



Domination number of the non-commuting graph of finite groups

Ebrahim Vatandoost*, Masoumeh Khalili

Department of Mathematics, Faculty of Science,
Imam Khomeini International University, Qazvin, Iran.

vatandoost@sci.ikiu.ac.ir, ava_kh17@yahoo.com

Abstract

Let G be a non-abelian group. The *non-commuting graph* of group G , shown by Γ_G , is a graph with the vertex set $G \setminus Z(G)$, where $Z(G)$ is the center of group G . Also two distinct vertices of a and b are adjacent whenever $ab \neq ba$. A set $S \subseteq V(\Gamma)$ of vertices in a graph Γ is a *dominating set* if every vertex $v \in V(\Gamma)$ is an element of S or adjacent to an element of S . The *domination number* of a graph Γ denoted by $\gamma(\Gamma)$, is the minimum size of a dominating set of Γ . Here, we study some properties of the non-commuting graph of some finite groups. In this paper, we show that $\gamma(\Gamma_G) < \frac{|G| - |Z(G)|}{2}$. Also we characterize all of groups G of order n with $t = |Z(G)|$, in which $\gamma(\Gamma_G) + \gamma(\overline{\Gamma}_G) \in \{n - t + 1, n - t, n - t - 1, n - t - 2\}$.

Keywords: non-commuting graph, dominating set, domination number

Mathematics Subject Classification : 05C25, 05C75

DOI: 10.5614/ejgta.2018.6.2.3

1. Introduction

Let G be a non-abelian group and $Z(G)$ be the center of G . Associate a graph Γ_G with G as follows: Take $G \setminus Z(G)$ as the vertices of Γ_G . Two vertices a and b are adjacent if $ab \neq ba$. This graph is called the *non-commuting graph of G* . Let Γ be a simple graph. A subset $S \subseteq V(\Gamma)$ is called a *dominating set* if each vertex $v \in V(\Gamma) \setminus S$ has at least one neighbor in S . The size of a smallest dominating set of Γ is called *domination number* of Γ and is denoted by $\gamma(\Gamma)$.

Received: 24 September 2016, Revised: 11 March 2018, Accepted: 16 April 2018.

*Corresponding author.

Erdős considered the non-commuting graph in 1975 for the first time. In 2004, Abdollahi, Akbari and Maimani studied some properties of the non-commuting graph of a group. For more results, see [1],[2], [6], [5] and [12].

Before starting, let us introduce some necessary notation and definitions. For every graph Γ , we denote the set of the vertices of Γ by $V(\Gamma)$. The minimum degree of a graph Γ denoted by $\delta(\Gamma)$. The *complete graph*, *path* and *cycle* on n vertices are denoted by K_n , P_n and C_n , respectively. The open neighborhood of a vertex v in Γ is the set $N_\Gamma(v)$ of vertices joined to v by an edge. The closed neighborhood of v is the set $N_\Gamma[v] = N_\Gamma(v) \cup \{v\}$. The complement of Γ denoted by $\bar{\Gamma}$. If u and v are vertices in Γ , then $d(u, v)$ denotes the length of the shortest path between u and v . A graph Γ is *connected* if there is a path between each pair of the vertices of Γ . The maximum value of $d(u, v)$ between all pair of the vertices of connected graph Γ is called the *diameter* of Γ and denoted by $\text{diam}(\Gamma)$. Let Γ_1 and Γ_2 be two graphs. The *corona* $\Gamma = \Gamma_1 \circ \Gamma_2$ is the graph formed from one copy of Γ_1 and $|V(\Gamma_1)|$ copies of Γ_2 such that the i th vertex of Γ_1 is adjacent to every vertex in the i th copy of Γ_2 .

For each $x \in G$, $C_G(x) = \{g \in G \mid gx = xg\}$. We denote the *symmetric group* and the *alternating group* on n letters by S_n and A_n , respectively. Also $Q_8 = \langle A, B \mid A^4 = 1, A^2 = B^2, B^{-1}AB = A^{-1} \rangle$ and $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$ are the *quaternion group* with 8 elements and the *dihedral group* of order $2n$, respectively.

In this paper, we study the domination number of the non-commuting graphs. In particular, we characterize all groups G of order n with $|Z(G)| = t$, in which $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) \in \{n - t + 1, n - t, n - t - 1, n - t - 2\}$.

2. Preliminaries

In this section, we provide some useful results which will be applied in the next section.

Theorem 2.1. (Ore) [10] Let Γ be a graph with no isolated vertices. Then $\gamma(\Gamma) \leq \frac{n}{2}$.

Theorem 2.2. [7] For a graph Γ with even order n and no isolated vertices, $\gamma(\Gamma) = \frac{n}{2}$ if and only if the components of Γ are the cycle C_4 or the Corona $H \circ K_1$ for any connected graph H .

Theorem 2.3. [1], 2.1. For any non-abelian group G , $\text{diam}(\Gamma_G) = 2$. In particular, Γ_G is connected. Also the girth of Γ_G equal 3.

Theorem 2.4. [11] Let Γ be a graph of order n . Then the following holds.

$$i) \gamma(\Gamma) + \gamma(\bar{\Gamma}) \leq n + 1.$$

$$ii) \gamma(\Gamma)\gamma(\bar{\Gamma}) \leq n.$$

Lemma 2.1. Let G be a finite non-abelian group. Then $\delta(\Gamma_G) \geq 3$.

Proof. Suppose that $\deg_{\Gamma_G}(v) \leq 2$, for some $v \in V(\Gamma_G)$. Since $\deg_{\Gamma_G}(v) \geq \frac{|G|}{2}$, then $|G| \leq 4$. So, G is an abelian group, which is a contradiction. $\square$

Remark 2.1. Here, we get figures of the non-commuting graph of some groups. These figures are useful in proving some theorems in the third section. (See Figures 1, 2, 3 and 4.)

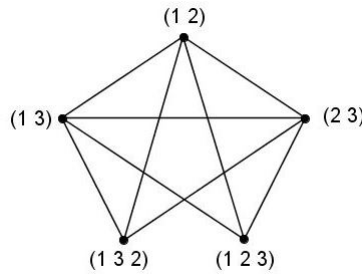


Figure 1. The non-commuting graph of S_3 .

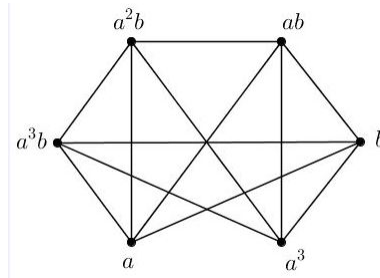


Figure 2. The non-commuting graph of D_8 .

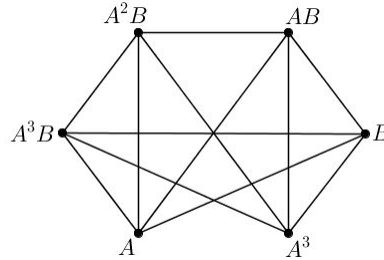


Figure 3. The non-commuting graph of Q_8 .

Lemma 2.2. *Let G be a non-abelian group of odd order. Then, the graph $\overline{\Gamma}_G$ contains no isolated vertex.*

Proof. Assume to the contrary, a is an isolated vertex of $\overline{\Gamma}_G$. Then, for each $x \in G \setminus Z(G)$ we have $ax \neq xa$. Hence $o(a) = 2$, which is a contradiction. $\square$

Lemma 2.3. *Let G be a non-abelian group and $|Z(G)| = 1$. Then, the vertices of degree one in $\overline{\Gamma}_G$ occur only at the edges. Furthermore, if $\deg_{\overline{\Gamma}_G} a = 1$ then $o(a) = 3$.*

Proof. Let $a \in V(\overline{\Gamma}_G)$, $\deg_{\overline{\Gamma}_G} a = 1$ and a be adjacent to b . Then, $C_G(a) = \{1, a, b\}$ and so $b = a^{-1}$. Hence, $C_G(b) = \{1, b, a\}$. Thus, $\deg_{\overline{\Gamma}_G} b = 1$. Furthermore, $o(a) = o(b) = 3$. $\square$

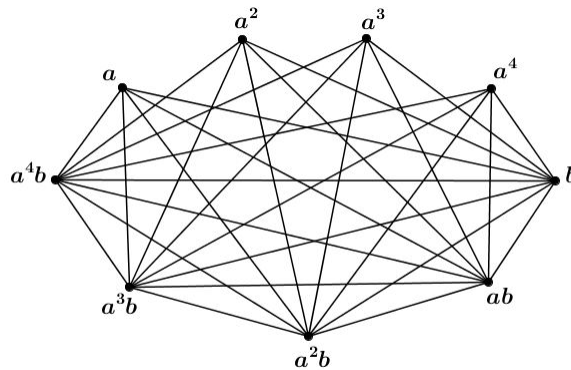


Figure 4. The non-commuting graph of D_{10} .

Lemma 2.4. Let G be a non-abelian group. Then $\overline{\Gamma}_G$ does not have C_n , ($n > 3$) as component.

Proof. Let C_n be a component of $\overline{\Gamma}_G$ and $\{a, b\} \subset V(C_n)$ such that $b \in N_{\overline{\Gamma}_G}(a)$. Then $N_{\overline{\Gamma}_G}[a] = \{a, b, f\}$ and $N_{\overline{\Gamma}_G}[b] = \{a, b, c\}$. So $C_G(a) = \{a, b, f\} \cup Z(G)$ and $C_G(b) = \{a, b, c\} \cup Z(G)$. Since $|C_G(a) \cap C_G(b)|$ divides $|C_G(a)|$, then $2 + |Z(G)|$ divides $3 + |Z(G)|$, that is a contradiction. $\square$

Definition 2.1. Let G be a group. Write S and T for the set of elements of G of order two and three, respectively. Then G is acceptable when neither S nor T is empty and $G = S^* \cup T$, where $S^* = S \cup \{e\}$.

Theorem 2.5. [3] If G is acceptable, then either $S^* \leq G$ or $T^* \leq G$.

3. Main results

In this section, we prove our main results.

Theorem 3.1. Let G be a non-abelian group. Then, the following holds.

- i) $\delta(\Gamma_G) = 3$ if and only if G is isomorphic to S_3 .
- ii) $\delta(\Gamma_G) = 4$ if and only if G is isomorphic to D_8 or Q_8 .
- iii) $\delta(\Gamma_G) = 5$ if and only if G is isomorphic to D_{10} .

Proof. We prove as follows:

- i) Let $\delta(\Gamma_G) = 3$ and $\deg_{\Gamma_G}(v) = 3$, for some $v \in V(\Gamma_G)$. Since $\deg_{\Gamma_G}(v) \geq \frac{|G|}{2}$, then $|G| \leq 6$. The only non-abelian group of order less than 7 is S_3 . Conversely, Suppose that G is isomorphic to S_3 . By considering the figure of the non-commuting graph associated to symmetric group S_3 (See Figure 1), we obtain $\delta(\Gamma_G) = 3$.

- ii) Let $\delta(\Gamma_G) = 4$. Then $|G| \leq 8$ and so $G \in \{S_3, D_8, Q_8\}$. By (i), $G \not\cong S_3$. So $G \cong D_8$ or Q_8 . Conversely, Suppose that G is isomorphic to D_8 or Q_8 . By considering the figure of the non-commuting graph associated to D_8 and Q_8 (See Figures 2 and 3), we obtain $\delta(\Gamma_G) = 4$.
- iii) Let $\delta(\Gamma_G) = 5$. Then $|G| \leq 10$. By (i) and (ii), $|G| = 10$. Since G is not an abelian group, then G is isomorphic to D_{10} . Conversely, Suppose that G is isomorphic to D_{10} . By considering the figure of the non-commuting graph associated to the dihedral group D_{10} (See Figure 4), we obtain $\delta(\Gamma_G) = 5$.

□

Corollary 3.1. *All of 3-regular and 5-regular graphs cannot be non-commuting graphs.*

Theorem 3.2. *Let Γ be a $(n - 2)$ -regular graph of order n . Then Γ is the non-commuting graph associated to a non-abelian group G if and only if $n = 6$ and G is isomorphic to D_8 or Q_8 .*

Proof. Let Γ be a $(n - 2)$ -regular graph of order n and G be a group such that $\Gamma_G = \Gamma$. Then n is even and $\bar{\Gamma}_G$ is a disjoint union of $\frac{n}{2}$ edges. If a and b are adjacent in $\bar{\Gamma}_G$, then $C_G(a) = Z(G) \cup \{a, b\}$. It is clear that $|Z(G)| \leq 2$.

If $|Z(G)| = 1$, then $C_G(a) = \{1, a, a^{-1} = b\}$ and $o(a) = 3$. Since $\bar{\Gamma}_G$ is a disjoint union of some edges, then for each $x \in G$ we have $o(x) = 3$. Thus, there is a positive integer s such that $|G| = 3^s$. Hence $|Z(G)| > 1$, which is a contradiction.

If $|Z(G)| = 2$ and $Z(G) = \{1, x\}$, then for each $a \in G \setminus Z(G)$, $C_G(a) = \{1, x, a, b\}$ and so we have $a^2 = 1$ or $a^2 = x$. Therefore for each $a \in G \setminus Z(G)$ we have $a^2 \in Z(G)$. Hence $\frac{G}{Z(G)}$ is an elementary abelian 2-group. So $G' \leq Z(G)$, which implies $|G'| = 1$ or 2 . Since G is not an abelian group, then $G' \neq \{1\}$. Thus $G' = Z(G)$. Also we have $cl(a) = \{g^{-1}ag : g \in G\} \subseteq aG'$ and so $|cl(a)| \leq 2$. Since $|cl(a)| = \frac{|G|}{|C_G(a)|}$, then $|G| \leq 8$. Hence $G \cong S_3, D_8$ or Q_8 . Since $Z(S_3) = 1$, then $G \cong D_8$ or $G \cong Q_8$.

Conversely, suppose that $G \cong D_8$ or $G \cong Q_8$. Then by considering the figures of the non-commuting graphs of these two groups (See Figures 2 and 3), we obtain Γ_G is a 4-regular graph of order 6. □

Theorem 3.3. *Every $(n - 3)$ -regular graph of order n is not the non-commuting graph.*

Proof. Let Γ be a graph of order n and $(n - 3)$ -regular. Also, suppose that G is a group and $\Gamma_G = \Gamma$. Thus, $\bar{\Gamma}_G$ is a 2-regular graph. It means that $\bar{\Gamma}_G$ is a disjoint union of cycles.

By Lemma 2.4, $\bar{\Gamma}_G$ is a disjoint union of triangles. Hence for every $a \in G \setminus Z(G)$, $C_G(a) = Z(G) \cup \{a, b, c\}$ such that $bc = cb$. Thus, $|Z(G)| \leq 3$.

Case 1. If $|Z(G)| = 1$, then $|C_G(a)| = 4$. So there is a positive integer s such that $|G| = 2^s$. Thus, $|Z(G)| > 1$, which is a contradiction.

Case 2. If $|Z(G)| = 2$, then $|C_G(a)| = 5$. Since $|Z(G)|$ divides $|C_G(a)|$, then $2 \mid 5$, which is a contradiction.

Case 3. If $|Z(G)| = 3$, then $|C_G(a)| = 6$. Thus, $C_G(a) \cong Z_6$ and so for each $x \notin Z(G)$, $o(x) \in \{2, 6\}$. So for each $a \in G \setminus Z(G)$, $a^2 \in Z(G)$ which implies $G/Z(G)$ is an elementary abelian 2-group. Therefore, $G' \leq Z(G)$. Since G is not an abelian group, then $G' = Z(G)$. We

know that $cl(a) \subseteq aG'$. So $|cl(a)| \leq 3$. Hence $6 < |G| \leq 18$. Since $|Z(G)| = 3$ and $|Z(G)|$ divides $|G|$, then $|G| \in \{9, 12, 15, 18\}$. Furthermore, since G is not an abelian group, then we have $|G| \in \{12, 18\}$.

If $|G| = 12$, then $G \cong A_4, D_{12}$ or $\langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$. Since G has only 2 elements of order three, then $G \not\cong A_4$. Also since $|Z(D_{12})| = 2$, then $G \not\cong D_{12}$. Hence $G \cong \langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$. So G has an element of order 4, which is a contradiction.

If $|G| = 18$, then $\bar{\Gamma}_G \cong 5K_3$. So G has exactly 5 elements of order 2. Let n_p be the number of Sylow p -subgroup of G . By Sylow Theorem, $n_2(G) \in \{1, 3, 9\}$. Hence G has 1, 3 or 9 elements of order 2, which is a contradiction. $\square$

In [10], Ore proved that if Γ is a graph with no isolated vertices, then $\gamma(\Gamma) \leq \frac{n}{2}$. In Theorems 3.4 and 3.5, we show that if G is a non-abelian group with $|Z(G)| = t$, then $\gamma(\Gamma_G)$ and $\gamma(\bar{\Gamma}_G)$ are less than $\frac{n-t}{2}$.

Theorem 3.4. *Let G be a non-abelian group of order n and $|Z(G)| = t$. Then $\gamma(\Gamma_G) < \frac{n-t}{2}$.*

Proof. We know $|V(\Gamma_G)| = n-t$. Since Γ_G is a connected graph, it contains no isolated vertex. By Theorem 2.1, $\gamma(\Gamma_G) \leq \frac{n-t}{2}$. Now, we show $\gamma(\Gamma_G) < \frac{n-t}{2}$. Assume to the contrary, $\gamma(\Gamma_G) = \frac{n-t}{2}$. By Theorem 2.2, each component of the graph Γ_G is the cycle C_4 or the Corona product K_1 and a connected graph H , that is $H \circ K_1$. By Lemma 2.1, is a contradiction. Hence $\gamma(\Gamma_G) < \frac{n-t}{2}$. $\square$

Theorem 3.5. *Let G be a non-abelian group of odd order n and $|Z(G)| = t$. Then $\gamma(\bar{\Gamma}_G) < \frac{n-t}{2}$.*

Proof. Since n is odd, then t is odd. By Lemma 2.2, the graph $\bar{\Gamma}_G$ contains no isolated vertex. By Theorem 2.1, $\gamma(\bar{\Gamma}_G) \leq \frac{n-t}{2}$. Now, suppose that $\gamma(\bar{\Gamma}_G) = \frac{n-t}{2}$. Then, by Theorem 2.2, $\bar{\Gamma}_G$ has connected components of kind of C_4 or $H \circ K_1$. By Lemma 2.4, all components of $\bar{\Gamma}_G$ are the corona product K_1 and a connected graph H . Let a be a vertex of degree 1 in $\bar{\Gamma}_G$ and b is adjacent to a . Then $C_G(a) = \{a, b\} \cup Z(G)$ and so $|Z(G)| \leq 2$. Since $|G|$ is odd, then $|Z(G)| = 1$. By Lemma 2.3, $|G| = 3^s$. Hence $|Z(G)| > 1$, which is a contradiction. $\square$

In the following theorem, we characterize all groups G of order n with $|Z(G)| = t$, in which $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) \in \{n-t+1, n-t, n-t-1, n-t-2\}$.

Theorem 3.6. *Let G be a non-abelian group of order n and $|Z(G)| = t$. Then the following holds.*

- i) $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) < n-t+1$.
- ii) $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = n-t$ if and only if $G \cong S_3$.
- iii) $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = n-t-1$ if and only if $G \cong D_8$ or $G \cong Q_8$.
- iv) $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = n-t-2$ if and only if $G \cong D_{10}$.

Proof. Since $|G| = n$ and $|Z(G)| = t$, then $|V(\Gamma_G)| = n-t$. By Theorem 3.4, $\gamma(\Gamma_G) < \frac{n-t}{2}$.

- i) By Theorem 2.4, $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) \leq n - t + 1$. Let $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = n - t + 1$. Then $\gamma(\bar{\Gamma}_G) > n - t + 1 - \frac{n-t}{2} = \frac{n-t+2}{2}$. By Theorem 2.1, $\bar{\Gamma}_G$ contains at least one isolated vertex. So $t = 1$ and $\gamma(\Gamma_G) = 1$. Thus $\gamma(\bar{\Gamma}_G) = n - 1$. Therefore $\Gamma_G = K_{n-1}$. By Theorem 2.3, it is impossible. Hence $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) < n - t + 1$.
- ii) Let $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = n - t$. Then $\gamma(\bar{\Gamma}_G) > \frac{n-t}{2}$. By Theorem 2.1, $\bar{\Gamma}_G$ contains at least one isolated vertex. So $t = 1$, $\gamma(\Gamma_G) = 1$ and $\gamma(\bar{\Gamma}_G) = n - 2$. Thus, $\bar{\Gamma}_G$ is a disjoint union of P_2 and isolated vertices $\{x_1, x_2, \dots, x_{n-3}\}$. All isolated vertices x_i are of order 2. Let a and b be the vertices of the path P_2 . By Lemma 2.3, $b = a^{-1}$ and $o(a) = o(b) = 3$. We claim that if x_1 and x_2 are two elements of order 2, then $o(x_1x_2) = 3$. To see this, suppose $o(x_1x_2) = 2$. Then $x_1x_2x_1x_2 = 1$ and so $x_1x_2 = x_2x_1$. Thus x_1 is adjacent to x_2 in $\bar{\Gamma}_G$, which is a contradiction. It is easy to see that $o(x_ib) = 2$, for $i = 1, 2, \dots, n - 3$. If $T = \langle x_1, b \mid x_1^2 = b^3 = 1, (x_1b)^2 = 1 \rangle$, then $T \cong S_3$. We prove $G \cong T$. If $x_2 \in G \setminus T$ and $x_1x_2 = b$, then $x_2 = x_1b$. Since $x_1b \in T$, $x_2 \in T$. Also if $x_1x_2 = b^{-1}$, then $x_2 = x_1b^{-1} \in T$. However it is a contradiction. Hence $G \cong T \cong S_3$. Conversely, if $G \cong S_3$, then by Figure 1, $\gamma(\Gamma_G) = 1$ and $\gamma(\bar{\Gamma}_G) = 4$ and the proof is complete.
- iii) Let $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = n - t - 1$. Then $\gamma(\bar{\Gamma}_G) > \frac{n-t-2}{2}$. We consider two cases:

Case 1. Let $\gamma(\bar{\Gamma}_G) = \frac{n-t}{2}$. Then $\gamma(\Gamma_G) = \frac{n-t}{2} - 1$. If $\bar{\Gamma}_G$ has an isolated vertex, then $t = 1$ and so n is odd. By Theorem 3.5, $\gamma(\bar{\Gamma}_G) < \frac{n-1}{2}$, which is false. If $\bar{\Gamma}_G$ does not have an isolated vertex, then by Theorem 2.2 and Lemma 2.4, $\bar{\Gamma}_G$ has a vertex of degree one. So $t = 1$ or $t = 2$. If $t = 1$, then n is odd. By Theorem 3.5, $\gamma(\bar{\Gamma}_G) < \frac{n-1}{2}$, which is false. If $t = 2$, then $\gamma(\bar{\Gamma}_G) = \frac{n-2}{2}$ and $\gamma(\Gamma_G) = \frac{n-4}{2}$. By Theorem 2.4, $\frac{n-2}{2} \cdot \frac{n-4}{2} \leq n - 2$. Since $n \neq 2$, then $n \leq 8$. So G is isomorphic to S_3 , D_8 or Q_8 . But $Z(S_3) = 1$. Hence G is isomorphic to D_8 or Q_8 .

Case 2. If $\gamma(\bar{\Gamma}_G) > \frac{n-t}{2}$, then $\bar{\Gamma}_G$ contains at least one isolated vertex. So $t = 1$ and so $\gamma(\Gamma_G) = 1$. Therefore $\gamma(\bar{\Gamma}_G) = n - 3$. By Lemmas 2.3, 2.4 and $\gamma(\bar{\Gamma}_G) = n - 3$, we have the following subcases.

Subcase 1. Let $\bar{\Gamma}_G$ be a union of the isolated vertices $\{x_1, x_2, \dots, x_{n-4}\}$ and K_3 with vertices a, b, c . Then $C_G(a) = C_G(b) = C_G(c) = \{1, a, b, c\}$. So orders of a, b and c are 2 or 4. If $o(a) = o(b) = o(c) = 2$, then order of each element of G is 2 and so G is an abelian group, which is a contradiction. If $o(a) = 2$ and $o(b) = 4$, then $a = b^2$ and for each i ($i = 1, 2, \dots, n - 4$), $x_ibx_i = b$ or $x_ibx_i = b^{-1}$. If $x_ibx_i = b$, then $bx_i = x_ib$. So b is adjacent to x_i , which is a contradiction. If $x_ibx_i = b^{-1}$, then $\langle b, x_i \rangle \cong D_8$. We claim that $G \cong \langle b, x_1 \rangle$. Suppose that $x \in G \setminus \langle b, x_1 \rangle$. Since $xbx = b^{-1}$ and $x_1bx_1 = b^{-1}$, then $xbx = x_1bx_1$ and so $x_1x \in C_G(b)$. We know that $C_G(b) = \langle b \rangle$. So $x \in \{x_1b, x_1b^2, x_1b^3\}$. Hence $x \in \langle b, x_1 \rangle$. Therefore $G \cong D_8$. Since in this case $Z(G) = \{1\}$, then we have $Z(D_8) = \{1\}$, which is a contradiction.

Subcase 2. Let $\bar{\Gamma}_G$ be a union of the isolated vertices $\{x_1, x_2, \dots, x_{n-5}\}$ and two edges with vertices $a \sim b$ and $c \sim d$. By Lemma 2.3, $o(a) = o(b) = o(c) = o(d) = 3$. Since $\{1, a, b, c, d\}$ is not a subgroup of G , by Theorem 2.5, $\{1, x_1, x_2, \dots, x_{n-5}\}$ is a subgroup of G . So $o(x_1 x_2) = 2$. Hence $x_1 x_2 = x_2 x_1$ and so x_1 is adjacent to x_2 in $\bar{\Gamma}_G$, which is a contradiction. Conversely, by using the figures 2 and 3 we can obtain the proof.

iv) Let $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = n - t - 2$. By Theorem 3.4, $\gamma(\Gamma_G) < \frac{n-t}{2}$. We have the following cases.

Case 1. If $\gamma(\Gamma_G) = \frac{n-t-2}{2}$, then $\gamma(\bar{\Gamma}_G) = \frac{n-t-2}{2}$. By Theorem 2.4, $\gamma(\Gamma_G) \cdot \gamma(\bar{\Gamma}_G) \leq n - t$. Hence, $1 \leq n - t \leq 7$. Since $n - t$ is even and $t \mid n$, then $n - t \in \{4, 6\}$. If $n - t = 4$, then $(n, t) = (6, 2)$ and so $G \cong S_3$. Which is a contradiction to the fact that $Z(S_3) = 1$. If $n - t = 6$, then $(n, t) \in \{(10, 4), (8, 2)\}$. Since $t \mid n$, we have $(n, t) = (8, 2)$. So $G \cong D_8$ or $G \cong Q_8$. By Figures 2 and 3, we have a contradiction.

Case 2. Let $\gamma(\Gamma_G) = \frac{n-t-4}{2}$ and $\gamma(\bar{\Gamma}_G) = \frac{n-t}{2}$. If $\bar{\Gamma}_G$ has an isolated vertex, then $t = 1$ and so n is odd. By Theorem 3.5, $\gamma(\bar{\Gamma}_G) < \frac{n-1}{2}$, which is false.

If $\bar{\Gamma}_G$ does not have an isolated vertex, then by Theorem 2.2 and Lemma 2.4, $\bar{\Gamma}_G$ has a vertex of degree 1. So $t = 1$ or $t = 2$. If $t = 1$, then n is odd, which is false. If $t = 2$, then $\gamma(\bar{\Gamma}_G) = \frac{n-2}{2}$ and $\gamma(\Gamma_G) = \frac{n-6}{2}$. Thus $n \leq 10$. By Figures 2, 3 and 4, we have a contradiction.

Case 3. Let $\gamma(\Gamma_G) < \frac{n-t-4}{2}$. Then $\gamma(\bar{\Gamma}_G) > \frac{n-t}{2}$ and so $\bar{\Gamma}_G$ has at least one isolated vertex. Thus $t = 1$, $\gamma(\Gamma_G) = 1$ and $\gamma(\bar{\Gamma}_G) = n - 4$. Let $u \in V(\bar{\Gamma}_G)$. If $\deg_{\bar{\Gamma}_G}(u) > 3$, then $\gamma(\bar{\Gamma}_G) < n - 4$, which is not true. Hence for every $u \in V(\bar{\Gamma}_G)$, $\deg_{\bar{\Gamma}_G}(u) \leq 3$ and $o(u) \leq 5$. We have the following subcases.

Subcase 1. Let $\deg_{\bar{\Gamma}_G}(u) \leq 1$, where $u \in V(\bar{\Gamma}_G)$. Then $\bar{\Gamma}_G$ is isomorphic to union of 3 copies of P_2 and $n - 7$ isolated vertices. It is clear that isolated vertices are of order 2. By Lemma 2.3, G is an abelian acceptable group. By Theorem 2.5, $T \cup \{1\} \leq G$ or $S \cup \{1\} \leq G$. Since 3 does not divide $|T \cup \{1\}|$, then $T \cup \{1\}$ is not a subgroup of G . Hence $S^* = S \cup \{1\}$ is a subgroup of G . Since $|S^*| \leq \frac{|G|}{2}$, then $|G| \leq 12$. We know that there is no group of order less than 12 with exactly 6 elements of order 3, which implies that $|G| = 12$. Hence $G \cong A_4, D_{12}$ or $L = \langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$. Since G has exactly 6 elements of order 3, we have $G \notin \{A_4, D_{12}\}$. Also in L we have $o(b) = 4$. So $G \not\cong L$.

Subcase 2. For each $u \in V(\bar{\Gamma}_G)$, $\deg_{\bar{\Gamma}_G}(u) \leq 2$. Let $u, v \in V(\bar{\Gamma}_G)$ and $\deg_{\bar{\Gamma}_G}(u) = \deg_{\bar{\Gamma}_G}(v) = 2$. If u and v are not adjacent in $\bar{\Gamma}_G$, then by Lemmas 2.3 and 2.4, $\gamma(\bar{\Gamma}_G) < n - 4$, which is not true. If u and v are adjacent in $\bar{\Gamma}_G$, then by Lemmas 2.3 and 2.4, u and v are vertices of a K_3 . Since $\gamma(\bar{\Gamma}_G) = n - 4$, then $\bar{\Gamma}_G$ is isomorphic to union of exactly one copy of K_3 , one copy of P_2 and some isolated vertices. Suppose $V(K_3) = \{u_1, u_2, u_3\}$, $V(P_2) = \{v_1, v_2\}$ and isolated vertices are denoted by x_i , where $1 \leq i \leq n - 6$. We have

$C_G(u_i) = \{1, u_1, u_2, u_3\}$. So $o(u_i) \in \{2, 4\}$. By Lemma 2.3, $o(v_1) = o(v_2) = 3$. Also we have $o(x_i) = 2$. So $|G| = 2^\ell 3$, where ℓ is natural. By Sylow Theorem, there are two subgroups H and K of G such that $|H| = 2^\ell$ and $K = \{e, v_1, v_1^{-1} = v_2\}$.

If $h \in H$ and $o(h) = 2$, then $o(hv_1h) = o(v_1) = 3$. So $hv_1h = v_1$ or v_1^{-1} . If $hv_1h = v_1$, then $v_1h = hv_1$ and so $o(v_1h) = 6$, which is false. If $hv_1h = v_1^{-1}$, then $v_1h = hv_1^{-1}$ or $hv_1 = v_1^{-1}h$. It is well known that $|Z(H)| > 1$.

Let $z \in Z(H)$ and $o(z) = 2$. Then for every $h \in H \setminus \{z\}$, we have

$(zh)v_1 = (hz)v_1 = h(zv_1) = h(v_1^{-1}z) = (hv_1^{-1})z = (v_1h)z = v_1(hz) = v_1(zh)$ Hence $zh \in C_G(v_1)$. So $zh \in \{v_1, v_1^{-1}\}$.

If $|H| > 4$, then $|H| \geq 8$. Thus there are $h_1, h_2 \in H$, $h_1 \neq h_2$ such that $zh_1 = zh_2$. Hence $h_1 = h_2$, which is a contradiction. Thus $|H| = 4$ and so $|G| = 12$. Since G has exactly two elements of order 3, then $|cl(v_1)| = 1$ or 2. Since $|C_G(v_1)| = 3$ and $[G : C_G(v_1)] = |cl(v_1)|$, we have $4 = 1$ or $4 = 2$, which is not true.

Subcase 3. Let $u \in V(\bar{\Gamma}_G)$, $deg_{\bar{\Gamma}_G}(u) = 3$ and $N_{\bar{\Gamma}_G}(u) = \{x, y, z\}$. Then $C_G(u) = \{1, u, x, y, z\}$ and so $o(u) = 5$. Hence induced subgraph on $N_{\bar{\Gamma}_G}[u]$ is isomorphic to K_4 . Since $\gamma(\bar{\Gamma}_G) = n - 4$, then $\bar{\Gamma}_G \cong K_4 \cup (n - 5)K_1$. On the other hand if x is a isolated vertex in $\bar{\Gamma}_G$, then $o(x) = 2$. Since $xu \notin \{1, u, u^2, u^3, u^4\}$, we have $o(xu) = 2$. Thus $xux = u^{-1}$ and so $\langle x, u \rangle \cong D_{10}$. Now let $y \in G \setminus \langle x, u \rangle$ and y be an isolated vertex in $\bar{\Gamma}_G$. Then $o(yu) = 2$. Hence $yuy = xux$. This implies that $xy \in C_G(u) = \{1, u, u^2, u^3, u^4\}$. Therefore $y \in \{x, xu, xu^2, xu^3, xu^4\}$ and so $y \in \langle x, u \rangle$, which is a contradiction. Hence $G \cong \langle x, u \rangle \cong D_{10}$.

Conversely, if $G \cong D_{10}$, then $t = 1$ and by Figure 4, $\gamma(\Gamma_G) + \gamma(\bar{\Gamma}_G) = 7$ and the proof is complete. □

Acknowledgments

The authors are very grateful to the referee for his/her useful comments. This research was partially supported by the Imam Khomeini International University.

References

- [1] A. Abdollahi, S. Akbari and H.R. Maimani, Non-commuting graph of a group, *J. Algebra* **298** (2006), 468–492.
- [2] N. Ahanjideh and A. Iranmanesh, On the relation between the non-commuting graph and the prime graph, *Int. J. Group Theory* **1** (2012), 25–28.
- [3] E.D. Bolker, Bryn Mawr College, Groups whose elements are ordered two or three, *Amer. Math. Monthly* **79** (9) (1972), 1007–1010.
- [4] J.A. Bondy, J.S.R. Murty, *Graph theory with applications*, Elsevier, (1977).

- [5] M.R. Darafsheh, H. Bigdely, A. Bahrami, Some results on non-commuting graph of a finite group, *Ital. J. Pure Appl. Math.* **27** (2010), 107–118.
- [6] M.R. Darafsheh, P. Yousefzadeh, A characterization of the group A_{p+3} by its non-commuting graph, *Int. Electron. J. Algebra* **11** (2012), 160–164.
- [7] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Robert, On graph having domination number half their order, *Period. Math. Hangar.* **16** (1985), 287–293.
- [8] J.C. Lennox, J. Wiegold, Extensions of a problem of Paul Erdős on groups, *J. Aust. Math. Soc. Ser. A* **31** (1981), 459–463.
- [9] B.H. Neumann, A problem of Paul Erdős on groups, *J. Aust. Math. Soc. Ser. A* **21** (1976), 467–472.
- [10] O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ. **38** (Amer. Math. Soc, Providence, RI), 1962.
- [11] C. Payan and N.H. Xuong, Domination-balanced graphs, *J. Graph Theory* **6** (1982), 23–32.
- [12] B. Tolue, A. Erfania, A. Jafarzadeh, A kind of non-commuting graph of finite groups, *J. Sci. Islam. Repub. Iran* **25** (2014), 379–384.
- [13] D.B. West, *Introduction to graph theory*, Prentice Hall, (1998).