# Comparison of Step Response Characteristics of Simple Fractional Order Systems and Second Order Systems

Mehmet Emir Koksal

Department of Mathematics, Ondokuz Mayis University, 55139 Samsun, Turkey
Email: mekoksal@omu.edu.tr

**Abstract--***The step response characteristics of first and second order systems are well known. On the other hand, the step response of fractional order systems (FOSs) with 2-term fractional denominator is like those of first and second order systems. But there are important differences between the two types of characteristics. Considering the step response, the behavior of simple FOS with a denominator polynomial having unity term and the other involves fractional power is investigated in this paper comparatively with $1^{st}$ and $2^{nd}$ order systems. The results bring light for the design of fractional order control systems (FOCSs).*

**Keywords— *Fractional order, control system, step response, time constant, rise time, overshoot.***

## I. INTRODUCTION

With increasing technology and high-speed computers, new numerical methods for modelling and solving physical systems have been presented. FOSs have founded many applications not only in this respect, but they also constitute more realistic models for physical phoneme. Therefore, plenty of literature has appeared for analyzing and designing these systems in the last two decades [1-5]. Especially, fractional order proportional integral derivative (FOPID) controllers have appeared extensively in control system design [6-8].

As some examples on very recent literature, [9] presents a FO positive position feedback compensator and it is compared to the standard integer-order one for active vibration Control of a rectangular free-edged carbon fiber composite plate. In [10], FO control of dissimilar redundant actuating system is used in large passenger aircraft; two dissimilar actuators used for position control surface are modelled by fractional orders and three FO controllers are used to improve transient response of the system. In [11], it is aimed to solve the stability problem for some FO nonautonomous systems by linear state feedback control

and adaptive control; a new property for Caputo fractional derivative is used to derive some sufficient conditions for the global asymptotical stabilization.

Considering sophisticated and rather special applications [12-17], there are a huge number of publications about FOSs; so only very few of them are referred in this paper. On the other hand, a compact publication investigating the step response characteristics of even simple FOSs, namely rise time, settling time, delay time, overshoot, oscillation period, damping time constant of oscillations, and some others which do not exhibit similarity with those of integer order systems [18] is not yet present. The purpose of this paper is to draw attention to the differences between the step response characteristics of simple FOSs and $1^{st}$ and 2nd order systems and thus to supply some lights for FOCSs' designers.

For the mentioned purpose, time domain characteristics of first and second order systems are reviewed in Section 2. In Section 3, the step response characteristics of o a FOS with 2-term denominator are presented depending on the fractional power. Section 4 discusses and lists the differences between $2^{nd}$ order and fractional order characteristics. Section 5 covers the conclusions.

## II. REVIEW OF FIRST AND SECOND INTEGER ORDER SYSTEMS

It is well known that the simple first order transfer function

$$H_1(s) = \frac{1}{p_1 s + 1} \qquad (1a)$$

has a step response

$$y_1(t) = 1 - e^{-t/p_1} \qquad (1b)$$

monotonically increasing from 0 to 1 with a time constant $\tau = p_1$ [18]. The dc gain is assumed to be 1 which is not important for relative time domain characteristics.

For the second order transfer function,

$$H_2(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad (2a)$$

is the standard form where $\omega_n$ is undamped natural frequency and $\xi \geq 0$ is the damping ratio [18].

Overdamped step response

$$y(t) = 1 - \frac{1}{\tau_2 - \tau_1}\left(\tau_2 e^{-\frac{t}{\tau_2}} - \tau_1 e^{-\frac{t}{\tau_1}}\right), \qquad (2b)$$

occurs for for $\xi > 1$, where two time constants are $\tau_1 = 1/\omega_n(\xi + \sqrt{\xi^2 - 1})$ and $\tau_2 = 1/\omega_n(\xi - \sqrt{\xi^2 - 1})$ so that it increases monotonically from 0 to its final value 1. For $\xi = 1$, the response is similar in shape to the overdamped case, but the variation is described by

$$y(t) = 1 - e^{-\omega_n t}(\omega_n + 1) \qquad (2c)$$

which is known as critically damped case.

Underdamped response occurs for $0 < \xi < 1$. The response

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\sqrt{1-\xi^2}\omega_n t + \sin^{-1}\sqrt{1-\xi^2}\right) (2d)$$

is stable and approaches to the reference value 1 with damped oscillations with frequency $\sqrt{1-\xi^2}\omega_n$ and damping time constant $1/\xi\omega_n$. The special case of oscillatory response occurs for $\xi = 0$, the system is undamped, and the step response is

$$y(t) = 1 - \cos\omega_n t \qquad (2e)$$

which represents sustained oscillations with frequency $\omega_n$ and undamped oscillation period $T_n = 2\pi/\omega_n$.

All the step responses considered so far stars from 0 at time $t = 0$ and approaches to 1 monotonically or oscillatory manner as $Lim\ t \to \infty$ (except the undamped case $\xi = 0$). The following time domain characteristics are defined for these responses:

*Time constant* $\tau$: Time required for the response to reach $1 - (1/e)$ of its final value.

*Rise time* $T_r$: Time required for the response to reach from 10 % to 90 % of its final value.

*Settling time* $Ts$: Time required for the response to stay around its final value with an error less than 2 %.

*Oscillation period* $T_o$: Period of oscillations for underdamped case.

*Peak time* $T_{max}$: The time the response reaches its maximum value.

*Peak value* $y_{max}$: Maximum value of the response; $y(T_{max})$.

*Overshoot* $y_{osh}$: $y_{max} - 1$; How much the response exceeds the final reference 1.

*Percent overshoot POSH:* It is defined by $(y_{osh}/y_{ref})100$.

*Reduction Ratio: RR:* Ratio of successive overshoots.
From Eq. (1b), it is straightforward to show that

$$\tau = p_1, \qquad (3a)$$
$$T_r = ln9\, p_1 = 2.197225\, p_1, \qquad (3b)$$
$$T_s = ln50\, p_1 = 3.912023\, p_1. \qquad (3c)$$

In summary for a first order system; i) The step response increases exponentially to its steady state value without any oscillations, ii) Rise time and settling time are some multiples of time constant $\tau = p_1$.

From Eq. (2b), the over damped response of a second order systems has

$$\tau: e^{-\tau\omega_n\xi} \sinh\left[\tau\omega_n\sqrt{\xi^2 - 1} + \cosh^{-1}\xi\right]$$
$$= e^{-1}\sqrt{\xi^2 - 1}, \qquad (4a)$$
$$T_r = t_2 - t_1, where \qquad (4b)$$
$$e^{-t_1\omega_n\xi}\sinh\left[t_1\omega_n\sqrt{\xi^2 - 1} + \cosh^{-1}\xi\right] = 0.9\sqrt{\xi^2 - 1},$$
$$e^{-t_2\omega_n\xi}\sinh\left[t_2\omega_n\sqrt{\xi^2 - 1} + \cosh^{-1}\xi\right] = 0.1\sqrt{\xi^2 - 1}$$
$$T_s: e^{-T_S\omega_n\xi}\sinh\left[T_S\omega_n\sqrt{\xi^2 - 1} + \cosh^{-1}\xi\right]$$
$$= 0.02\sqrt{\xi^2 - 1}. \qquad (4c)$$

From Eq. (2c), the critically damped response of a second order systems has

$$\tau = [1 + \ln(1 + \omega_n)]/\omega_n, \qquad (5a)$$
$$T_r = ln9/\omega_n, T_s = \ln[50(1 + \omega_n)]/\omega_n, \qquad (5b, c)$$
$$T_s = \frac{1}{\xi\omega_n}ln\frac{50}{\sqrt{1-\xi^2}} = \frac{\tau_e}{2\xi^2}ln\frac{50}{\sqrt{1-\xi^2}}. \qquad (5d)$$

From Eq. (2d), the under damped response of a second order systems has

$$\tau: e^{-\tau\omega_n\xi}\sin\left[\tau\omega_n\sqrt{1-\xi^2} + \sin^{-1}\sqrt{1-\xi^2}\right]$$
$$= e^{-1}\sqrt{1-\xi^2}, \qquad (6a)$$
$$T_r = t_2 - t_1, where \qquad (6b)$$
$$e^{-t_1\omega_n\xi}\sin\left[t_1\omega_n\sqrt{1-\xi^2} + \sin^{-1}\sqrt{1-\xi^2}\right]$$
$$= 0.9\sqrt{1-\xi^2}, e^{-t_2\omega_n\xi}\sin\left[t_2\omega_n\sqrt{1-\xi^2}\right.$$
$$\left. + \sin^{-1}\sqrt{1-\xi^2}\right] = 0.1\sqrt{1-\xi^2},$$
$$T_s \cong \frac{1}{\xi\omega_n}ln\frac{50}{\sqrt{1-\xi^2}}, T_o = \frac{2\pi}{\sqrt{1-\xi^2}\omega_n}, \qquad (6c, d)$$
$$T_{max} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}, \quad y_{max} = 1 + \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}, \qquad (6e, f)$$
$$y_{osh} = \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}, \quad POSH = \frac{100 e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}, \qquad (6g, h)$$
$$RR = e^{\frac{-2\xi\pi}{\sqrt{1-\xi^2}}}. \qquad (6i)$$

For the succeeding peaks, Eqs. (6e-i), should be modified by using $(2k - 1)\pi$ instead of $\pi$ where $k$ represents the

peak numbers; for the first peak $k = 1$ and Eqs. (6e-i) are valid.

From Eq. (2e), the un-damped response of a second order systems has

$$\tau = 1 + \frac{cos^{-1}\left(\frac{1}{e}\right)}{\omega_n}, \qquad (7a)$$

$$T_r = \frac{cos^{-1}(0.1) - cos^{-1}(0.9)}{\omega_n}, \qquad (7b)$$

$$T_s = \infty, \quad T_o = \frac{2\pi}{\omega_n}, \quad T_{max} = \frac{\pi}{\omega_n}, \quad (7c, d, e)$$

$$y_{max} = 2, y_{osh} = 1, \qquad (7f, g)$$

$$POSH = 100, \quad RR = 1. \qquad (7h, i)$$

To be able to make comparisons with the step response characteristics of the fractional order system dealt with in the next section, and as well as for the sake of completeness, the step responses of the second order system in Eq. (2a) are shown in Fig. 1 for a few values of $\xi$. In the same respect, some of the time domain characteristics are plotted in Fig. 2. The derived transcendental equations for $\tau$ and $T_r$ (see Eqs. (4a,6a) and Eqs. (4a,6b), respectively) are solved numerically to obtain the plots for relevant characteristics in Fig. 2. This approach is not needed for obtaining the remaining characteristics in the figure for the necessary formulas are presented explicitly.
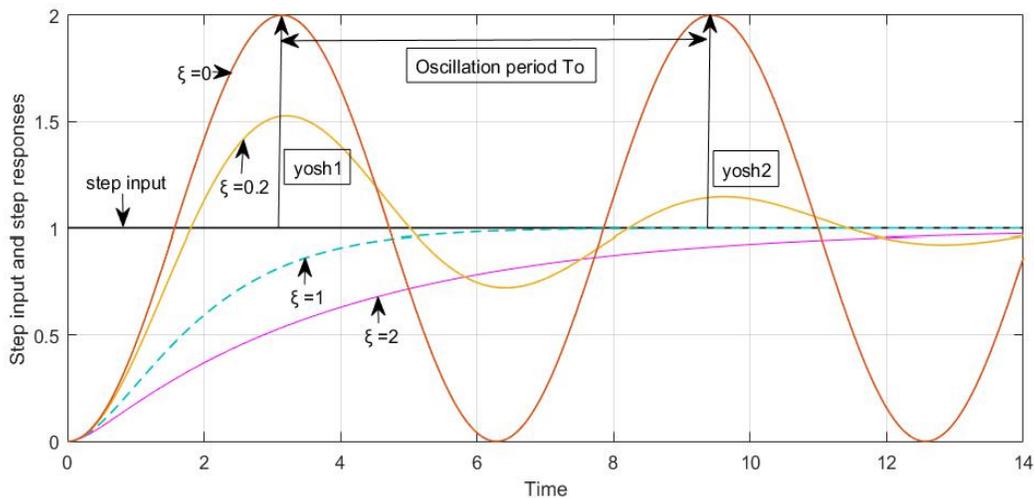


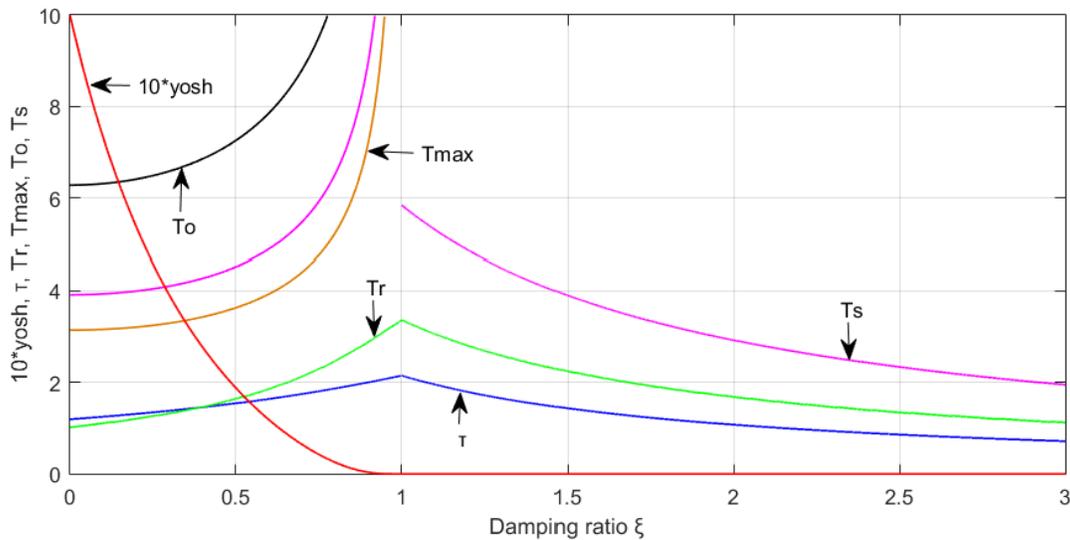*Fig.1: Step responses of the system in Eq. (2a) for values of $\xi$ =2,1,0.2, 0; $\omega_n = 1$.*



*Fig.2: Some time domain characteristics of the system in Eq. (2a) for $\xi \in [0,3]$.*

### III.    INVESTIGATION OF THE SIMPLE FRACTIONAL ORDER SYSTEM

Consider the fractional order transfer function with a constant numerator and 2-term fractional denominator

$$H(s) = \frac{1}{s^\alpha + 1}. \qquad (8)$$

When $\alpha = 1\ (2)$, the transfer function is a 1st order (2nd order with $\omega_n = 1, \xi = 0$) integer type system.; hence, the step response is an increasing exponential (sustained oscillation). But there are differences in the step responses for the general values of $\alpha$:

Fig. 3 shows the step responses for $\alpha \in [0.01, 1.99]$. All the responses tend to the reference value 1 as $t \to \infty$, since inserting $s = 0$ and assuming $\alpha \neq 0$ in Eq. (8), dc gain of this system is seen as 1. For small values of $\alpha$ (for eg. $\alpha$ =0.01), the step response is almost equal to ½ (the transfer function is equal to a constant gain of ½ for $\alpha = 0$); except $t = 0$ and $t \to \infty$; for these values, the

response starts from 0 and tends to the reference value 1 (this can not be seen in the figure since the final value of $t$ is chosen 30 in simulations). In the figure, exponential like increase of step response for $0 < \alpha \leq 1$, exponential-like decaying oscillations for large $\alpha$ (for eq. $\alpha = 1.99$) are obviously seen. For $\alpha > 1$ but not too much, there is an overshoot in the response then after exponential-like decaying to 1 without oscillations are also seen. For $\alpha > 2$, the system is unstable, and the step response increases exponentially and oscillatory like manner, which is a case not shown in the figure.

We note that those responses for $1 < \alpha \leq 2$ and $0 < \alpha \leq 1$, although resemble to those of a second order and of a first order integer order systems, respectively, there are important differences.
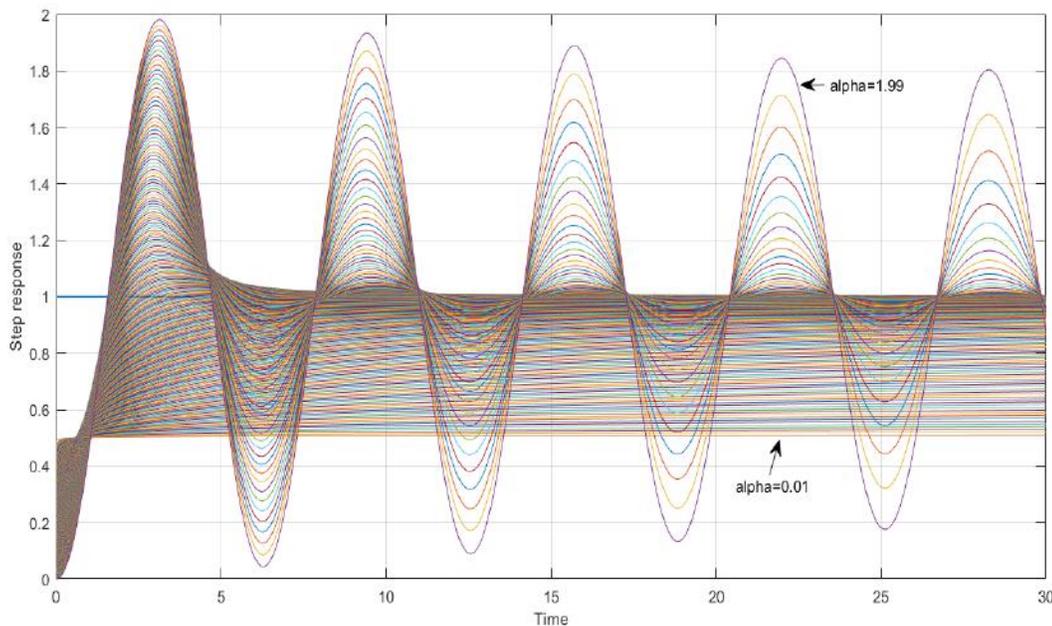


*Fig.3: Step responses of the FOS in Eq. (8) for different values of α.*

For example, if there exist oscillations, the oscillation period is not constant and naturally the peak times do not occur uniformly as shown in Fig. 4. This figure shows the change of oscillation periods starting from each $T_{max}$. Numerical data show that the first oscillatory like response (number of maximums = 2) occurs for $\alpha = 1.34$, for which the period of oscillation is $6.1750$. But this case is not observed on the graph. The plot starts from the case where there exists 3 maximums and

resultantly 2 oscillation periods; this case occurs for $\alpha = 1.49$ for which the first period is 7.188 and the second one is $5.4470$. The plot for this case is indicated in the figure. It is seen that the first oscillation period for each $\alpha$ is maximum and it decreases with increasing $\alpha$. Further, for each $\alpha$ the oscillation period decreases for subsequent oscillations. This decrease reaches to its highest value just before the oscillations stop. For $\alpha = 1.99 \cong 2$, period of oscillations is almost constant being 6.284 for first

oscillation, $6.283, 6.283, 6.284$ for the following 3 oscillations. The change in the $3^{rd}$ digit after decimal,

though expected since $\alpha = 1.99$, may originate from numerical errors.
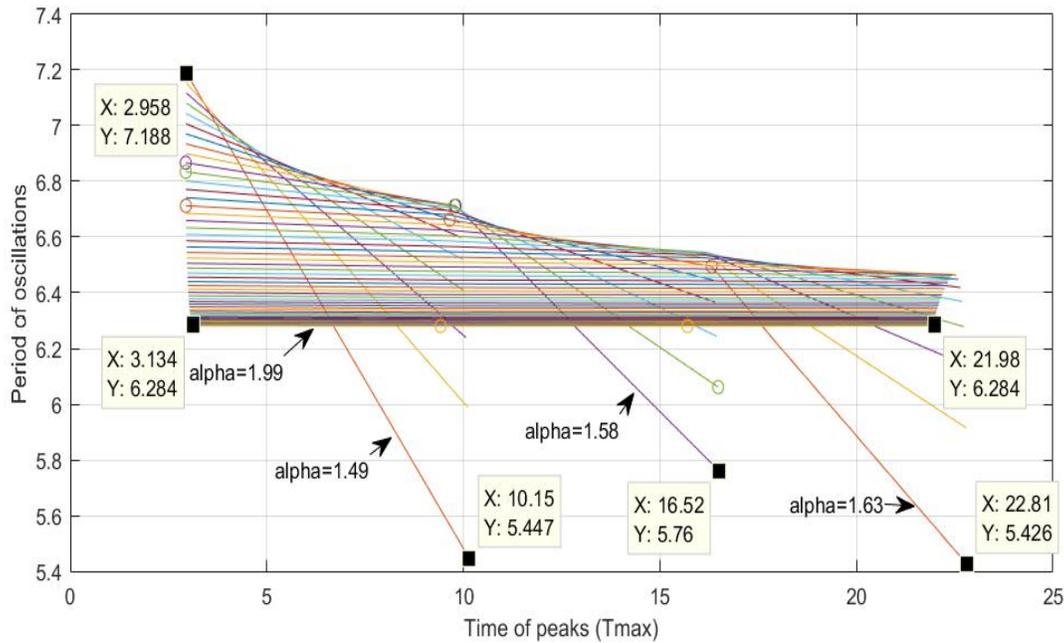


*Fig.4: Change of oscillation period with $T_{max}$.*

The exponential decays or increases do not have fixed time constants; hence reduction ratio decreases for succeeding maximum values as seen in Fig. 5. This figure shows the reduction ratio of subsequent peaks. Reduction ratio starts from $\alpha = 1.34$ where at least 2 peaks (maximums) occur for the first time (Reduction ratio = $14.4192$) and ends at $\alpha = 1.99$ where 5 peaks occur until $t = 30\ s$. Hence, reduction ratio is defined for the first 4 peaks. In the figure, the plot for $\alpha =$ $1.34, 1.35, ...,1.48$, reduction ratio can't be plotted (since point plot is not used). Therefore, the first curve starts from $\alpha = 1.49$ for which 3 peaks occur, and 2 reduction ratios are defined ($20.0430$ and $3.3023$). It is seen that the reduction ratio from the first peak to the second one decreases as $\alpha$ is changing from $1.49$ to $1.99$. Further, the ratio of the subsequent peaks for each $\alpha$ decreases with time (or the oscillation number).
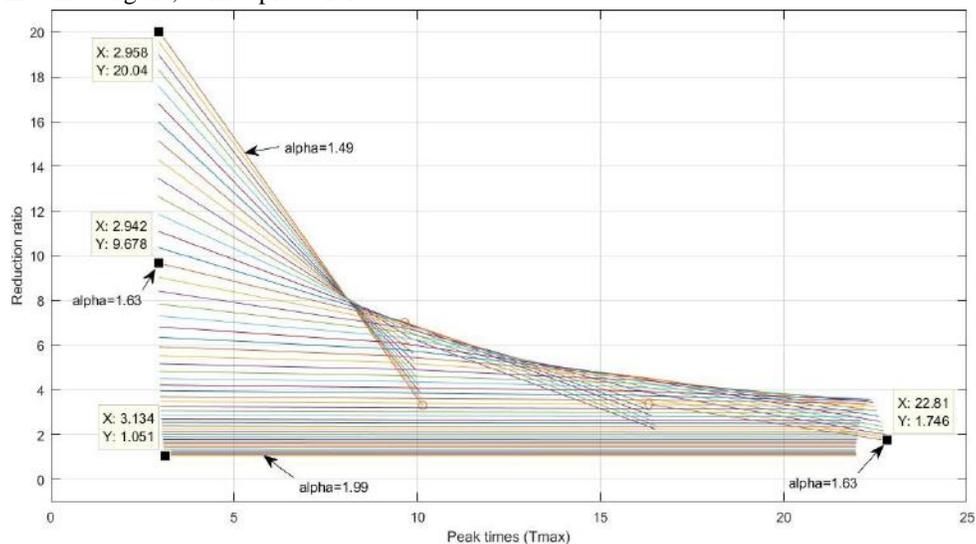


*Fig.5: Reduction ratio of subsequent peaks against peak times $T_{max}$.*

The size of overshoots is shown in Fig. 6. For $\alpha = 1.2$, there is a single overshoot which is $0.0744$; this is not shown in the graph since there is not a second one so that response is not oscillatory.

In Fig. 6, the values of overshoots are plotted against the time of maximums. The first maximum occurs for $\alpha = 1.01$; since a single point plot is not shown, the plot in the figure is started from $\alpha = 1.34$ where the second peak occurs for the first time; (for $\alpha = 1.34$, the first overshoot and the second one are 0.1640 and 0.0114, respectively). Hence, for $\alpha \in [1.01, 1.33]$ there is one maximum in the step response and no other maximums occur until $\alpha = 1.34$. Note that this last statement is confined to the present numerical data. More elaborate numerical simulation covering values of $\alpha$ to four decimal digits show that this interval is $\alpha \in [1.0001, 1.3395]$, and no other maximums occur until $\alpha = 1.3396$.

In the simulation results shown in Fig. 6, for $\alpha \in [1.63, 1.99]$, there are five peaks in the step response. The overshoots, as a general rule, change so that for each $\alpha$ subsequent peaks decrease, while the rate of decrease gets stronger. Note that among the peaks occurring for $\alpha = 1.67$, the minimum one is the fifth one and it is equal to $0.0001018660$; this point could not be data tipped due to the crowdedness of curves near to it, the minimum peak value is found from the workspace data.
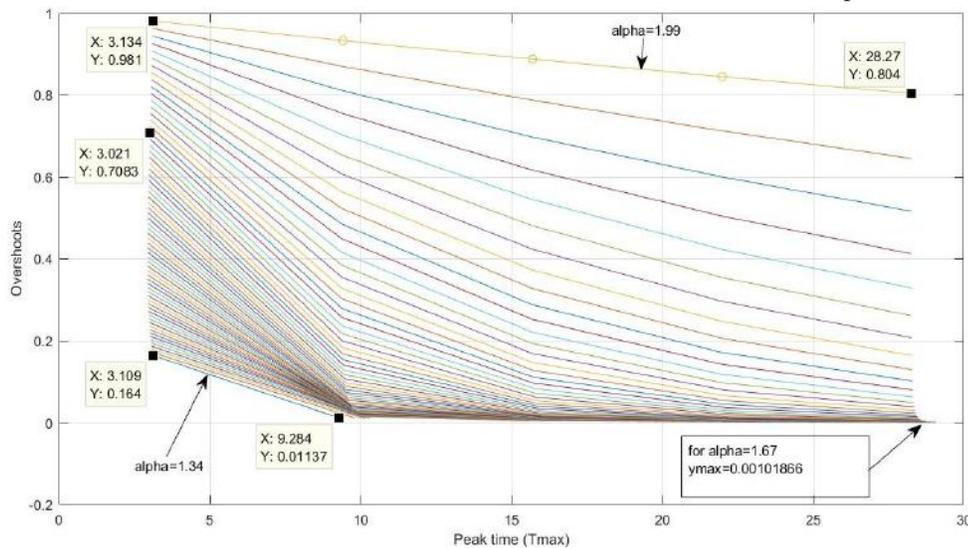


*Fig.6: Overshoot against peak times $T_{max}$.*

Fig. 7 shows the maximum value of step response against its occurrence time. This figure is plotted for the case of responses that have at least 2 maximum values for $t \in [0, 30]$. These responses are for $\alpha = 1.34, 1.35, \ldots, 1.99$. $\alpha = 1.34$ is the first value of $\alpha$ where at least 2 maximums (hence oscillatory like response) occur. More elaborate numerical simulation shows that this value of $\alpha$ is $\alpha = 1.3396$ (as mentioned) for which oscillatory damping starts with period of oscillation $6.0680$. After the oscillations start at $\alpha = 1.34$, the number of oscillations increases with increasing $\alpha$ so that 5 maximums occur for $\alpha = 1.99$ in $[0, 30]$. The time of occurrence of the maximums are not the same; they get near to each other as $\alpha \to 1.99$, and they get more different values as $\alpha \to 1.34$.
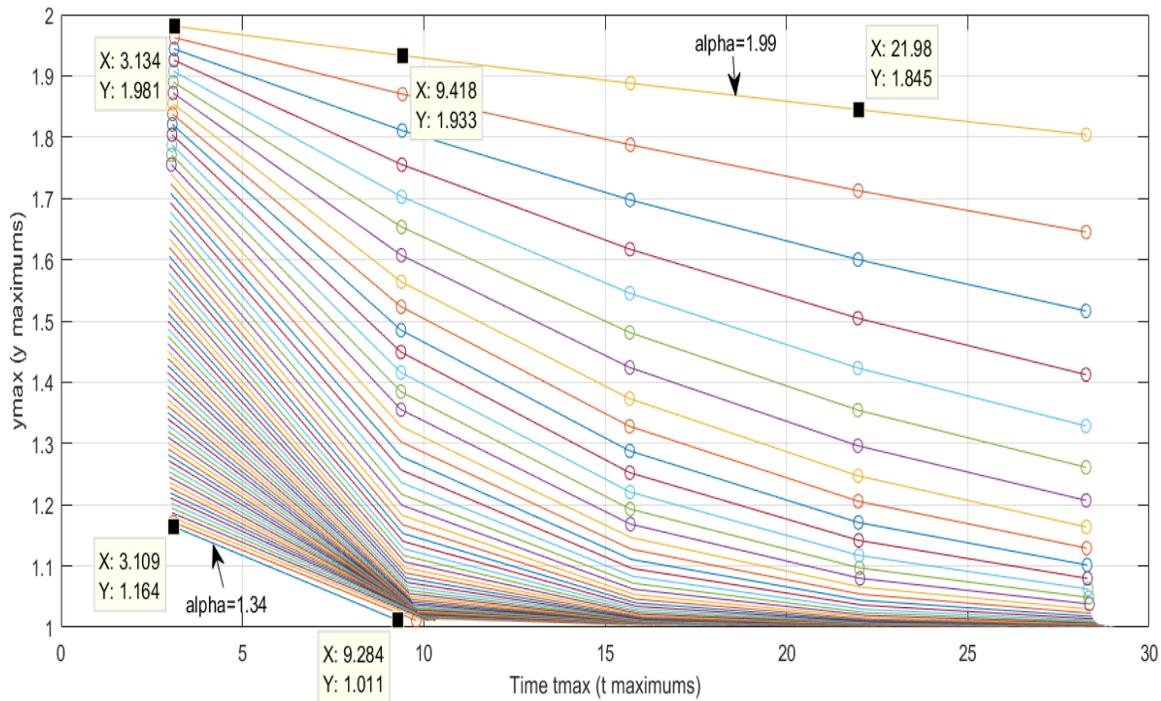
*Fig.7: Maximum value of step response against the time of occurrence.*

Next, it is worth to observe the change of first oscillation period and the number of peak values in the first $30\ s$. These are plotted in Fig. 8. It is observed that there are no oscillations in the step response for $\alpha = 0, 0.5, 1$. For $\alpha = 1.2$, there are no oscillations but there exist 1 maximum with an overshoot of $0.0744$ as mentioned before. For $\alpha = 1.7$, $\alpha = 2$, and $\alpha = 2.018$ there are 5 maximums in the interval $t \in [0, 30]$; whilst period of first oscillation changes with $\alpha$, i.e., it is $6.543$ for $\alpha = 1.7$, and it is $6.283$ for both $\alpha = 2$ and $\alpha = 2.018$. Note also that period of succeeding oscillations is not constant for the same $\alpha$.
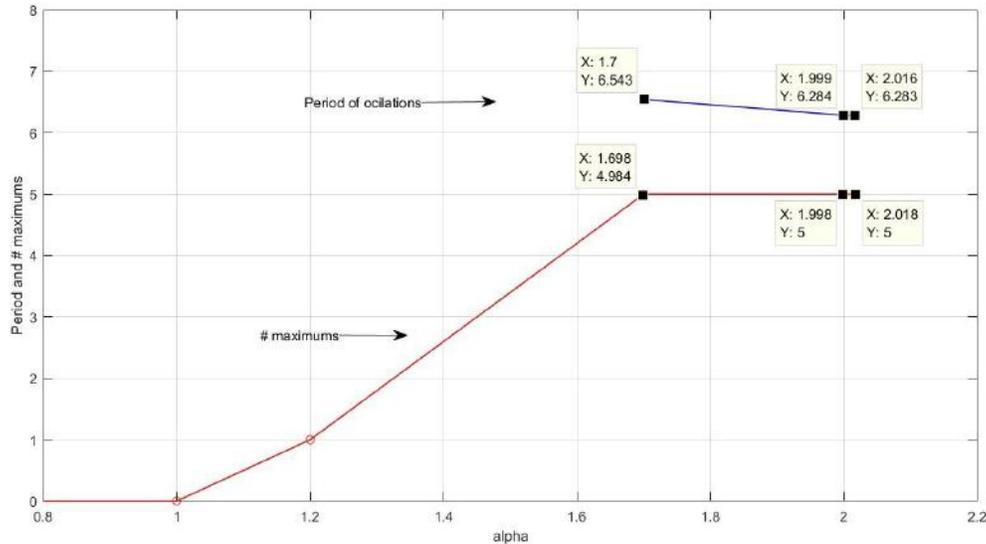


*Fig.8: Period of first oscillation and number of peaks in 30 s against $\alpha$.*

In Fig. 9, the variation of the time constant $\tau$, rise time $T_r$, and settling time $T_s$ are plotted for $\alpha = 0.5, 1, 1.2, 1.7$. It

is seen that $\tau$ is the least effected by $\alpha$. $T_r$ could not be recorded for $\alpha = 0.5$ since it is greater than total run time $t = 30\ s$. For $\alpha = 1, \tau < T_r < T_s$, with values $1.001 < 2.198 < 3.913$. It is important to note that the settling

time gets large very rapidly with increasing $\alpha$; infact for $\alpha \geq 2$ it is $\infty$. Therefore, the data for $\alpha = 2$ and $\alpha = 2.018$ is not included in the plot.
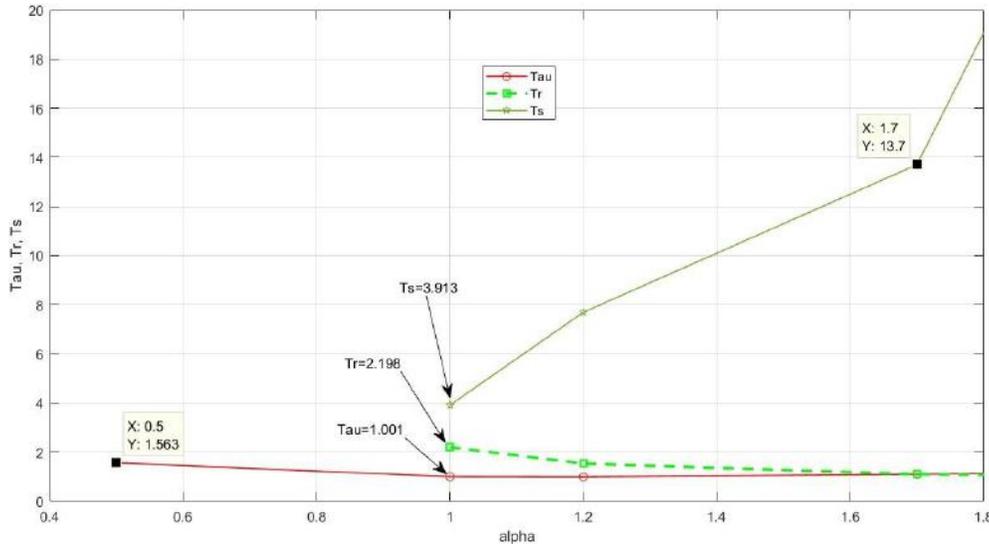


*Fig.9: Variation of time constant ($\tau = Tau$ ), risetime ($T_r$) and settling time ($T_s$) with $\alpha$ included in the plot.*

Finally, in Fig. 10, the damping ratio $\xi$ of a 2nd order system and the fractional power $\alpha$ of the fractional order system are compared with respect to the overshoot. The

figure can be used for finding the values of $\xi$ and $\alpha$ for the desired overshoot.
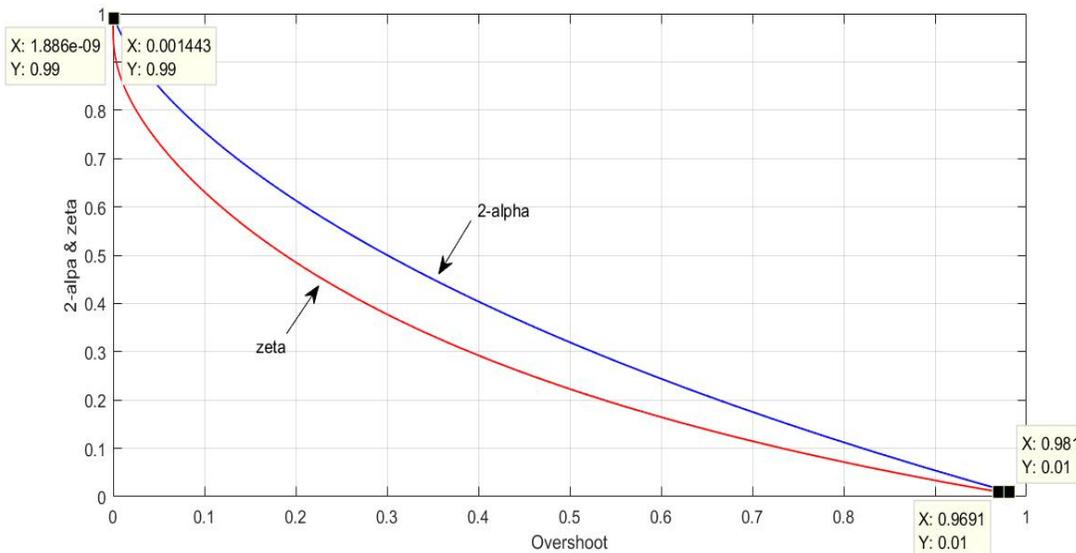


*Fig.10: $\beta = 2 - \alpha$ versus overshoot and $\xi$ versus overshoot.*

The step response characteristic investigated so far are already enough to highlight the important properties and differences between some simple integer order and fractional order systems. Therefore, this section is ended at this step. However, for a continuous variation of step

response characteristics, such as period of oscillations, number of peaks, $\tau$, $T_r$, $T_s$, POSH with the system parameters $\alpha$ and $\xi$, as well as variation of $\alpha$ for fractional order, and $\xi$ for second order systems with the overshoot can be obtained, these are not included for the length

limitations of the paper. Further, they are not vital to observe the following comparisons.

## IV. DIFFERENCES BETWEEN 2$^{ND}$ ORDER AND SIMPLE FRACTIONAL ORDER CHARACTERISTICS

We have the apparent observation from the previous two sections that the following important similarities and differences appear between the step responses of a second order system and the simple FOS considered. The comparison is done on the base of the transfer functions in Eq. (2a) for a second order integer system and Eq. (8) for a FOS. Being considered before, the extreme cases $\alpha = 0$, $\alpha = 0$, and $\alpha = 2$ are not included in the comparison.

1) All the time domain characteristics of a second order system are expressed by analytic formulas as given in Eqs. (4-7). For FOS, there does not exist such analytical formulas; only graphical dependences on the fractional power $\alpha$ are available. See Figs. 4-9.

2) For $1 \leq \xi$ and $0 < \alpha \leq 1$, both of the systems exhibit monotonically increasing step responses starting from 0 and tending to the final value 1 as $Lim \ t \to \infty$. There is no overshoot in the responses. See Fig. 1 for 2$^{nd}$ order system, and Fig. 2 for fractional one.

3) For $0 \leq \xi < 1$ and $1 < \alpha \leq 2$, both systems generate overshoots which decay to zero and the step responses tend to the final value 1 as $Lim \ t \to \infty$. See Fig. 1 and Fig. 2, respectively.

4) For $0 < \xi < 1$ and $1 < \alpha \leq 2$, although the step responses of both the second order system and that of the FOS are similar in the sense that they are not monotonically increasing, they have overshoots, and they approach to unity at infinite time, they have completely different characteristics:

   a) There are infinitely many overshoots in the step response of the 2$^{nd}$ order system, whilst the number of overshoots in the step response of the fractional system increases with $\alpha$. For example, there is only one overshoot and naturally one maximum for $1 < \alpha \leq 1.3395$, and the second overshoot starts for $\alpha = 1.3396$. Refer to Fig. 8 to see how the number of maximums (oscillations) increases with $\alpha$.

   b) The period of sustained oscillations and their damping rate, that is RR, are constant for each $0 < \xi < 1$ and they do not change with time;

they are given by Eqs. (6d) and (6i), respectively. On the other hand, these quantities change with time for FOS; see Fig. 4 for the period of decaying oscillations (oscillation period decreases with time), and Fig. 5 for RR (RR decreases with time).

   c) As seen in Fig. 2, the period of oscillations monotonically increases with $\xi$ for the second order system. On the other hand, as seen in Fig. 5 for the FOS, starting from 6.175 at $\alpha = 1.34$ the duration of the first oscillations first increases and it reaches to its maximum value 7.327 at $\alpha \leq 1.43$ and then decreases to 6.284 at $\alpha = 1.99$.

## V. CONCLUSIONS

Time domain step response characteristics of the fractional order system with unity numerator and a 2-term denominator polynomial involving a single fractional power is studied in this paper. Many step response characteristics which are important for control engineering such as rise time, settling time, delay time, overshoot, oscillation period, damping time constant of oscillations, and some others are investigated, and the results are presented in graphical forms by figures. The study is conducted comparatively by considering integer order systems of 1$^{st}$ and 2$^{nd}$ order types. It is shown that the same simplicity and explicitness present for second order systems do not exist between the transfer function parameters and the step response characteristics for low order fractional order control systems. The results bring light for the design such systems which has been a vacancy fulfilled by this paper.

The derived transcendental equations for $\tau$ and $T_r$ in Eqs. (4a,6a) and Eqs. (4a,6b), respectively), which have, as far as author knowledge, not been appeared in the literature before, are derived and solved numerically to obtain the plots for relevant characteristics in Fig. 2. This approach is not needed for obtaining the remaining characteristics in the figure for the necessary formulas are presented explicitly. For the simulation of the FOS, the subprograms of FOMCON toolbox of Aleksei Tepljakov [1] integrated with MATLAB R2017 [7] is used.

## REFERENCES

[1] A. Tepljakov, "FOMCON: Fractional-order modeling and control toolbox for MATLAB", Proc. of the 18$^{th}$ International Conference "Mized Design

of Integrated Circuits and Systems", Poland, pp. 684-689, 2011.

[2] A. Elwakil, "Fractional order circuits and systems: An engineering interdisiplinary reserch are", IEEE Circuits and System Magazine, vol. 10, pp. 40-50, 2010.

[3] T.M. Marinov, N. Ramicez, and F. Santamaria, "Fractional integration toolbox", Fractional Calculus and Applied Analysis, vol. 16, no. 3, pp. 670-681, 2013.

[4] D.P. Mata, O. Valerio, V. Ninteger, "Fractional Control Toolbox for MATLAB: User and Programmer Manual", Institute Superiok Tecnico, Universidade Tecnica de Lisboa, August 2005.

[5] Brief Presentation of the Object Oriented Crone toolbox, Crone Research group, IMS, Universite' de Bordeo, 2010.

[6] S. Dormido, E. Pisoni, and A. Visioli, "Interactive tools for designing fractional-order PID controllers", International Journal of Innovative Computing, Informaiton and Control, vol. 8, no. 7(A), pp. 4579-4590, 2012.

[7] C.A. Monje, B.M. Vinagre, V. Feliv, and Y.Q. Chen, "Tuning and auto-tuning of fractional order controllers for industry applications", Control Engineering Practice, vol. 16, pp. 798-812, 2008.

[8] Y.Q. Chen, I. Petras, and D. Xue, "Fractional-order control-a tutorial", 2009 American Control Conference, USA, pp. 1397-1411, 2009.

[9] L. Marinangeli, F. Alijani, S.H. HosseinNia, "Fractional-order positive position feedback compensator for active vibration control of a smart composite plate", Journal of Sound and Vibration, vol. 412, pp. 1-16, 2018.

[10] S. Ijaz, L. Yan, M.T. Hamayun, "Fractional order modeling and control of dissimilar redundant actuating system used in large passenger aircraft", Chinese Journal of Aeronautics, vol. 31, no. 5, pp. 1141-1152, 2018.

[11] Q. Xu, S. Zhuang, X. Xu, C. Che, "Stabilization of a class of fractional-order nonautonomous systems using quadratic Lyapunov functions", Advances in Difference Equations, vol. 2018, pp. 1-14, 2018.

[12] S. Mahata, S. Saha, R. Kar, D. Mandal, "Optimal design of wideband fractional order digital integrator using symbiotic organisms search algorithm", IET Circuits, Devices & Systems, vol. 12, no. 4, pp. 362-373, 2018.

[13] D.A. John, K. Biswas, "Analysis of disturbance rejection by PI$\lambda$ controller using solid state fractional capacitor", IEEE Proc. of the International Symposiums of Circuits and Systems, pp. 1-5, 2018.

[14] D. Kubanek, T. Freeborn, "(1+$\alpha$) Fractional-order transfer functions to approximate low-pass magnitude responses with arbitrary quality factor", AEU-International Journal of Electronics and Communications, vol. 83, pp. 570-578, 2018.

[15] M. Sowa, "Application of SubIval in solving initial value problems with fractional derivatives", Applied Mathematics and Computation, vol. 319, pp. 86-103, 2018.

[16] K. Lu, W. Zhou, G. Zeng, W. Du, "Design of PID controller based on a self-adaptive state-space predictive functional control using extremal optimization method", Journal of the Franklin Institute, vol. 355, no. 5, pp. 2197-2220, 2018.

[17] M. Zagorowska, "Analysis of performance indicators for optimization of noninteger-order controllers", Proc. of the 39th International Conference on Telecommunications and Signal Processing, pp. 615-619, 2016.

[18] F. Golnaraghi, B.C. Kuo, Automatic Control Systems, 9th Edition, Wiley John Wiley & Sons, Inc., ISBN-13 978-0470-04896-2, 2010.