2s & \text{for } 1\le s\le S, 2n-1\le i\le \min\{r_1,\ 4n-5\} \text{ and } 1\le j\le 2n-2\left\lfloor\frac{i-2n+2}{2}\right\rfloor-2\\ 2s+1 & \text{for } 1\le s\le S, 2n-1\le i\le \min\{r_1,\ 4n-5\} \text{ and } 2n-2\left\lfloor\frac{i-2n+2}{2}\right\rfloor-1\le j\le 2n-2\end{cases}$$

$$\lambda\left(v(t)_i^j\right)=\begin{cases}2t-1, & \text{for } 1\le t\le T, 1\le i\le \min\{r_2,\ 2n-2\} \text{ and } 1\le j\le 2n-i-2;\\ 2t, & \text{for } 1\le t\le T, 1\le i\le \min\{r_2,\ 2n-2\} \text{ and } 2n-i-1\le j\le 2n-3;\\ 2t, & \text{for } 1\le t\le T, 2n-1\le i\le \min\{r_2,\ 4n-5\} \text{ and } 1\le j\le 2n-2\left\lfloor\frac{i-2n+2}{2}\right\rfloor-3;\\ 2t+1, & \text{for } 1\le t\le T, 2n-1\le i\le \min\{r_2,\ 4n-5\} \text{ and } 2n-2\left\lfloor\frac{i-2n+2}{2}\right\rfloor-2\ \le j\le 2n-3;\\ 2t-2, & \text{for } 1\le t\le T, i=1 \text{ and } j=2n-2;\\ 2t-1, & \text{for } 1\le t\le T, 2\le i\le \min\{r_2,\ 2n-1\} \text{ and } j=2n-2;\\ 2t, & \text{for } 1\le t\le T-1, 2n\le i\le 4n-5 \text{ and } j=2n-2.\\ 2t, & \text{for } t=T, 2n-1\le i\le \min\{r_2-1,\ 4n-6\} \text{ and } j=2n-2\end{cases}$$

Case 2. For odd m with $r_2=1$ or $2n\le r_2\le 4n-5$;

Define an entire k-labeling $\lambda^*: V\cup E\cup F\to\{1,2,\cdots,k\}$ of B_m^n as follows.

$\lambda^*(x)=\lambda^*(y)=\lambda^*(xy)=\lambda^*(f_{ext}^n)=1;$

$\lambda^*(f_{int}^n)=2;$

$\lambda^*\big(u(s)_i^j\big)=\lambda\big(u(s)_i^j\big)$

$$\lambda^*\left(v(t)_i^j\right)=\begin{cases}2T-2, & \text{for } r_2=1,\ t=T,\ i=1,\ j=1;\\ 2T-1, & \text{for } r_2=1,\ t=T-1,\ i=4n-5,\ j=2n-2;\\ \lambda\big(v(t)_i^j\big)+1, & \text{for } r_2 \text{ odd}, 2n\le r_2\le 4n-5,\ t=T,\ i=r_2,\ j=1;\\ \lambda\big(v(t)_i^j\big)-1, & \text{for } r_2 \text{ odd}, 2n\le r_2\le 4n-5,\ t=T,\ i=r_2-1,\ j=2n-2;\\ \lambda\big(v(t)_i^j\big)-1, & \text{for } r_2 \text{ even}, 2n\le r_2\le 4n-5,\ t=T,\ i=r_2-1,\ j=2n-3;\\ \lambda\big(v(t)_i^j\big)+1, & \text{for } r_2 \text{ even}, 2n\le r_2\le 4n-5,\ t=T,\ i=r_2-1,\ j=2n-2;\\ \lambda\big(v(t)_i^j\big), & \text{for otherwise.}\end{cases}$$

It is easy to check that the labeling λ is an entire z-labeling. Then we have evaluate the face –weights set $\left\{w(f_{ext}^n), w(f_{int}^n), w\big(u(s)_i^{2n-2}\big), w\big(v(t)_i^{2n-2}\big) \mid 1\le s\le S,\ 1\le t\le T,\ 1\le i\le 4n-5\right\}$ as follows.

$w(f_{ext}^n)=2n+1;$

$w(f_{int}^n)=2n+2;$

$$w\big(u(s)_i^{2n-2}\big)=\begin{cases}(2s-1)(4n-5)+2i, & \text{for } 1\le s\le S-1, 1\le i\le 4n-5;\\ (2s-1)(4n-5)+2i, & \text{for } s=S-1, 1\le i\le r_1;\\ (2s-1)(4n-5)+2r_1, & \text{for even } m,\ s=S-1, i=r_1;\\ (2s-1)(4n-5)+2r_1-1, & \text{for odd } m,\ s=S-1, i=r_1.\end{cases}$$

$$w\big(v(t)_i^{2n-2}\big)=\begin{cases}(2t-1)(4n-5)+2i+1, & \text{for } 1\le t\le T-1, 1\le i\le 4n-5;\\ (2T-1)(4n-5)+2i+1, & \text{for } t=T, 1\le i\le r_2-1.\end{cases}$$

Since all face-weights are distinct, then λ is a face irregular entire z-labeling of B_m^n where m is odd with $2\le r_2\le 2n-1$ or m is even; and λ^* is a face irregular entire z-labeling of B_m^n where m is odd with $r_2=1$ or $2n\le r_2\le 4n-5$. Thus, $z=\left\lceil\frac{4n+m-7}{4n-5}\right\rceil$ is the entire face irregularity strength of B_m^n. ■

Note that our result in Theorem 2.2 show that the $efs(B_m^n)$ is greater than the lower bound in Lemma 2.1. Hence, we propose the following open problem.

Open Problems

1. Find a class of graph which satisfy a condition where the lower bound in Lemma 2.1 is sharp;
2. Generalize the lower bound for any condition.

References

[1] M. Baca, S. Jendrol, M. Miller and J. Ryan, “On Irregular Total Labelings,” *Discrete Mathematics,* vol. 307, pp. 1378-1388, 2007.

[2] J. A. Galian, “A Dynamic Survey of Graph Labeling,” *Electronic Journal of Combinatorics,* vol. 18 #DS6, 2015.

[3] J. Ivanco and S. Jendrol, “The Total Edge Irregularity Strength of Trees,” *Discuss. Math. Graph Theory,* vol. 26, pp. 449-456, 2006.

[4] C. C. Marzuki, A. N. M. Salman and M. Miller, “On The Total Irregularity Strengths of Cycles and Paths,” *Far East Journal of Mathematical Sciences,* vol. 82 (1), pp. 1-21, 2013.

[5] R. Ramdani and A. N. M. Salman, “On The Total Irregularity Strengths of Some Cartesian Products Graphs,” *AKCE Int. J. Graphs Comb.,* vol. 10 No. 2, pp. 199-209, 2013.

[6] R. Ramdani, A. N. M. Salman, H. Assiyatun, A. Semanicova-Fenovcikova and M. Baca, “Total Irregularity Strength of Three Family of Graphs,” *Math. Comput. Sci,* vol. 9, pp. 229-237, 2015.

[7] M. I. Tilukay, A. N. M. Salman and E. R. Persulessy, "On The Total Irregularity Strength of Fan, Wheel, Triangular Book, and Friendship Graphs," *Procedia Computer Science,* vol. 74, pp. 124-131, 2015.

[8] M. Baca, S. Jendrol, K. Kathiresan and K. Muthugurupackiam, “Entire Labeling of Plane Graphs,” *Applied Mathematics and Information Sciences,* vol. 9, no. 1, pp. 263-207, 2015.