Electronic Journal of Graph Theory and Applications **6** (2) (2018), 258–268

Electronic Journal of Graph Theory and Applications

Some diameter notions in lexicographic product of graphs

Chithra M R[a,b], Manju K. Menon[c], A. Vijayakumar[a]

[a]*Department of Mathematics, Cochin University of Science and Technology, India*
[b]*Current Affiliation: Department of Mathematics, Amrita School of Arts and Sciences, Kochi, Amrita Vishwa Vidyapeetham, India*
[c]*Department of Mathematics, St. Paul's College, India*

chithramohanr@gmail.com, manjumenonk@gmail.com, vambat@gmail.com

Abstract

Many graphs such as hypercubes, star graphs, pancake graphs, grids, tori etc are known to be good interconnection network topologies. In any network topology, the vertices represent the processors and the edges represent links between the processors. Two most important criteria - efficiency and reliability of network models - can be studied with the help of graph theoretical techniques. The lexicographic product is a well studied graph product. The distance notions such as various diameters of a graph help to analyze the efficiency of any interconnection network. In this paper, we study some distance notions such as wide diameter, diameter variability and diameter vulnerability of lexicographic products that could be used in the design of interconnection networks.

Keywords: lexicographic product, wide diameter, diameter vulnerability, fault diameter, diameter variability
Mathematics Subject Classification : 05C12, 05C76, 05C99
DOI: 10.5614/ejgta.2018.6.2.6

1. Introduction

The processors of a parallel and distributed system and the connections between the processors can be represented as an interconnection network. The topological structure of an interconnection

Received: 19 October 2017, Revised: 6 May 2018, Accepted: 10 June 2018.

network can be modelled by a connected graph where the vertices and edges represent sites of the network and the physical communication links respectively. Many graph theoretic parameters that are useful to study the efficiency and reliability of an interconnection network are discussed in [6].

A simple graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ is denoted as $G = (n, m)$. The *degree* of a vertex u in G, $d_G(u)$ or simply $d(u)$, is the number of edges incident with u in G. The minimum degree and the maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path joining u and v in G. The *diameter* of a graph G, $diam(G)$, is the maximum distance between any two vertices in G. The diameter often measures efficiency of a network with maximum time - delay or signal degradation. The *diametral vertices* of G are two vertices $u, v \in V(G)$ such that $d(u, v) = diam(G)$. A subset $S \subseteq V(G)$ of vertices is an *independent set* if no two vertices of S are joined by an edge in G. The *independent domination number* of a graph G, $\gamma_i(G)$, is the minimum cardinality of a maximal independent set in G. The *vertex connectivity,* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G makes the graph either disconnected or K_1. The *edge connectivity,* $\kappa'(G)$ of a graph G is the minimum number of edges whose removal makes the graph disconnected. The network fault tolerance capacity can be measured by studying the connectivity of the corresponding graph. A good network must be hard to disrupt even if some vertices or edges are being attacked and the transmissions between the processors must remain connected. For all notions not given here, see [13].

The *lexicographic product* $H_1 \circ H_2$ of any two graphs H_1 and H_2 is the graph with the vertex-set $V(H_1) \times V(H_2)$ and two vertices (u_i, v_x) and (u_j, v_y) of $H_1 \circ H_2$ are adjacent if either $u_i - u_j \in E(H_1)$, or $u_i = u_j$ and $v_x - v_y \in E(H_2)$. The necessary and sufficiency condition for the lexicographic product of two graphs $H_1 \circ H_2$ to be connected is that H_1 is connected. If $H_1 \neq K_n$, then diam($H_1 \circ H_2$) = diam(H_1) and diam$(K_n \circ H) = 2$, [7]. In [14], Yang et al. studied the connectivity of the lexicographic product of graphs and they have proved that if $H_1 = (n_1, m_1)$ is a connected simple graph and $H_2 = (n_2, m_2)$ is any simple graph then:

- $\kappa(H_1 \circ H_2) = \kappa(H_1)\,|n_2|$, if H_1 is not complete,
- $\kappa(K_n \circ H_2) = (n-1)\,|n_2| + \kappa(H_2)$,
- $\kappa'(H_1 \circ H_2) = min\{\kappa'(H_1)n_2^2, \delta(H_2) + \delta(H_1)n_2\}$.

Let $H_1 * H_2$ be any of the graph products. For any vertex $u \in H_1$, the subgraph of $H_1 * H_2$ induced by $\{u\} \times V(H_2)$ is the H_2-*layer at* u and is denoted by ${}^uH_2$. For any vertex $v \in H_2$, the subgraph of $H_1 * H_2$ induced by $V(H_1) \times \{v\}$ is the H_1-*layer at* v and is denoted by H_1^v.

For every integer w, $1 \leq w \leq \kappa(G)$, any collection of 'w' internally vertex disjoint paths between two vertices u and v of G is termed as the w-*container* and it is denoted by $C_w(u, v)$. In $C_w(u, v)$, the parameter w is the *width* of the container. The *length* of the container is the length of the longest path in $C_w(u, v)$. The w-*wide diameter* $D_w(G)$ of a graph G is the minimum number l such that there is a $C_w(u, v)$ of length at most l between any pair of distinct vertices u and v in G. The *wide diameter* of a graph is $D_{\kappa(G)}(G)$. This concept was introduced by Hsu [6] to unify the concepts of diameter and connectivity. The wide diameter of some networks are studied in [9] and [5].

Vulnerability measures maximum routing delay that can happen because of vertex or edge faults. Diameter can be used to measure the maximum delay in routing. In this context, the vertex fault diameter and the edge fault diameter are defined and studied by several authors. The *vertex fault diameter* is $f(G) = \max\{diam(G - S)|S \subseteq V(G), |S| = \kappa(G) - 1\}$ and the *edge fault diameter* is $f'(G) = \max\{diam(G - F)|F \subseteq E(G), |F| = \kappa'(G) - 1\}$, [8]. Chung and Garey [3] proposed the problem of determining the diameter vulnerability of a graph. In [15] Ye et al. improves the result of Peyrat [10] and gave a bound as $4\sqrt{2t}-6 < f'(G) \leq \max\{59, 5\sqrt{2t}+7\}$ for $t \geq 4$. The concept of fault diameter was introduced by Krishnamoorthy and B. Krishnamurthy [8]. The problem of diameter vulnerability is proved to be NP-complete by Schoone et al. [11].

The diameter of a graph may change by the addition or the deletion of edges. The following notations denote the *diameter variability* of a graph G. Let $k \geq 1$ be any positive integer. $D^{-k}(G)$ is the minimum number of edges to be added to G to decrease the diameter by (at least) k and $D^0(G)$ is the maximum number of edges that can be deleted from G so that the diameter is not altered. In [1], [2], the diameter variability of the product graphs are discussed. In [12], Wang et al. studied the diameter variability of cycles and tori. Graham and Harary studied the diameter variability of hypercubes in [4].

In this paper, we study the wide diameter, the diameter vulnerability and the diameter variability of the lexicographic product of graphs. We consider both H_1 and H_2 to be connected graphs with $V(H_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(H_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Then $G \cong H_1 \circ H_2$ has $V(G) = \{(u_1, v_1), (u_1, v_2), ..., (u_1, v_{n_2}), ..., (u_{n_1}, v_1), ..., (u_{n_1}, v_{n_2})\}$. Since $H_1 \circ K_1 \cong K_1 \circ H_1 \cong H_1$, we assume that both H_1 and H_2 are different from K_1.

2. Wide diameter of the lexicographic product of graphs

Lemma 2.1. *Let $G' \cong G \circ H$. If there exists a container of width $w, 1 \leq w \leq \kappa(G)$, in G with the length l then there exists a container of width $\kappa(G) \times |V(H)|$ in G' with the same length l.*

Proof. The proof is divided into three cases.

Case 1: Consider (u_i, v_j) and (u_k, v_j) in G' where $i \neq k$ and $i, k \in \{1, 2, ...n_1\}$.
There exists a container of length at most l between any two vertices u_i and u_k in G, since there exists a container of length l in G, If $P_1 = u_i - u_{i+1} - u_{i+2} - ... - u_{k-1} - u_k$ is a path in the container $C_w(u_i, u_k)$ of G, then $(u_i, v_j) - (u_{i+1}, v_j) - (u_{i+2}, v_j) - ... - (u_{k-1}, v_j) - (u_k, v_j)$ is a path connecting (u_i, v_j) and (u_k, v_j) in G' and $(u_i, v_j) - (u_{i+1}, v_a) - (u_{i+2}, v_a)...(u_{k-1}, v_a) - (u_k, v_j)$ are also paths connecting (u_i, v_j) and (u_k, v_j) where $a \neq j$ and $a \in \{1, 2, ..., n_2\}$ in G'. Thus, corresponding to the w internally vertex disjoint paths in $C_w(u_i, u_k)$ of G, there exist $w\,|V(H)|$ internally vertex disjoint paths between (u_i, v_j) and (u_k, v_j) in G which are of length at most l. Since the length of the container in G is l, there exists a pair of vertices u_x and u_y in G such that the path joining u_x and u_y is of length exactly equal to l. Then $C_{w|V(H)|}((u_x, v_j), (u_y, v_j))$ in G' is of length exactly equal to l.

Case 2: Consider (u_i, v_j) and (u_i, v_k) in G' where $j \neq k$ and $j, k \in \{1, 2, ...n_2\}$.
If u_i is adjacent to u_a in G, then both (u_i, v_j) and (u_i, v_k) will be adjacent to $(u_a, v_1), (u_a, v_2), \cdots$

(u_a, v_{n_2}) in G'. Thus, there exists at least $d(u_i)\,|V(H)|$ internally vertex disjoint paths between (u_i, v_j) and (u_i, v_k) which are of length two. So we can say that for any vertex u_i in G, there exists $C_{\delta(G)|V(H)|}((u_i, v_j), (u_i, v_k))$ of length two in G'.

Case 3: Consider (u_i, v_j) and (u_a, v_b) in G' where $i \neq a$ and $j \neq b$.
Consider the vertices u_i and u_a in H_1. By the assumption there exists a container of length at most l in between u_i and u_a in G. If $P_1 = u_i - u_{i+1} - u_{i+2} - ... - u_{a-1} - u_a$ is a path in the container $C_w(u_i, u_a)$, then $(u_i, v_j) - (u_{i+1}, v_j) - (u_{i+2}, v_j) - ... - (u_{a-1}, v_j) - (u_a, v_b)$ is a path connecting (u_i, v_j) and (u_a, v_b) in G' which is of length same as that of P_1. Again, by the structure of the lexicographic product, there exists $w\,|V(H)|$ internally vertex disjoint paths between (u_i, v_j) and (u_a, v_b) which is of length at most l. Since the length of the container in G is l, there exists a pair of vertices u_x and u_y in G such that the path joining u_x and u_y is of length exactly equal to l. So $C_{w|V(H)|}((u_x, v_j), (u_y, v_b))$ in G' is of length exactly equal to l.

Since $1 \leq w \leq \kappa(G')$ and $\kappa(G') \leq \delta(G')$, the result follows. □

Theorem 2.1. *If G is a connected non-complete graph and H is a connected graph, then* $D_{\kappa(G)\times|V(H)|}(G \circ H) D_{\kappa(G)}(G)$.

Proof. Suppose that $G' \cong G \circ H$. Then $\kappa(G') = \kappa(G) \times |V(H)|$.
Let $D_{\kappa(G)}(G) = k$. Then there exists a container of width $\kappa(G)$ between any two vertices of G which is of length at most k. Then, by Lemma 2.1, there exists a container of width $\kappa(G) \times |V(H)|$ between any two vertices of G' which is of length at most k.
Hence, $D_{\kappa(G)\times|V(H)|}(G \circ H) \leq D_{\kappa(G)}(G)$.

Let $D_{\kappa(G)\times|V(H)|}(G \circ H) = k$. There exists a container of length at most k joining (u_i, v_1) and (u_j, v_1). More over there exists a container of width at least $\kappa(G)$ between (u_i, v_1) and (u_j, v_1) where all the internal vertices are of the form (u_a, v_1), $a \in \{1, 2, ..., x, y, ..., n_1\}$. If $(u_i, v_1), (u_x, v_1), (u_y, v_1), ..., (u_j, v_1)$ is a path in the container of G', then $u_i - u_x - u_y - ... - u_j$ is a path in G. Thus there exist a container of width $\kappa(G)$ which is of length at most k joining u_i and u_j in G. Hence, $D_{\kappa(G)}(G) \leq D_{\kappa(G)\times|V(H)|}(G \circ H)$. □

3. Diameter vulnerability of the lexicographic product of graphs

Theorem 3.1. *Let $G' \cong G \circ H$ where G and H are connected graphs with $n_1, n_2 \geq 3$. Then,* $f'(G') \leq f'(G) + diam(H)$.

Proof. Let $G' \cong G \circ H$. Then $\kappa'(G') = min\{\kappa'(G)n_2^2, \delta(H) + \delta(G)n_2\}$. Let u_x, u_y be a pair of diametral vertices of G, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$. Let G'' be the subgraph obtained from G' after the deletion of $\kappa'(G')-1$ edges from G'. Let us consider the following cases.

Case 1: $\kappa'(G') = \delta(H) + n_2\delta(G)$.
Case 1a: Let $\kappa'(G') - 1$ edges be deleted from G-layer of G' at v_k. Then, the deleted edges are of the form $(u_i, v_k) - (u_j, v_k)$ where $i, j \in \{1, 2, ...n_1\}$.
Consider any two vertices (u_a, v_k) and (u_b, v_k) in G'. If $u_a - u_{a1} - u_{a2} - ...u_{ai} - u_b$ is a path

joining u_a and u_b in H_1 then $(u_a, v_k) - (u_{a1}, v_x) -(u_{a2}, v_x)-$ ... - $(u_{ai}, v_x) - (u_b, v_k)$ where $k \neq x \in \{1, 2, ...n_2\}$ is a path joining (u_i, v_k) and (u_j, v_k) in G''. Clearly, this length is at most diam(G).

Case 1b: Let $\kappa'(G') - 1$ edges be deleted from H-layer of G' at u_i. Then, the deleted edges are of the form $(u_i, v_j) - (u_i, v_k)$ where $j, k \in \{1, 2, ..., n_2\}$.
If u_{i+1} is a vertex adjacent to u_i in G then $(u_i, v_j) - (u_{i+1}, v_j) - (u_i, v_k)$ is a path of length two in G''. Thus the diam(G') is unaltered by this type of deletion.

Case 1c: Let $\kappa'(G') - 1$ edges deleted from G' be any arbitrary collection of edges.
Consider a pair of diametral vertices (u_x, v_w) and (u_y, v_w) in G'. Let the $\kappa'(G') - 1$ edges adjacent to the vertex (u_x, v_w) except (u_x, v_{w+1}) be deleted from G' to get G''. Then, $d_{G''}((u_x, v_w), (u_y, v_w))$ = diam$(G') + 1$ by a path $(u_x, v_w) - (u_x, v_{w+1}) - (u_{x+1}, v_w) - (u_{x+2}, v_w) - ... - (u_y, v_w) - (u_y, v_w)$, where $d_{G''}((u_x, v_w), (u_x, v_{w+1})) = 1$ and $d_{G''}((u_x, v_{w+1}), (u_y, v_w)) = diam(G')$ (see Figure 1).

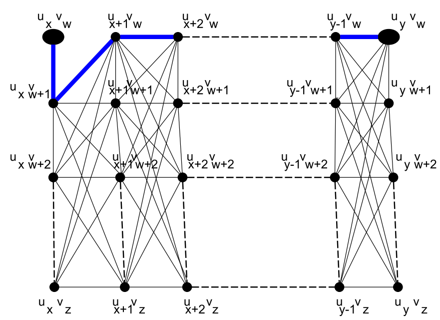

Figure 1. $\kappa'(G') - 1$ edges adjacent to the vertex (u_x, v_w) are deleted from G'.

Consider a pair of diametral vertices (u_x, v_w) and (u_y, v_z) in G'. Since, we have already considered Cases 1a and 1b, there exist a path of length diam(G') between (u_x, v_w) and (u_y, v_z) in G'', $(u_x, v_w) - (u_{x+1}, v_p) - (u_{x+2}, v_q) - (u_{x+3}, v_r)... - (u_{y-1}, v_s) - (u_y, v_z)$, where the vertex (u_x, v_w) in ${}^{u_x}H$-layer will be adjacent to at least one vertex (say)(u_{x+1}, v_p) in ${}^{u_{x+1}}H$-layer, the vertex (u_{x+1}, v_p) in ${}^{u_{x+1}}H$-layer will be adjacent to at least one vertex (say) (u_{x+2}, v_q) in ${}^{u_{x+2}}H$-layer and so on (see Figure 2).

Case 2: $\kappa'(G') = \kappa'(G)n_2^2$.
Let E' be the minimal edge cut of G. Then corresponding to each edge $u_i - u_j \in E'$, $(u_i, v_r) - (u_j, v_r)$, $(u_i, v_p) - (u_j, v_q)$ where $r \in \{1, 2, \dots, n_2 - 1\}$, $q \neq p \in \{1, 2, \dots, n_2\}$ are deleted. Also, $\kappa'(G) - 1$ edges are deleted from the G - layer at v_{n_2} in G'. Now, $d_{G''}((u_a, v_{n_2}), (u_b, v_{n_2})) \leq f'(G)$ by a path $(u_a, v_{n_2}) - (u_{a+1}, v_{n_2}) - \dots - (u_{b-1}, v_{n_2}) - (u_b, v_{n_2})$, since the deletion of $\kappa(G) - 1$ edges from G increases the diam(G) to at most $f'(G)$. If $v_w \in V(H)$ then $d_{G''}((u_a, v_w), (u_b, v_w)) \leq f'(G)$ + diam(H) by a path $(u_a, v_w) - (u_a, v_{w+1}) - \dots - (u_a, v_{n_2}) - (u_{a+1}, v_{n_2}) - \dots - (u_{b-1}, v_{n_2}) - (u_b, v_w)$ where $d_{G''}((u_a, v_w), (u_a, v_{n_2})) \leq$ diam(H) and $d_{G''}((u_a, v_{n_2}), (u_b, v_w)) \leq f'(G)$. Similarly, the distance between any two vertices in G' is at most $f'(G)$ + diam(H). □

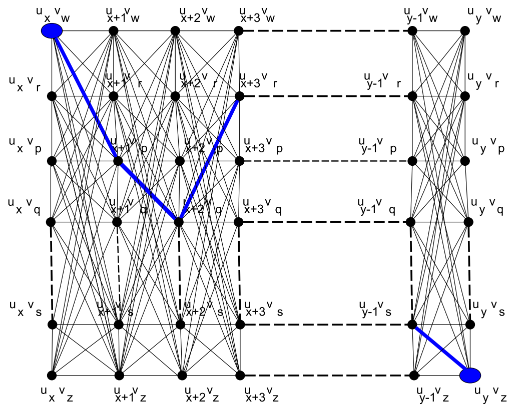

Figure 2. Arbitrary collection of $\kappa'(G') - 1$ edges are deleted from G'.

Remark: Consider $H \circ P_3$ where H is the graph obtained by taking two copies of $K_n, n > 3$ which is joined by an edge. For this graph $f'(H \circ P_3) = 5$, since $f'(H) = 3$ and diam(P_3) = 2. Thus the above bound is strict for an infinite family of graphs.

Theorem 3.2. *If* $G' \cong G \circ H$ *is a connected graph, then* $f(G') \leq max\{f(G), f(H)\}$.

Proof. Let S be a collection of $\kappa(G') - 1$ vertices in G'. When S is deleted from G' the new subgraph obtained is denoted as G''. Let u_x, u_y be a pair of diametral vertices of G, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$. Let us consider the following cases.

Case 1: $G' \cong K_{n_1} \circ H$.
Then diam$(G') = 2$ and $\kappa(G') = (n_1 - 1)n_2 + \kappa(H)$. Let the $\kappa(G') - 1$ vertices adjacent to (u_i, v_j) in the H-layer at u_i except (u_i, v_s), be deleted. Then, $d((u_i, v_p), (u_i, v_q)) \leq f(H)$, since the deletion of $\kappa(H) - 1$ vertices from H increases the diam(H) to at most $f(H)$. Thus $f(G') \leq f(H)$.

Case 2: $G' \cong G \circ H$ where $\kappa(G) = 1$ and $G \neq K_2$.
Then diam(G') = diam(G) and $\kappa(G') = \kappa(G)\,|V(H)| = n_2$. Now, let us consider the following sub cases.

Case 2a: Let $S = \{(u_{x+1}, v_p)\}$, where u_{x+1} is a neighbour of u_x and $p \in \{1, 2, 3, ..., n_2\}$.
Consider a pair of diametral vertices (u_x, v_a) and (u_y, v_a) in G'. Let the $n_2 - 1$ vertices except (u_{x+1}, v_{n_2}) from S be deleted. Then, $d_{G''}((u_x, v_a), (u_y, v_a))$ = diam(G') by a path $(u_x, v_a) - (u_{x+1}, v_{n_2}) - (u_{x+2}, v_a) - (u_{x+3}, v_a) - ... - (u_{y-1}, v_a) - (u_y, v_a)$. Thus, the diam$(G')$ remains the same after removing vertices in S.

Case 2b: Let $S = \{(u_i, v_p)\}$ where $p \in \{1, 2, 3, ..., n_1\}$.
Let $n_2 - 1$ vertices from S be deleted. Clearly the distance between any two vertices in G' is not affected by the removal of these vertices.

Case 2c: Let S be any arbitrary collection of vertices.
Consider a pair of diametral vertices (u_x, v_p) and (u_y, v_q) in G. Let the $\kappa(G')-1$ vertices from G' be deleted. Then, $d_{G''}((u_x, v_p), (u_y, v_q))$ = diam(G) by a path $(u_x, v_p) - (u_{x+1}, v_a) - (u_{x+2}, v_b) - ... - (u_y, v_q)$, since we have already considered the case of the deletion of vertices of the form (u_i, v_p) where $i \in \{1, 2, 3, ..., n_1\}$, there exist at least one vertex (say) (u_i, v_j) for each $j \in \{1, 2, \ ... \ , n_2\}$ and are adjacent to the vertices (u_r, v_p) where $p \in \{1, 2, 3, ..., n_2\}$. Thus the diam$(G')$ remains the same.

Case 3: $G' \cong G \circ H$ where $\kappa(G) > 1$.
Then $\kappa(G') \geq 2n_2$. We shall prove the theorem by considering the following sub cases.

Case 3a: Let $S = \{(u_i, v_p)\}$ where $i \in \{1, 2, 3, ..., n_1\}$.
Then, the diam(G'') = diam(G').

Case 3b: Let S be any arbitrary collection of vertices.
Consider (u_p, v_w) and (u_q, v_w) in G'. Let the vertices (u_i, v_p), where $\{u_i\}$ is a collection of $\kappa(G)$ vertices which form a vertex cut of G and $p \in \{1, 2, 3, ..., n_2 - 1\}$, be deleted. Now, from the G - layer at v_{n_2} in G', only $\kappa(G) - 1$ vertices can be deleted, otherwise G'' becomes disconnected. Then, $d_{G''}((u_p, v_{n_2}), (u_q, v_{n_2})) \leq f(G)$ by a path $(u_p, v_{n_2}) - (u_{p+1}, v_{n_2}) - (u_{p+2}, v_{n_2}) - \ ... \ - (u_{q-1}, v_{n_2}) - (u_q, v_{n_2})$, since the deletion of $\kappa(G) - 1$ vertices from G increases the diameter to at most $f(G)$. Now, $d((u_p, v_w), (u_q, v_w)) \leq f(G)$ by a path $(u_p, v_w) - (u_{p+1}, v_{n_2}) - (u_{p+1}, v_{n_2}) - ... - (u_{q-1}, v_{n_2}) - (u_q, v_w)$ (see Figure 3). Thus, $f(G') \leq f(G)$.

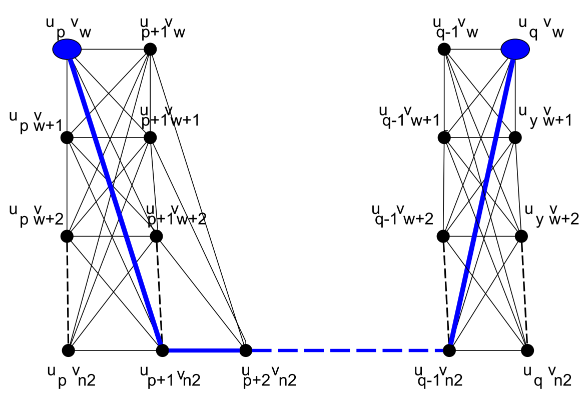

Figure 3. The vertices (u_i, v_p), where $\{u_i\}$ is a vertex cut of G and $p \in \{1, 2, 3, ..., n_2 - 1\}$ are deleted.

Consider a pair of diametral vertices (u_x, v_w) and (u_y, v_z) in G'. Let the $\kappa(G') - 1$ vertices be deleted. Since, we have already considered Cases 3a, there exist a path of length diam(G') between (u_x, v_w) and (u_y, v_z) in G'', $(u_x, v_w) - (u_{x+1}, v_p) - (u_{x+2}, v_q) - (u_{x+3}, v_r) - ... - (u_{y-1}, v_s) - (u_y, v_z)$, where the vertex (u_x, v_w) in ${}^{u_x}H$-layer will be adjacent to at least one vertex (say)(u_{x+1}, v_p) in ${}^{u_{x+1}}H$-layer, the vertex (u_{x+1}, v_p) in ${}^{u_{x+1}}H$-layer will be adjacent to at least one vertex (say) (u_{x+2}, v_q) in ${}^{u_{x+2}}H$-layer and so on (see Figure 4). Thus, the diam(G') remains the same after removing vertices in S.

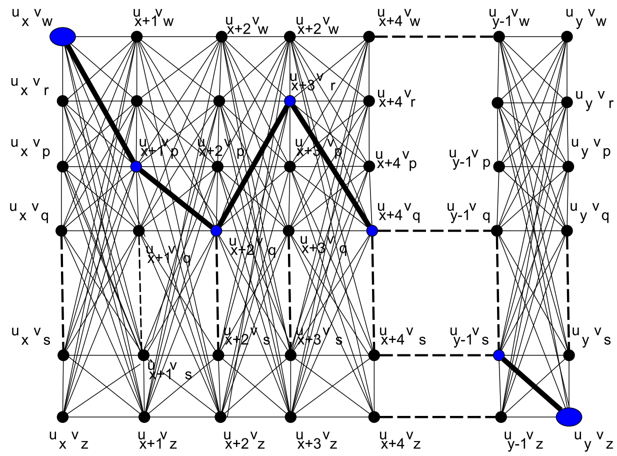

Figure 4. Arbitrary collection of $\kappa(G') - 1$ vertices are deleted from G'.

From the above cases, the result follows. ☐

4. Diameter variability of the lexicographic product of graphs

Theorem 4.1. *Let $G' \cong G \circ H$ where G and H are connected graphs. Then, $D^0(G') \geq n_1 m_2$.*

Proof. Consider a pair of diametral vertices (u_x, v_w) and (u_y, v_z) in G' where u_x and u_y in G are joined by a path $u_x - u_{x+1} - u_{x+2} ... u_{y-1}, u_y$. Let the edges $(u_i, v_p) - (u_i, v_q)$ where $p, q \in \{1, 2, ..., n_2\}$ and $i \in \{1, 2, ..., n_1\}$ in G be deleted to get G''. Then, $d_{G''}((u_x, v_w), (u_y, v_z))$ = diam(G') by a path $(u_x, v_w) - (u_{x+1}, v_w) - (u_{x+2}, v_w) - ... - (u_{y-1}, v_w) - (u_y, v_z)$. Also, $d_{G''}((u_i, v_p), (u_i, v_q)) = 2$ by a path $(u_i, v_p) - (u_{i+1}, v_p) - (u_i, v_q)$. Thus, the distance between any two vertices in G'' is not affected by the removal of these edges. ☐

Theorem 4.2. *Let $G' \cong G \circ H$ where G and H are connected graphs with diam$(H) <$ diam(G). Then, $D^0(G') \geq n_2^2 m_1 - (m_1 n_2 + 2m_1 m_2)$.*

Proof. Let u_x, u_y be a pair of diametral vertices of G by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$.

Suppose that $d_H(v_p, v_q) = L$ by a path $v_p - v_{p+1} - v_{p+2} - ... - v_{q-1} - v_q$. Consider a pair of vertices (u_x, v_p) ,(u_y, v_q) in G'. By Theorem 4.1, even if the $n_1 m_2$ edges $(u_i, v_p) - (u_i, v_q)$ where $p, q \in \{1, 2, ...n_2\}$ and $i \in \{1, 2, ...n_1\}$ are deleted, the diam(G') remains the same. Now, let the $n_2^2 m_1 - (m_1 n_2 + 2m_1 m_2)$ edges $(u_i, v_p) - (u_j, v_q)$ where $i, j \in \{1, 2, ...n_1\}$, $p, q \in \{1, 2, ...n_2\}$, v_p's and v_q's are nonadjacent vertices in H, be deleted to get G''. Then $d_{G''}((u_x, v_p), (u_y, v_q)) =$ diam(G') by a path $(u_x, v_p) - (u_{x+1}, v_p) - (u_{x+2}, v_p)$ ... $(u_i, v_p) - (u_{i+1}, v_{p+1})$ ... $(u_{y-2}, v_{q-2}) - (u_{y-1}, v_{q-1}) - (u_y, v_q)$ where $d_{G''}((u_x, v_p), (u_i, v_q))$ = diam$(G) - L$, and $d_{G''}((u_i, v_p), (u_y, v_q)) = L$. Also, $d_{G''}((u_i, v_w), (u_i, v_z))$ = diam(H) and $d_{G''}((u_i, v_w), (u_i, v_z))$ = diam$(H) + 1$ according as

$d_H(v_w, v_z)$ is even or odd respectively. Thus, diam(G') = diam(G'').
Hence, $D^0(G') \geq n_1m_2 + n_2^2m_1 - (n_1m_2 + m_1n_2 + 2m_1m_2) = n_2^2m_1 - (m_1n_2 + 2m_1m_2)$. ☐

Theorem 4.3. *If $G' \cong G \circ H$ then $D^{-k}(G') \leq \gamma_i(H)D^{-(k+2)}(G)$.*

Proof. Let $d_G(u_x, u_y)$ = diam(G) and let e_l edges are added to G to decrease the diameter of G by $k+2$. Consider a pair of diametral vertices (u_x, v_q) and (u_y, v_r) in G'. Let the e_l edges $u_xv_a - u_jv_a$ where $a \in \gamma_i(H_2)$, be added in G'. Then, clearly $d_{G'}((u_x, v_a), (u_y, v_a))$ = diam$(G') - (k+2)$. Also, $d_{G'}((u_x, v_q), (u_y, v_r))$ = diam$(G') - k$ by a path $(u_x, v_q) - (u_x, v_a) - (u_j, v_a) - \ ... \ - (u_{y-1}, v_a) - (u_y, v_r)$ where $d_{G'}((u_x, v_q), (u_x, v_a)) = d_{G'}((u_{y-1}, v_a), (u_y, v_r)) = 1$ and $d_{G'}((u_x, v_a), (u_{y-1}, v_a))$ = diam$(G) - (k+2)$. Thus, $D^{-k}(G') \leq \gamma_i(H)D^{-(k+2)}(G)$. ☐

Corollary 4.1. *Let $G' \cong G \circ H$. Then, $D^{-k}(G') \leq D^{-k}(G)$ where the edges added are not incident on the diametral vertices of G.*

Proof. Let $d_G(u_x, u_y)$ = diam(G) and let e_l edges are added to G to decrease the diameter of G by k, where added edges are not incident on the diametral vertices of G. Consider a pair of diametral vertices (u_x, v_q) and (u_y, v_r) in G'. Let the e_l edges whose end vertices are of the form $(u_i, v_1), (u_j, v_1)$, be added in G'. Then, $d_{G'}((u_x, v_1), (u_y, v_1))$ = diam$(G') - k$. Also, $d_{G'}((u_x, v_q), (u_y, v_r))$ = diam$(G') - k$ by a path $(u_x, v_q) - (u_{x+1}, v_1) - \ ... \ - (u_{y-1}, v_1) - (u_y, v_q)$ where $d_{G'}((u_x, v_q), (u_{x+1}, v_1)) = d_{G'}((u_{y-1}, v_1), (u_y, v_q)) = 1$ and $d_{G'}((u_{x+1}, v_1), (u_{y-1}, v_1))$ = diam$(H) - 2 - k$. Thus, the distance between any two vertices is at most diam$(G') - k$. ☐

Corollary 4.2. *If $G \cong P_{n_1} \circ P_{n_2}$ then $D^{-k}(G) = 1$ where $k \neq n_1/2$.*

Corollary 4.3. *If $G \cong C_{n_1} \circ C_{n_2}$ then $D^{-k}(G) = 2$ when $n_1 \geq 8$ and $1 \leq k \leq \lfloor n_1/2 \rfloor - D^*(C_{n_1})$.*

Proof. In [12], Wang et al. proved that $D^{-k}(C_m) = 2$ for all $m \geq 8$ and $1 \leq k \leq \lfloor m/2 \rfloor - D^*(C_m)$, where $D^*(C_m)$ denote the minimum diameter among those graphs obtained by adding two edges [$e_1 = (0, \lfloor m/2 \rfloor)$ and $e_2 = (\lfloor m/4 \rfloor, \lceil 3m/4 \rceil)$ for $m \equiv 2$ mod 4 or $e_1 = (0, \lfloor m/2 \rfloor)$ and $e_2 = (\lfloor m/4 \rfloor, \lfloor 3m/4 \rfloor)$ for $m \equiv$ 0,1,3 mod 4] to C_m. Note that, in this case,

$$D^*(C_m) = \begin{cases} \lfloor m/4 \rfloor + 1 & m \equiv 0,1,2 \mod 4, \\ \lfloor m/4 \rfloor + 2 & m \equiv 3 \mod 4. \end{cases}$$

Then, the corollary follows from the above result. ☐

5. Concluding Remarks and Further Scope

Two main interconnection network models - grids and tori- motivated us to study the graph product structures from the view point of interconnection models. We have seen several papers in which the distance notions have been studied and the graph product considered mainly in those papers was the Cartesian product. In [14], connectivity of Lexicographic product is studied and this motivated us to think the Lexicographic product as a network model. In this paper, we have studied wide diameter, diameter variability and fault diameter of the lexicographic product of graphs since it is important in the design of interconnection networks and we established some bounds for these parameters. We have noted that $H_1 \circ H_2$ has better wide diameter, diameter variability, fault diameter as compared to that of H_1. Hence $H_1 \circ H_2$ can be a better network model as compared to that of H_1. One can extend this work by characterizing the graphs for which the equality of the bounds is attained. We have discussed the diameter notions based on connectivity. One may think of these notions based on some other graph parameter which may be helpful to study the reliability and efficiency of the model.

Acknowledgement

- The authors thank the referee for the suggestions for the refinement of the paper.
- The first author thanks the PURSE programme of Department of Science and Technology, Government of India for awarding a research fellowship.
- The second author thanks the University Grants Commission for granting financial assistance for the minor research project MRP(S)-970/10-11/KLMG015/ UGC-SWRO.

References

[1] Chithra M.R., A. Vijayakumar, Diameter variability of the Cartesian product of graphs, *Discrete Math. Algorithms Appl.* **6**, (1) (2014), 1450001–1450009.

[2] Chithra M.R., A. Vijayakumar, The diameter variability of the product graphs, *J. Comb. Math. Comb. Comput.* ***103*** *(2017)*, 159 - 170.

[3] F.R.K. Chung, M.R. Garey, Diameter bounds for altered graphs, *J. Graph Theory* **8** (4) (1984), 511–534.

[4] N. Graham, F. Harary, Changing and unchanging the diameter of a hypercube, *Discrete Appl. Math.* **37/38** (1992), 265–274.

[5] F. Hsu, On container width and length in graphs, groups, and networks, *IEICE transactions on fundamentals* **E77-A** (1994), 668–680.

[6] L.H. Hsu, C.K. Lin, *Graph theory and interconnection networks*, CRC Press, 2009.

[7] W. Imrich, S. Klavžar, R. Hammack, *Handbook of Product Graphs*, CRC Press, 2011.

[8] M.S. Krishnamoorthy, B. Krishnamurthy, Fault diameter of interconnection networks, *Comput. Math. Appl.* **13** (1987), 577–582.

[9] B. Liu, X. Zhang, Wide diameter and diameter of networks, *Ars Combin.* **89** (2008).

[10] C. Peyrat, Diameter vulnerability of graphs, *Discrete Appl. Math.* **9** (1984) 245–250.

[11] A.A. Schoone, H.L. Bodlaeder, J. van Leeuwen, Diameter increase caused by edge deletion, *J. Graph Theory* **11** (3) (1987), 409–427.

[12] J.J. Wang, T.Y. Ho, D. Ferrero, T.Y. Sung, Diameter variability of cycles and tori., *Inform. Sci.* **17** (2008), 2960–2967.

[13] D.B. West, *Introduction to Graph Theory*, Prentice Hall of India, 2003.

[14] C. Yang, J.M. Xu, Connectivity of lexicographic product and direct product of graphs, *Ars Combin.* **111** (2013), 3–12.

[15] H.-X. Ye, C.Yang, J.-M. Xu, Diameter vulnerability of graphs by edge deletion, *Discrete Math.* **309** (4) (2009), 1001–1006.