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On topological integer additive set-labeling of star graphs

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Abstract

For integer $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. In this paper, we determine the order of a star graph $K_{1,n}$ of $n + 1$ vertices, such that $K_{1,n}$ admits a topological integer additive set-labeling (TIASL) with respect to a set X. We also give a condition for a star graph $K_{1,n}$ such that $K_{1,n}$ is not a TIASL-graph on set X.

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1. Introduction

Research on graph labeling was started after Rosa introduced the concept of β-valuation of graphs [2]. The concept of set-assignment [1], which is defined as follows, is analogous to the number valuations of graphs. Let $G(V, E)$ be a graph, X be a non-empty set, and $\mathcal{P}(X)$ be the power set of X. Then the set-valued function $f : V(G) \to \mathcal{P}(X)$ is called the *set-assignment* of vertices of G. We can also define a set-assignment of edges or both elements (vertices and edges)

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in a similar way. A set-assignment of a graph G is called a *set-labeling* (or a *set-valuation*) of G if it is injective.

In this paper, we combine the concept of the vertex set-labeling and the set topology. A *topology* on a non-empty set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. The set X and $\emptyset$ are in $\mathcal{T}$.
2. The union of the elements of any sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.

Let G be a connected, simple, and finite graph. Let X be a finite non-empty set of non-negative integers. A vertex set-labeling $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ is called a *topological integer additive set-labeling* (TIASL) of G if f is an injective function, $\{f(V(G)) \cup \{\emptyset\}\}$ is a topology of X, and there exists the corresponding function $f^+ : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ such that for every edge $uv \in E(G)$, $f^+(uv) = f(u) + f(v)$. We recall that the *sumset* (or *Minkowski sum* [4]) of two non-empty sets A and B, denoted by $A + B$, is defined by $A + B = \{a + b \mid a \in A; \, b \in B\}$. A graph G which admits TIASL is called a *topological integer additive set-labelled graph* (in short, TIASL-graph).

The topological integer additive set-labeling was introduced by Sudev and Germina [3]. They give a tight condition for a TIASL-graph. They proved that G is a TIASL-graph if and only if G has at least one pendant vertex. They also characterized all TIASL-graphs with respect to either the indiscrete topology or Sierpenski's topology.

Let G be a graph having a pendant vertex. For integer $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. It seems that every graph G admits a topological integer additive set-labeling on set X if the cardinality of X is big enough. In [3], Sudev and Germina proved that an (n, m)-tadpole graph is a TIASL-graph. An (n, m)*-tadpole graph* is a graph obtained from one copies of cycle C_n, $n \geq 3$, and path P_m, $m \geq 2$, by identifying an end point of the path P_m to a vertex of cycle C_n. They have shown that an (n, m)-tadpole graph of $n + m - 1$ vertices admits a topological integer additive set-labeling on set $X = \{0, 1, 2, \ldots, k\}$ where $k = 2(m + n) - 5$.

In this paper, we consider a star graph $K_{1,n}$ of $n+1$ vertices and a given set $X = \{0, 1, 2, \ldots, k\}$ where $k \geq 2$. We obtain two main results. The first result is related to the order of a star graph $K_{1,n}$ such that $K_{1,n}$ is a TIASL-graph on the set X.

Theorem 1.1. *Let $K_{1,n}$ be a star graph with $n + 1$ vertices. For $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. If n is one of the positive integers below, then $K_{1,n}$ is a TIASL-graph on set X.*

(a) $n \in \{1, 2, \ldots, 4k - 4\}$*, or*
(b) $n = 2^{r_1} + r_2 - 2$ *for* $r_1 \in \{2, 3, \ldots, k - 1\}$ *and* $r_2 \in \{1, 2\}$.

In the second result, we give a condition for a star graph $K_{1,n}$ such that $K_{1,n}$ is not a TIASL-graph on set X.

Theorem 1.2. *Let $K_{1,n}$ be a star graph with $n + 1$ vertices. For $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. If $3 \cdot 2^{k-1} - 2 \leq n \leq 2^{k+1} - 2$, then $K_{1,n}$ is not a TIASL-graph on set X.*

In order to prove both theorems above, we also consider the following useful proposition.

Proposition 1.1. *Let S be a finite non-empty set of non-negative integers with s elements. Then $\mathcal{P}(S)$ is a topology of S with 2^s elements.*

2. Proof of Theorem 1.1

For an integer $k \geq 2$, let $X = \{0, 1, 2, \ldots, k\}$. First we must consider the following proposition which has been proved by Sudev and Germina [3].

Proposition 2.1. *Let $f : V(G) \to \mathcal{X} - \{\emptyset\}$ is a TIASL of a graph G. Then, the vertices whose set-labels containing the maximal element of the ground set X are pendant vertices which are adjacent to the vertex having the set-label $\{0\}$.*

From Proposition 2.1, if f is a TIASL of a graph G, then there exists a vertex v of G such that $f(v) = \{0\}$. Therefore, we must construct a topology of X containing $\{0\}$.

Proposition 2.2. *There exists a topology $\mathcal{T}$ containing $\{0\}$ on set X such that $|\mathcal{T}| = t$, where t is one of the positive integers as follows.*

(a) $3 \leq t \leq 4k - 2$, *or*
(b) $t = 2^{r_1} + r_2$ *for* $r_1 \in \{2, 3, \ldots, k-1\}$ *and* $r_2 \in \{1, 2\}$.

Proof. We distinguish two cases.

Part 2.2.1. $3 \leq t \leq 4k - 2$

Let $I_0 = X$. For $i \in \{1, 2, \ldots, k\}$, we define recursively

$$I_i = I_{i-1} - \max(I_{i-1})$$

and

$$\mathcal{I}_i = \{I_k\} \cup \{I_s \mid 0 \leq s \leq i-1\}.$$

Note that $|\mathcal{I}_i| = i + 1$. We also define $I_i^* = I_{k-i} - \{0\}$ and $\mathcal{I}_i^* = \{I_s^* \mid 1 \leq s \leq i\}$. In this case, $|\mathcal{I}_i^*| = i$. For $j \in \{1, 2, \ldots, k-2\}$, we define

$$\widehat{I}_j = I_{j+2} \cup \{k-1\}$$

and

$$\widehat{I}_j^* = \widehat{I}_j - \{0\}.$$

We also define

$$\mathcal{I}_j^{**} = \widehat{\mathcal{I}}_j \cup \widehat{\mathcal{I}}_j^*,$$

where $\widehat{\mathcal{I}}_j = \{\widehat{I}_s \mid 1 \leq s \leq j\}$ and $\widehat{\mathcal{I}}_j^* = \{\widehat{I}_s^* \mid 1 \leq s \leq j\}$. Note that $|\mathcal{I}_j^{**}| = 2j$.

By some definitions above, we define a collection-set $\mathcal{T}_1$ with t elements as follows.

$$\mathcal{T}_1 = \{\emptyset\} \cup \begin{cases} \mathcal{I}_{t-2}, & \text{if } 3 \leq t \leq k+2, \\ \mathcal{I}_k \cup \mathcal{I}_{t-k-2}^*, & \text{if } k+3 \leq t \leq 2k+2, \\ \mathcal{I}_k \cup \mathcal{I}_{k-1}^* \cup \mathcal{I}_{\frac{t-1}{2}-k}^{**}, & \text{if } 2k+3 \leq t \leq 4k-3 \text{ and } t \text{ is odd}, \\ \mathcal{I}_k \cup \mathcal{I}_k^* \cup \mathcal{I}_{\frac{t-2}{2}-k}^{**}, & \text{if } 2k+4 \leq t \leq 4k-2 \text{ and } t \text{ is even}. \end{cases}$$

Note that $I_k = \{0\} \in \mathcal{T}_1$. Now, we will show that $\mathcal{T}_1$ is a topology of X.

Let A and B be two distinct elements of $\mathcal{T}_1$ where $|A| \leq |B|$. If $A \subset B$, then $A \cap B = A \in \mathcal{T}_1$ and $A \cup B = B \in \mathcal{T}_1$. Otherwise, we distinguish six cases.

1. $A \in \mathcal{I}_k$ and $B \in \mathcal{I}_i^*$ for $i \in \{1, 2, \ldots, k\}$ (or $B \in \mathcal{I}_k$ and $A \in \mathcal{I}_i^*$)
   Then $A \cap B \in \mathcal{I}_i^*$ and $A \cup B \in \mathcal{I}_k$.
2. $A \in \mathcal{I}_k$ and $B \in \widehat{\mathcal{I}}_j$ for $j \in \{1, 2, \ldots, k-2\}$ (or $B \in \mathcal{I}_k$ and $A \in \widehat{\mathcal{I}}_j$)
   Then $A \cap B \in \mathcal{I}_k$ and either $A \cup B \in \mathcal{I}_k$ or $A \cup B \in \widehat{\mathcal{I}}_j$.
3. $A \in \mathcal{I}_k$ and $B \in \widehat{\mathcal{I}}_j^*$ for $j \in \{1, 2, \ldots, k-2\}$ (or $B \in \mathcal{I}_k$ and $A \in \widehat{\mathcal{I}}_j^*$)
   Then $A \cap B \in \mathcal{I}_k^*$ and either $A \cup B \in \widehat{\mathcal{I}}_j$ or $A \cup B \in \mathcal{I}_k$.
4. $A \in \mathcal{I}_i^*$ and $B \in \widehat{\mathcal{I}}_j$ for $i \in \{k-1, k\}$ and $j \in \{1, 2, \ldots, k-2\}$ (or $B \in \mathcal{I}_i^*$ and $A \in \widehat{\mathcal{I}}_j$)
   Then either $A \cap B = \emptyset$ or $A \cap B \in \mathcal{I}_i^*$ or $A \cap B \in \widehat{\mathcal{I}}_j^*$. Also, we have either $A \cup B \in \widehat{\mathcal{I}}_j$ or $A \cup B \in \mathcal{I}_k$.
5. $A \in \mathcal{I}_i^*$ and $B \in \widehat{\mathcal{I}}_j^*$ for $i \in \{k-1, k\}$ and $j \in \{1, 2, \ldots, k-2\}$ (or $B \in \mathcal{I}_i^*$ and $A \in \widehat{\mathcal{I}}_j^*$)
   Then either $A \cap B \in \mathcal{I}_k$ or $A \cap B = \emptyset$. Also, we have either $A \cup B \in \mathcal{I}_i^*$ or $A \cup B \in \widehat{\mathcal{I}}_j^*$.
6. $A \in \widehat{\mathcal{I}}_j$ and $B \in \widehat{\mathcal{I}}_j^*$ for $j \in \{1, 2, \ldots, k-2\}$ (or $B \in \widehat{\mathcal{I}}_j$ and $A \in \widehat{\mathcal{I}}_j^*$)
   Then $A \cap B \in \widehat{\mathcal{I}}_j^*$ and $A \cup B \in \widehat{\mathcal{I}}_j$.

From the six cases above, we obtain that every two distinct elements A and B in $\mathcal{T}_1$ satisfy $A \cap B \in \mathcal{T}_1$ and $A \cup B \in \mathcal{T}_1$. Since $\mathcal{T}_1$ also contains $\emptyset$ and X, it implies that $\mathcal{T}_1$ is a topology of X.

Part 2.2.2. $t = 2^{r_1} + r_2$ for $r_1 \in \{2, 3, \ldots, k-1\}$ and $r_2 \in \{1, 2\}$

We define the sets $J_{r_1} = \{0, 1, \ldots, r_1\}$. Now, we consider an element a of X such that $a \neq \max(X)$. Let $X^- = X - \{a\}$. By these definitions, we define a collection-set $\mathcal{T}_2$ with t elements as follows.

$$\mathcal{T}_2 = \begin{cases} \mathcal{P}(J_{r_1}) \cup \{X\}, & \text{if } t = 2^{r_1} + 1, \\ \mathcal{P}(J_{r_1}) \cup \{\{X\}, \{X^-\}\}, & \text{if } t = 2^{r_1} + 2. \end{cases}$$

Now, we will show that $\mathcal{T}_2$ is a topology of X.

Note that $\emptyset, \{0\}, X \in \mathcal{T}_2$. Let A and B be two distinct elements of $\mathcal{T}_2$. We distinguish three cases.

1. $A, B \in \mathcal{P}(J_{r_1})$
   By Proposition 1.1, then $A \cap B \in \mathcal{P}(J_{r_1})$ and $A \cup B \in \mathcal{P}(J_{r_1})$.
2. $A \in \mathcal{P}(J_{r_1})$ or $A = X^-$, and $B = X$
   Then $A \cup B = B$ and $A \cap B = A$.
3. $A \in \mathcal{P}(J_{r_1})$ and $B = X^-$.
   Then $A \cap B \in \mathcal{P}(J_{r_1})$ and $A \cup B \in \{X, X^-\}$.

From three cases above, we obtain that $A \cap B, A \cup B \in \mathcal{T}_2$. ■

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$, where v_1 is the centre of $K_{1,n}$. Let $\mathcal{T}_t$ be a topology of X with t elements satisfying Proposition 2.2. Let $\mathcal{T}_t' = \mathcal{T}_t - \{\emptyset\}$. Now, we define a vertex injective labeling $f : V(S_n) \to \mathcal{T}_t'$ such that $f(v_1) = \{0\}$. Since for $2 \leq i \leq n$, v_1 is adjacent to v_i and $f(v_1) + f(v_i) = f(v_i) \in \mathcal{T}_t' \subseteq \mathcal{P}(X)$, we obtain that $K_{1,n}$ is a TIASL-graph on the set X. ■

3. Proof of Theorem 1.2

Let S be a finite non-empty set of non-negative integers. From Proposition 1.1, it is clear that $\mathcal{P}(S)$ is a topology on the set S. Let $\mathcal{A} \subset \mathcal{P}(S)$. On some cases of $\mathcal{A}$, the collection $\mathcal{P}(S) - \mathcal{A}$ is not a topology on the set S. In proposition below, we prove that if $L \in \mathcal{P}(S)$ is not an element of a topology $\mathcal{T}$ on the set S, then there exists an element $l \in L$ such that $\{l\} \notin \mathcal{T}$.

Proposition 3.1. *Let S be a finite non-empty set of non-negative integers with s elements, and $\mathcal{T}$ be a topology of S. Let $A \in \mathcal{P}(S)$ but $A \notin \mathcal{T}$. Then there exists an element a of A such that $\{a\} \notin \mathcal{T}$.*

Proof. By the definition of a topology, we have $A \neq \emptyset$. Let $A = \{a_1, a_2, \ldots, a_r\}$. If $r = 1$, then we are done. Now, we assume that $r \geq 2$. Suppose that $\{a_i\} \in \mathcal{T}$ for $1 \leq i \leq r$. Note that $\bigcup_{i=1}^{r}\{a_i\} = A \notin \mathcal{T}$, a contradiction. ■

Let the collection $\mathcal{T}$ be a topology on the set S which is satisfying Proposition 3.1 above and the set $L \in \mathcal{P}(S)$ but $L \notin \mathcal{T}$. Let $l \in L$ and $\{l\} \notin \mathcal{T}$. So, there are no two distinct sets A_1 and A_2 in $\mathcal{T}$ such that $A_1 \cap A_2 = \{l\}$. Therefore, we need to determine how many elements of $\mathcal{T}$ such that $\mathcal{T}$ may be a topology on the set S.

Proposition 3.2. *Let S be a finite non-empty set of non-negative integers with $s \geq 2$ elements. Let $\mathcal{A}$ be a non-empty collection-set, where every element of $\mathcal{A}$ is an element of $\mathcal{P}(S)$. If $\mathcal{P}(S) - \mathcal{A}$ is a topology of S, then $|\mathcal{P}(S) - \mathcal{A}| \leq 3 \cdot 2^{s-2}$.*

Proof. Let $S = \{v_1, v_2, \ldots, v_s\}$. By Proposition 1.1, $\mathcal{P}(S)$ is a topology of S with 2^s elements. Let $\mathcal{A}$ be a non-empty collection-set, where every element of $\mathcal{A}$ is element of $\mathcal{P}(S)$. Let $\mathcal{T} = \mathcal{P}(S) - \mathcal{A}$ be a topology of S.

Let $E \in \mathcal{A}$. Since $\mathcal{T}$ is a topology of S, it is clear that $E \neq \emptyset$ and $E \neq S$. By considering Proposition 3.1, without lost of generality, let $v_s \in E$ and $\{v_s\} \notin \mathcal{T}$. We can say that $\{v_s\} \in \mathcal{A}$.

Let $\mathcal{B} = \{\{v_s, v_i\} \mid 1 \leq i \leq s-1\}$. Note that $|\mathcal{B}| = s-1$. Since $\mathcal{T}$ is a topology of S, then at least $s-2$ elements of $\mathcal{B}$ are in $\mathcal{A}$. Without lost of generality, let $\widehat{\mathcal{B}} = \{\{v_s, v_i\} \mid 1 \leq i \leq s-2\} \subseteq \mathcal{A}$. Now, we define $B = \{v \mid \{v_s, v\} \in \widehat{\mathcal{B}}\}$. We also define $\mathcal{C} = \{\{v_s\} \cup C \mid C \in \mathcal{P}(B)\}$. Note that $|\mathcal{C}| = 2^{s-2}$, $\{v_s\} \in \mathcal{C}$, and $\mathcal{B} \subseteq \mathcal{C}$. Note that for any distinct elements $C_1, C_2 \in \mathcal{C}$, we have $C_1 \cup C_2$ and $C_1 \cap C_2$ are also in $\mathcal{C}$. However, every $C \in \mathcal{C}$ satisfy $C \cap \{v_s, v_{s-1}\} = \{v_s\} \in \mathcal{A}$. So, it must be $\mathcal{C} \subseteq \mathcal{A}$. Therefore, we obtain

$$|\mathcal{P}(S) - \mathcal{A}| \leq 2^s - 2^{s-2} = 3 \cdot 2^{s-2}.$$

■

Proof of Theorem 2. Theorem 1.2 is a direct consequence of Propositions 1.1 and 3.2. ■

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