Electronic Journal of Graph Theory and Applications **6** (2) (2018), 362–370

Electronic Journal of
Graph Theory and Applications

Characterization of perfect matching transitive graphs

Ju Zhou

Department of Mathematics
Kutztown University of Pennsylvania Kutztown, PA 19530, USA

zhou@kutztown.edu

Abstract

A graph G is perfect matching transitive, shortly PM-transitive, if for any two perfect matchings M and N of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f_e(M) = N$, where $f_e(uv) = f(u)f(v)$. In this paper, the author proposed the definition of PM-transitive, verified PM-transitivity of some symmetric graphs, constructed several families of PM-transitive graphs which are neither vertex-transitive nor edge-transitive, and discussed PM-transitivity of generalized Petersen graphs.

Keywords: vertex-transitive, edge-transitive, symmetric, PM-transitive
Mathematics Subject Classification : 68R10, 05C60
DOI: 10.5614/ejgta.2018.6.2.15

1. Introduction

An automorphism of a graph is a form of symmetry in which the graph is mapped onto itself while preserving the edge-vertex connectivity. Formally, an automorphism of a graph $G = (V(G), E(G))$ is a permutation f of the vertex set $V(G)$, such that the pair of vertices uv is an edge of G if and only if $f(u)f(v)$ is also an edge of G. That is, it is a graph isomorphism from G to itself. Every graph automorphism f induces a map $f_e : E(G) \mapsto E(G)$ such that $f_e(uv) = f(u)f(v)$. For any vertex set $X \subseteq V(G)$ and edge set $M \subseteq E(G)$, denote $f(X) = \{f(v) : v \in X\}$ and

Received: 20 March 2018, Revised: 2 September 2018, Accepted: 27 September 2018.

$f_e(M) = \{f_e(uv) = f(u)f(v) : uv \in M\}$. In this paper, we also use f to denote the map induced by the automorphism f.

A graph G is *vertex-transitive* [11] if for any two given vertices v_1 and v_2 of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f(v_1) = v_2$. In other words, a graph is vertex-transitive if its automorphism group acts transitively upon its vertices. A graph is vertex-transitive if and only if its graph complement is vertex-transitive (since the group actions are identical). For example, the finite Cayley graphs, Petersen graph, and $C_n \times K_2$ with $n \geq 3$, are vertex-transitive.

A graph G is *edge-transitive* if for any two given edges e_1 and e_2 of G, there is an automorphism of G that maps e_1 to e_2. In other words, a graph is edge-transitive if its automorphism group acts transitively upon its edges. The complete bipartite graph $K_{m,n}$, Petersen graph, and the cubical graph $C_n \times K_2$ with $n = 4$, are edge-transitive.

A graph G is *symmetric* or *arc-transitive* if for any two pairs of adjacent vertices $u_1|v_1$ and $u_2|v_2$ of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f(u_1) = u_2$ and $f(v_1) = v_2$. In other words, a graph is symmetric if its automorphism group acts transitively upon ordered pairs of adjacent vertices, that is, upon edges considered as having a direction. The cubical graph $C_n \times K_2$ with $n = 4$, and Petersen graph are symmetric graphs.

Every connected symmetric graph must be both vertex-transitive and edge-transitive, and the converse is true for graphs of odd degree [2]. However, for even degree, there exist connected graphs which are vertex-transitive and edge-transitive, but not symmetric [3]. Every symmetric graph without isolated vertices is vertex-transitive, and every vertex-transitive graph is regular. However, not all vertex-transitive graphs are symmetric (for example, the edges of the truncated tetrahedron), and not all regular graphs are vertex-transitive (for example, the Frucht graph and Tietze's graph).

A lot of work has been done about the relationship between vertex-transitive graphs and edge-transitive graphs. Some of the related results can be found in [3]-[17]. In general, edge-transitive graphs need not be vertex-transitive. The Gray graph is an example of a graph which is edge-transitive but not vertex-transitive. Conversely, vertex-transitive graphs need not be edge-transitive. The graph $C_n \times K_2$, where $n \geq 5$ is vertex-transitive but not edge-transitive.

A graph G is *perfect matching transitive*, shortly *PM-transitive*, if for any two perfect matchings M and N of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f_e(M) = N$, where f_e is the map induced by f.

Are there any PM-transitive graphs? What kind of properties do PM-transitive graphs have? What is the relationship between PM-transitive and edge-transitive? What is the relationship between PM-transitive and vertex-transitive? What is the relationship between PM-transitive and symmetric?

In section 2, the author verified that some well known symmetric graphs such as C_{2n}, K_{2n}, $K_{n,n}$ and Petersen graph are PM-transitive, and constructed several families of PM-transitive graphs which are neither vertex-transitive nor edge-transitive. In section 3, the author discussed some methods to generate new PM-transitive graphs. In section 4, the author proved that all the generated Petersen graphs except the Petersen graph are non-perfect matching transitive. In section 5, the author provided some examples which have one or more properties of vertex-transitive, edge-transitive, or PM-transitive.

2. PM-Transitive Graphs

In this section, we characterize some PM-transitive graphs.

Theorem 2.1. *Every even cycle $G = C_{2n}$ with $n \geq 2$ is PM-transitive.*

Proof. Let $C_{2n} = u_1u_2 \cdots u_{2n}u_1$ with $n \geq 2$. Note that C_{2n} has exactly two perfect matchings, denoted by $M = \{u_1u_2, u_3u_4, \cdots, u_{2n-1}u_{2n}\}$ and $N = \{u_2u_3, u_4u_5, \cdots, u_{2n}u_1\}$, respectively. Define $f : V(G) \mapsto V(G)$ such that $f(u_i) = u_{i+1}$. Since for any edge $u_iu_{i+1} \in E(C_{2n})$, $f(u_iu_{i+1}) = f(u_i)f(u_{i+1}) = u_{i+1}u_{i+2} \in E(C_{2n})$, f is an automorphism of C_{2n}. By the definition of M, N and f, $f_e(M) = N$ follows and G is PM-transitive. □

Theorem 2.2. *Every even complete graph $G = K_{2n}$ with $n \geq 2$ is PM-transitive.*

Proof. Let M and N be two perfect matchings of G. Let $M \triangle N$ denote the symmetric difference of M and N, and also the graph induced by $M \triangle N$. Since every vertex of G is incident with exactly one edge in M and exactly one edge in N, $M \triangle N$ is a disjoint union of even cycles. Define $f : V(G) \mapsto V(G)$ such that $f(u) = u$ if $u \in V(M \cap N)$ and $f(u_i) = u_{i+1}$ if $u_i \in V(C_{2s})$, where $C_{2s} = u_1u_2 \cdots u_{2s}u_1$ is an even cycle in $M \triangle N$. Since G is a complete graph, f is an automorphism of G. Let $uv \in M$. If $uv \in M \cap N$, then $f_e(uv) = f(u)f(v) = uv \in N$. If $uv \in M \setminus N$, then uv is an edge of some even cycle of $M \triangle N$. Let $uv = u_iu_{i+1}$. Then $f_e(uv) = f_e(u_iu_{i+1}) = f(u_i)f(u_{i+1}) = u_{i+1}u_{i+2} \in N$. Therefore, $f_e(M) = N$ follows and G is PM-transitive. □

Theorem 2.3. *Every complete bipartite graph $G = K_{n,n}$ is PM-transitive.*

Proof. Let M and N be two perfect matchings of G. Then $M \triangle N$ is a disjoint union of even cycles. Define f such that $f(u) = u$ if $u \in V(M \cap N)$, and $f(u_1) = u_1$ and $f(u_i) = u_{2s+2-i}$ for $2 \leq i \leq 2s$, where $C_{2s} = u_1u_2 \cdots u_{2s}u_1$ is an even cycle in $M \triangle N$. Let (X, Y) be the bipartition of G. By the definition of f, $f(X) = X$ and $f(Y) = Y$. Since G is a complete bipartite graph, f is an automorphism of G. Let $uv \in M$. If $uv \in M \cap N$, then $f_e(uv) = f(u)f(v) = uv \in N$. If $uv \in M \setminus N$, then uv is an edge of some even cycle of $M \triangle N$. Without loss of generality, suppose that $uv = u_iu_{i+1}$. Then $f_e(uv) = f_e(u_iu_{i+1}) = f(u_i)f(u_{i+1}) = u_{2s+2-i}u_{2s+2-(i+1)} \in N$. Therefore, $f_e(M) = N$ follows and G is PM-transitive. □

Theorem 2.4. *The Petersen graph is PM-transitive.*

Proof. Let G be the graph obtained from the union of two cycles $u_1u_2u_3u_4u_5u_1$ and $v_1v_3v_5v_2v_4v_1$ by connecting u_iv_i, where $i = 1, 2, \cdots, 5$. Then G is a Petersen graph. Let M be a perfect matching of G and let $N = \{u_1v_1, u_2v_2, u_3v_3, u_4v_4, u_5v_5\}$.

Claim: If $M \neq N$, then $|M \cap N| = 1$. Suppose that $M \neq N$. Then $G - V(M \cap N)$ is an even cycle or unions of even cycles. Since the Petersen graph does not have 2-cycles, 4-cycles, or 6-cycles, $|M \cap N| \neq 4, 3, 2$.

Without loss of generality, we assume that $N = \{u_1v_1, u_2u_3, u_4u_5, v_2v_4, v_3v_5\}$. Define $f : V(G) \mapsto V(G)$ such that $f(v) = v$ if $v \in \{u_1, u_2, u_5, v_1\}$, $f(u_3) = v_2$, $f(u_4) = v_5$, $f(v_2) = u_3$, $f(v_3) = v_4$, $f(v_4) = v_3$ and $f(v_5) = v_4$. Then f is an automorphism of G such that $f_e(M) = N$. □

Theorem 2.5. *Let G be the graph obtained from $C_{2n+2} = u_0u_1u_2 \cdots u_nv_0v_nv_{n-1} \cdots v_1u_0$ with $n \geq 2$ by connecting u_iv_{i+1} and v_iu_{i+1}, where $i = 1, 2, \cdots, n-1$. Then G is PM-transitive.*

Proof. We prove this result by induction. When n=2, G has four different perfect matchings. Denote them by $M_1 = \{u_0v_1, u_1v_2, u_2v_0\}$, $M_2 = \{u_0u_1, v_1v_2, u_2v_0\}$, $M_3 = \{u_0v_1, u_1u_2, v_0v_2\}$, and $M_4 = \{u_0u_1, u_2v_1, v_0v_2\}$, respectively. To prove that G is perfect matching transitive, it suffices to prove that for $M \in \{M_2, M_3, M_4\}$, there is an automorphism f of G such that $f_e(M) = M_1$. Furthermore, we can restrict that v_0 and u_0 are fixed under f.

If $M = M_2$, then define $f_2 : V(G) \mapsto V(G)$ such that $f_2(u_1) = v_1$, $f_2(v_1) = u_1$, and $f_2(v) = v$ for $v \in V(G)\backslash\{u_1, v_1\}$. Then f_2 is an automorphism such that $f_2(M_2) = M_1$, and v_0 and u_0 are fixed under f_2.

If $M = M_3$, then define $f_3 : V(G) \mapsto V(G)$ such that $f_3(u_2) = v_2$, $f_3(v_2) = u_2$ and $f_3(v) = v$ for $v \in V(G)\backslash\{u_2, v_2\}$. Then f_3 is an automorphism such that $f_3(M_3) = M$, and v_0 and u_0 are fixed under f_3.

If $M = M_4$, then define $f_4 : V(G) \mapsto V(G)$ such that $f_4(u_1) = v_1$, $f_4(v_1) = u_1$, $f_4(u_2) = v_2$, $f_4(v_2) = u_2$ and $f_4(v) = v$ for $v \in V(G)\backslash\{u_1, v_1, u_2, v_2\}$. Then f_4 is an automorphism such that $f_4(M_4) = M_1$, and v_0 and u_0 are fixed under f_4.

Now assume the result is true for $n = m \geq 2$. That is, for any two different perfect matchings M and N of G, there is an automorphism f of G such that $f_e(M) = N$, and v_0 and u_0 are fixed under f.

We want to prove the result is true for $n = m + 1$. Let M and N be two perfect matchings of G. Without loss of generality, assume that $u_0u_1 \in M$.

Case 1. $u_0u_1 \in N$. In this case, $M_1 = M - \{u_0u_1\}$ and $N_1 = N - \{u_0u_1\}$ are two perfect matchings of $G_1 = G - \{u_0, u_1\}$. By the induction hypothesis, there is an automorphism f_1 of G_1 such that $f_1(M_1) = N_1$, and v_0 and v_1 are fixed under f_1. Define $f : V(G) \mapsto V(G)$ such that $f(u_0) = u_0$, $f(u_1) = u_1$ and $f(v) = f_1(v)$ if $v \in V(G_1)$. Then f is an automorphism of G such that $f_e(M) = N$.

Case 2. $u_0u_1 \notin N$. In this case, $u_0v_1 \in N$. Define $f_1 : V(G) \mapsto V(G)$ such that $f_1(u_1) = v_1$, $f_1(v_1) = u_1$ and $f_1(v) = v$ for $v \in V(G) \setminus \{u_1, v_1\}$. Let $N_1 = f_1(N)$. Then $u_0u_1 \in N_1$. By Case 1, there is an automorphism f_2 of G such that $f_2(M) = N_1$, and v_0 and u_0 are fixed under f_2. Then $f = f_1^{-1}f_2$ is an automorphism of G such that $f_e(M) = N$. □

Theorem 2.6. *Let G be the graph obtained from $C_{2n+2} = u_0u_1u_2 \cdots u_nv_0v_nv_{n-1} \cdots v_1u_0$ with $n \geq 2$ by connecting u_iv_{i+1} and v_iu_{i+1} for $i = 1, 2, \cdots, n-1$, and u_iv_i for $i = 1, 2, \cdots, n$. Then G is PM-transitive.*

Proof. Since u_iv_i is not contained in any perfect matching of G, where $i = 1, 2, \cdots, n$, following exactly the same argument to the proof of Theorem 4, we can prove that G is PM-transitive. □

Theorem 2.7. *Let G be the graph obtained from $K_{n+1,n+1}$ by removing one edge u_0v_0. Then G is PM-transitive.*

Proof. We prove this result by induction. Let (X, Y) be the bipartition of $K_{n+1,n+1}$ and let $X = \{u_0, u_1, u_2, \cdots, u_n\}$ and $Y = \{v_0, v_1, v_2, \cdots, v_n\}$. When n=2, by Theorem 2.5, G is PM-transitive and furthermore, for any two different perfect matchings M and N of G, there is an automorphism f of G such that $f_e(M) = N$, and v_0 and u_0 are fixed under f. Now assume the result is true for $n = m \geq 2$. That is, for any two different perfect matchings M and N of G, there is an automorphism f of G such that $f_e(M) = N$, and v_0 and u_0 are fixed under f.

We want to prove the result is true for $n = m + 1$. Let M and N be two perfect matchings of G. Without loss of generality, assume that $u_0u_1, v_0v_1 \in M$.

Case 1: $u_0v_1, v_0u_1 \in N$. Then, $M_1 = M - \{u_0v_1, v_0u_1\}$ and $N_1 = N - \{u_0v_1, v_0u_1\}$ are two perfect matchings of $G_1 = G - \{u_0, u_1, v_0, v_1\} \cong K_{n-1,n-1}$. By Theorem 2.3, there is an automorphism f_1 of G_1 such that $f_1(M_1) = N_1$, $f_1(X \setminus \{u_0, u_1\}) = X \setminus \{u_0, u_1\}$ and $f_1(Y \setminus \{v_0, v_1\}) = Y \setminus \{v_0, v_1\}$. Define $f : V(G) \mapsto V(G)$ such that $f(v) = f_1(v)$ if $v \in V(G_1)$ and $f(v) = v$ if $v \in \{u_0, u_1, v_0, v_1\}$. Then f is an automorphism of G such that $f(M) = N$ and, v_0 and u_0 are fixed under f.

Case 2: $u_0v_1 \notin N$ or $v_0u_1 \notin N$. Without loss of generality, we can assume that $v_0u_i \in N$ and $u_0v_j \in N$. Define $f_1 : V(G) \mapsto V(G)$ such that $f_1(u_1) = u_i$, $f_1(u_i) = u_1$, $f_1(v_1) = v_j$, $f_1(v_j) = v_1$ and $f_1(v) = v$ if $v \notin \{u_1, u_i, v_1, v_j\}$. Let $N_1 = f_1(N)$. Then $u_0v_1, v_0u_1 \in N_1$. By Case 1, there is an automorphism f_2 of G such that $f_2(M) = N_1$, and v_0 and u_0 are fixed under f_2. Then $f = f_1^{-1}f_2$ is an automorphism of G such that $f_e(M) = N$. □

Theorem 2.8. *Let G be the graph obtained from K_{2n+2} by removing one edge u_0v_0. Then G is PM-transitive.*

Proof. Let M and N be two perfect matchings of G. Without loss of generality, assume that $u_0u_1, v_0v_1 \in M$.

Case 1: $u_0u_1, v_0v_1 \in N$. Then, $M_1 = M - \{u_0u_1, v_0v_1\}$ and $N_1 = N - \{u_0u_1\}$ are two perfect matchings of $G_1 = G - \{u_0, u_1, v_0, v_1\} \cong K_{2n-2}$. By Theorem 2.2, there is an automorphism f_1 of G_1 such that $f_1(M_1) = N_1$. Define $f : V(G) \mapsto V(G)$ such that $f(v) = f_1(v)$ if $v \in V(G_1)$ and $f(v) = v$ if $v \in \{u_0, u_1, v_0, v_1\}$. Then f is an automorphism of G such that $f(M) = N$.

Case 2: $u_0u_1 \notin N$ or $v_0v_1 \notin N$. Without loss of generality, we can assume that $u_0u_i \in N$ and $v_0v_j \in N$. Define $f_1 : V(G) \mapsto V(G)$ such that $f_1(u_1) = u_i$, $f_1(u_i) = u_1$, $f_1(v_1) = v_j$, $f_1(v_j) = v_1$ and $f_1(v) = v$ if $v \notin \{u_1, u_i, v_1, v_j\}$. Let $N_1 = f_1(N)$. Then $u_0u_1, v_0v_1 \in N_1$. By Case 1, there is an automorphism f_2 of G such that $f_2(M) = N_1$. Then $f = f_1^{-1}f_2$ is an automorphism of G such that $f_e(M) = N$. □

A *wheel* W_n is the graph obtained from a n-cycle by adding a new vertex and joining the new vertex to every vertex of the cycle.

Theorem 2.9. *Let $G = W_{2n+1}$. Then G is PM-transitive.*

Proof. Without loss of generality, let G be the graph obtained from $C_{2n+1} = v_1v_2 \cdots v_{2n+1}v_1$ by adding a new vertex v and joining v to every vertex of the cycle. Suppose that M and N are two perfect matchings of G. Without loss of generality, suppose that $vv_i \in M$ and $vv_j \in N$. Define $f : V(G) \mapsto V(G)$ such that $f(v) = v$, $f(v_i) = v_j$, and $f(v_{i+k}) = v_{j+k}$, where the subscripts are taken modular $2n + 1$. Then f is an automorphism of G such that $f_e(M) = N$. □

Theorem 2.10. *Let k be a positive even integer, $W_{2n_1}, W_{2n_2}, \cdots W_{2n_k}$ be k wheels and $v_1, v_2, \cdots, v_k$ be the centers of the wheels, respectively. If $G = W_{2n_1} \cup W_{2n_2} \cup \cdots \cup W_{2n_k} + \{v_1v_2, v_2v_3, \cdots, v_{k-1}v_k\}$, then G is PM-transitive.*

Proof. Let M and N be two perfect matchings of G. Then $v_1v_2, v_3v_4, \cdots, v_{k-1}v_k \in M \cap N$. Furthermore, $M_1 = M \backslash \{v_1v_2, v_3v_4, \cdots, v_{k-1}v_k\}$ and $N_1 = N \backslash \{v_1v_2, v_3v_4, \cdots, v_{k-1}v_k\}$ are two perfect matchings of disjoint union of even cycles generated by $G - \{v_1, v_2, \cdots, v_k\}$. Then there is an automorphism f of G such that $f_e(M) = N$. □

3. Properties of PM-Transitive Graphs

In this section, we generate new PM-transitive graphs from existing PM-transitive graphs.

Theorem 3.1. *If G_1 and G_2 are two perfect matching transitive graphs and $V(G_1) \cap V(G_2) = \emptyset$, then $G = G_1 \cup G_2$ is PM-transitive.*

Proof. Let M and N be two perfect matchings of G. For $i = 1, 2$, let $M_i = M \cap E(G_i)$ and $N_i = N \cap E(G_i)$. Then M_i and N_i are two perfect matchings of G_i. Since G_i is PM-transitive, there is an automorphism f_i of G_i such that $f_i(M_i) = N_i$. Define $f : V(G) \mapsto V(G)$ such that $f(v) = f_1(v)$ if $v \in V(G_1)$, and $f(v) = f_2(v)$ if $v \in V(G_2)$. Then f is an automorphism of G such that $f_e(M) = N$. □

Theorem 3.2. *Let G_1 and G_2 be two PM-transitive graphs and H be a path of odd length. Suppose that G is the graph obtained from G_1, G_2 and H by connecting one end vertex of H with every vertex of G_1 and the other end vertex of H with every vertex of G_2. Then G is PM-transitive.*

Proof. Let M and N be two perfect matchings of G. Then $M \cap E(H) = N \cap E(H)$ and $M_1 = M \setminus M \cap E(H)$ and $N_1 = M \setminus N \cap E(H)$ are two perfect matchings of $G' = G_1 \cup G_2$. By Theorem 4.2, there is an graph automorphism f' of G' such that $f'(M_1) = N_1$. We can easily extend the graph automorphism f' of G' to a graph automorphism f of G such that $f(M) = N$. □

Corollary 3.1. *Let G_1 be a perfect matching transitive graph and H be a path of odd length. Suppose that G is the graph obtained from G_1 and H by connecting an end vertex of H with every vertex of G. Then G is PM-transitive.*

Corollary 3.2. *Let $W_{2n_1+1}, W_{2n_2+2}, \cdots W_{2n_k+1}$ be k wheels and $v_1, v_2, \cdots, v_k$ be the centers of the wheels, respectively. Let G be the graph obtained from $W_{2n_1+1} \cup W_{2n_2+1} \cup \cdots \cup W_{2n_k+1} + \{v_1v_2, v_2v_3, \cdots, v_{k-1}v_k, v_kv_1\}$ by replacing v_iv_{i+1} with an odd path, where $i = 1, 2, \cdots, k$ and the addition is modular k. Then G is PM-transitive.*

4. Non PM-Transitive Graphs

In this section, we characterize some non-perfect matching transitive graphs. The generalized Petersen graph $GP(n,k)$ for $n \geq 3$ and $1 \leq k \leq (n-1)/2$ is a graph consisting of an inner star polygon $\{n,k\}$ (or circular graph) and an outer regular polygon $\{n\}$ (or cycle graph C_n) with corresponding vertices in the inner and outer polygons connected with edges.

Theorem 4.1. *([4] and [10]) The generalized Petersen graph $GP(n,k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1$ (mod n) or $(n,k) = (10,2)$, and symmetric only for the cases $(n,k) = (4,1),(5,2),(8,3),(10,2),(10,3),(12,5)$, and $(24,5)$.*

Theorem 4.2. *([1] and [14]) The generalized Petersen graph $GP(n,k)$ is non-hamiltonian if and only if $k = 2$ and $n \equiv 5$ (mod 6).*

Theorem 4.3. *The generalized Petersen graph $GP(n,k)$ is PM-transitive if and only if it is the Petersen graph.*

Proof. Let G be a generalized graph $GP(n,k)$, where $k \neq 2$ or $n \not\equiv 5$ (mod 6). By Theorem 4.2, G is Hamiltonian. Let C be a hamiltonian cycle of G and $M = G - C$. Since G is a 3-regular graph, M is a perfect matching of G. Let N denote the perfect matching consisting of all the edges between the inner star polygon and the outer regular graph of $GP(n,k)$. Note that $G - M$ is a Hamiltonian cycle and $G - N$ is the disjoint union of cycles. Therefore, there is no graph automorphism f of G such that $f(M) = N$.

Let G be a generalized graph $GP(n,k)$, where $k = 2$ and $n \equiv 5$ (mod 6) and $n \neq 5$. In this case, the inner star polygon is an n-cycle. Denote the inner cycle by $v_1v_3v_5\cdots v_nv_2v_4\cdots v_{n-1}v_1$ and the outer cycle by $u_1u_2u_3\cdots u_nu_1$. Let $M = \{u_1v_1\} \cup \{u_2u_3, u_4u_5, \cdots, u_{n-1}u_n\} \cup \{v_3v_5, v_7v_9, \cdots, v_nv_2, v_4v_6, \cdots, v_{n-3}v_{n-1}\}$ and $N = \{u_1v_1, u_2v_2, \cdots, u_nv_n\}$. Then M and N are two perfect matchings of G. Note that $G - N$ is the disjoint union of two n-cycles. If $n \equiv 3$(mod 4), then $G - N$ has a 14-cycle $u_1u_2v_2v_4u_4u_3v_3v_1v_{n-1}u_{n-1}u_{n-2}v_{n-2}v_nu_nu_1$ and some 8-cycles. If $n \equiv 1$(mod 4), then $G - N$ has a 5-cycle $u_1u_2v_2v_nu_uu_1$ and a $(2n-5)$-cycle. Therefore, there is no graph automorphism of G such that $f(M) = N$.

Coming the above discussion and Theorem 2.4 , the result follows. □

5. Further Discussion

In this section, we give some examples which are PM-transitive, vertex-transitive, or edge-transitive.

Theorem 5.1. *The generalized Petersen graph $PG(n,1) \cong C_n \times K_2$, where $n = 3$ or $n \geq 5$, is vertex-transitive, but not edge-transitive or PM-transitive.*

Theorem 5.2. *The cubical graph $C_4 \times K_2 \cong PG(4,1)$ is vertex-transitive and edge-transitive, but not PM-transitive.*

Theorem 5.3. *The graphs discussed in Theorem 2.1, Theorem 2.2, Theorem 2.3, and Theorem 2.4 are vertex-transitive, edge-transitive, and also PM-transitive.*

Theorem 5.4. *The graphs constructed in Theorem 2.5, Theorem 2.6, Theorem 2.7, Theorem 2.8, Theorem 2.9, and Theorem 2.10 are PM-transitive, but neither vertex-transitive nor edge-transitive.*

References

[1] B.R. Alspach, The classification of Hamiltonian generalized Petersen graphs, *J. Combin. Theory Ser. B* **34** (1983), 293–312.

[2] L. Babai, Automorphism groups, isomorphism, reconstruction, *Handbook of Combinatorics*, 1996.

[3] Z. Bouwer, Vertex and edge-transitive, but not 1-transitive graphs, *Canad. Math. Bull.* **13** (1970), 231–237.

[4] N. Biggs, *Algebraic Graph Theory*, Second Edition, Cambridge England, Cambridge University Press, 1993.

[5] M. Conder, Trivalent symmetric graphs on up to 768 vertices, *J. Combin. Math. Combin. Comput.* **20** (2000), 41–63.

[6] R. Diestel, and I. Leader, A conjecture concerning a limit of non-Cayley graphs, *J. Algebraic Combin.* **14** (1) (2001), 17–25.

[7] A. Eskin, D. Fisher, and K. Whyte, Quasi-isometries and rigidity of solvable groups, *Proceedings of the International Congress of Mathematicians*, Hyderabad, India 2010.

[8] R.M. Foster, Geometrical circuits of electrical networks, *Transactions of the American Institute of Electrical Engineers* **51** (1932), 309–317.

[9] R.M. Foster, I.Z. Bouwer, W.W. Chernoff, B. Monson and Z. Star, *The Foster Census: R.M. Foster's Census of Connected Symmetric Trivalent Graphs*, Winnipeg, Canada, Charles Babbage Research Center, 1988.

[10] R. Frucht, J.E. Graver, and M.E. Watkins, The groups of the generalized Petersen graphs, *Proc. Cambridge Philos. Soc.* **70** (1971), 211–218.

[11] C. Godsil, and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics **207**, New York, Springer-Verlag, 2001.

[12] J.L. Gross, J. Yellen, *Handbook of Graph Theory*, CRC Press, 2004.

[13] D.F. Holt, A graph which is edge-transitive but not arc-transitive, *J. Graph Theory* **5** (2) (1981), 201–204.

[14] D.A. Holton, and J. Sheehan, *The Petersen Graph*, Cambridge England, Cambridge University Press, 1993.

[15] J. Lauri, and R. Scapellato, *Topics in graph automorphisms and reconstruction*, London Math. Soc. Stud. Texts **54**, 1st Edition, Cambridge England, Cambridge University Press, 2003.

[16] P. Potočnik, P. Spiga, and G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, *J. Symbolic Comput.* **50** (2013), 465–477.

[17] Z. Ryjáček, On a closure concept in claw-free graphs, *J. Combin. Theory Ser. B* **70** (2) (1997), 217–224.