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The Global Attractors for the Higher-order Kirchhoff-type Equation with Nonlinear Strongly Damped Term

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***Abstract*— The paper studies the longtime behavior of solutions to the initial boundary value problem for a class of Higher-order Kirchhoff models:**

$$u_{tt}+\sigma(\|\nabla^m u^2\|)(-\Delta)^m u_t+\phi(\|\nabla^m u\|^2)(-\Delta)^m u+g(x,u)=f(x).$$

For strong nonlinear damping σ and ϕ, we make assumptions (H1)-(H3). $g(x,u)$ are nonlinear function specified later , we make assumptions (G1)-(G3). Under of the proper assume, the main results are existence and uniqueness of the solution are proved , and deal with the global attractors in natural energy space $X=H_0^m\times H$.

***Index Terms*— strongly nonlinear damped, Higher-order Kirchhoff equation, the global attractors. 2010 Mathematics Classification: 35B41, 35G31**

I. INTRODUCTION

We consider the following Higher-order Kirchhoff-type equation with nonlinear strongly damping:

$$u_{tt}+\sigma(\|\nabla^m u\|^2)(-\Delta)^m u_t+\phi(\|\nabla^m u\|^2)(-\Delta)^m u+g(x,u)=f(x),(x,t)\in\Omega\times[0,+\infty), \tag{1.1}$$

$$u(x,t)=0,\ \frac{\partial^i u}{\partial\nu^i}=0,\ i=1,2,\dots,m-1,\ x\in\partial\Omega,t\in(0,+\infty), \tag{1.2}$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega. \tag{1.3}$$

where Ω is a bounded domain of R^n with a smooth dirichlet boundary $\partial\Omega$ and initial value, and $m>1$ is an integer constant. Moreover, ν is the unit outward normal on $\partial\Omega$. $\sigma(s)$ and $\phi(s)$ are scalar functions specified later, $g(x,u)$ are nonlinear function specified later. And $f(x)$ is an external force term.

This kind of wave models goes back to G. Kirchhoff **Error! Reference source not found.**. In 1883,Kirchhoff proposed the following model in the study of elastic string free vibration:

$$u_{tt}-\alpha\Delta u-M\left(\|\nabla u\|^2\right)\Delta u=f(x,u) \tag{1.4}$$

where α is associated with the initial tension, M is related to the material properties of the rope, and $u(x,t)$ indicates the vertical displacement at the x point on the t. The equation is more accurate than the classical wave equation to describe the motion of an elastic rod.

This kind of wave models has been studied by many authors under different types of hypotheses. There have been many researchers on the global attractors existence of Kirchhoff equation, we can refer 0 0 **Error! Reference source not found. Error! Reference source not found. Error! Reference source not found.**. What's more, the global attractors for the Higher-order Kirchhoff-type equation are investigated and we refer to **Error! Reference source not found. Error! Reference source not found.**.

Igor Chueshov **Error! Reference source not found.** studied the longtime dynamics of Kirchhoff wave models with strong nonlinear damping:

$$u_{tt}-\sigma\left(\|\nabla u\|^2\right)(\Delta)u_t-\phi\left(\|\nabla u\|^2\right)(\Delta)u+f(u)=h(u),x\in\Omega,t>0. \tag{1.5}$$

in natural energy space $H(\Omega)=H_0^1(\Omega)\cap L^{p+1}(\Omega)\times L^2(\Omega)$. His results allow that the growth exponent p of the nonlinearity $g(u)$ is supercritical, that is, $p^*<p<p^{**}$. Here the growth exponent p^* is called critical for $H^1(\Omega)\subset L^{p+1}(\Omega)$ as $p\le p^*$. He established a finite-dimensional global attractor in the sense of partially strong topology in $H(\Omega)$. In particular, in nonsupercritical case: (i) the partially strong topology becomes strong; (ii) an exponential attractor is obtained in $H(\Omega)$ by virtue of the strong quasi-stability estimates. Moreover, Chueshov[3] also studied the global well-posedness and the longtime dynamics for the Kirchhoff equations with a structural damping of the form $\sigma\left(\|\nabla u\|^2\right)(-\Delta)^\theta u_t$, with $1/2\le\theta<1$, at an abstract level. For the related works on the quasilinear wave equations (rather than the semilinear ones) with strong damping.

Recently, Yang, Ding and Liu[4] put forward a functional analysis method and used it to construct a a bounded absorbing set in $H(\Omega)$, which is of higher

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global-regularity. They removed the restriction of "partially strong topology" for $H(\Omega)$ in [2] and established a strong global attractor in supercritical nonlinearity case.

About Higher-order Kirchhoff-type equation, Ling Chen, Wei Wang and Guoguang Lin[7] investigate the global well-posedness and the longtime dynamics of solution for the higher-order Kirchhoff-type equation with nonlinear strongly dissipation:

$$u_{tt}+(-\Delta)^m u_t+\phi(\|D^m u\|^2)(-\Delta)^m u+g(u)=f(x) \quad (1.6)$$

Under of the proper assume, the main results are that existence and uniqueness of the solution is prove by using priori estimate and Galerkin method, the existence of the global attractor with finite-dimension, and estimation Hausdorff and fractal dimension of the global.

Subsequently, Yuting Sun, Yunlong Gao and Guoguang Lin[8] investigate the global well-posedness and the global attractors of the solutions for the Higher-order Kirchhoff-type wave equation with nonlinear strongly damping:

$$u_{tt}+\sigma\left(\|\nabla^m u\|^2\right)(-\Delta)^m u_t+\phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u=f(x). \quad (1.7)$$

For strong nonlinear damping σ and ϕ, we make assumptions (H_1) - (H_4). Under of the proper assume, the main results are existence and uniqueness of the solution in $H^{2m}(\Omega)\times H_0^m(\Omega)$ are proved by Galerkin method, and deal with the global attractors.

At present, most Higher-order Kirchhoff-type equations investigate the global. On the basis of Yang, we investigate the global attractor of the higher-order Kirchhoff-type equation (1.1) with strong nonlinear damping. Such problems have been studied by many authors, but $\sigma(\|\nabla^m u\|^2)$ is a definite constant and even $\sigma(\|\nabla^m u\|^2)=0$. Generally, the equation exist a nonlinear $g(u)$. But in the paper, $\sigma(\|\nabla^m u\|^2)$ is a scalar function and exist a nonlinear $g(x,u)$. In section 2, we give some proper assumes. In section 3, the existence and uniqueness of solution is proved. In section 4, we deal with the global attractor.

II. Preliminaries

For brevity, we denote the simple symbol, $H=L^2(\Omega)$, $H_0^m=H_0^m(\Omega), H_0^{m-1}=H_0^{m-1}(\Omega)$, $X=H_0^m\times H$, $f=f(x)$, $\|\cdot\|$ represents norm, $(\cdot,\cdot)$ represents inner product. λ is the first eigenvalue of the operator $(-\Delta)$. C is a constant.

In this section, we present some assumptions need in the proof of our results. For this reason, we assume nonlinear term $g(\cdot,0)\in L^2$,satisfies that

(G1) $N=2,\ \left|\frac{\partial g}{\partial u}\right|\le c(1+|u|^{p-1}),\ \ p\ge 1;$

(G2) $N=3,\ \left|\frac{\partial g}{\partial u}\right|\le c\left(1+|u|^{p-1}\right),\ 1\le p\le\frac{N+2}{N-2};$

(G3) $\tilde{G}(u)=\int_\Omega G(x,u)dx,\quad G(x,u)=\int_0^u g(x,\tau)d\tau$

We make the following hypotheses on the function $\sigma(s)$ and $\phi(s)$:

(H1) $\phi(s)\in C^1\ \ and\ \ \sigma(s)\in C^1$;

(H2) $\phi'(s)\ \ge\ 0\ \ amd\ \ \sigma'(s)\ \ge\ 0$;

(h3) $\phi(s)\ge 0,\ \ and\ \ \sigma(s)\ge 0$.

III. The Existence and Uniqueness of Solution

Lemma 3.1 [5](Gronwall-type lemma) Let X be a Banach space, and let $Z\subset C(R^+,X)$. Let $\Phi\colon X\to R$ be a continuous function such that

$$\sup_{t\in R^+}\Phi(z(t))\ge-\eta,\ \ \Phi(z(0))\le K. \quad (3.1)$$

for some $\eta, K>0$ and every $z\in Z$. In addition, assume that for every $z\in Z$ the function $t\mapsto\Phi(z(t))$ is continuously differentiable, and satisfies the differentiable inequality

$$\frac{d}{dt}\Phi(z(t))+\delta\|z(t)\|_X^2\le k \quad (3.2)$$

for some $\delta>0$ and $k>0$ independent of $z\in Z$. Then, for every $\gamma>0$, there is $t_0=\frac{\eta+K}{\gamma}>0$, such that

$$\Phi(z(t))\le\sup_{\varsigma\in X}\left\{\Phi(\varsigma):\delta\|\varsigma\|_X^2\le k+\gamma\right\}\quad t\ge t_0. \quad (3.3)$$

Lemma 3.2[5] Let $y\colon R^+\to R^+$ be an absolutely continuous function satisfying

$$\frac{d}{dt}y(t)+2\varepsilon y(t)\le h(t)y(t)+z(t),\quad t>0 \quad (3.4)$$

where $\varepsilon>0,\ z\in L^1_{loc}(R^+),\ \int_s^t h(\tau)d\tau\le\varepsilon(t-s)+m$

for $t \ge s \ge 0$ and some $m > 0$. Then

$$y(t) \le e^m (y(0)e^{-\varepsilon t} + \int_0^t |z(t)| e^{-\varepsilon(t-\tau)} d\tau), \quad t > 0 \qquad (3.5)$$

Theorem 3.1[5] Let assumption (G1)-(G3) and (H1)-(H3) be in force. Then problem (1.1)-(1.3) admits a unique solution $u \in L^\infty(0,\infty; H_0^m)$, with $u_t \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; H_0^m)$. This solution possesses the following propertiess:

(1)

$$\|u_t\|^2 + \|\nabla^m u\|^2 + \int_0^t \|\nabla^m u_t\|^2 d\tau \le c(R_0, \|f\|), \quad t \ge 0 \qquad (3.6)$$

(2) For any $a > 0$,

$$u_t \in L^\infty(a,T; H_0^m), \quad u_{tt} \in L^\infty(a,T; H_0^{-m}) \qquad (3.7)$$

and there exists a small contant $a_2 > 0$, such that

$$\|\nabla u_t\|^2 + \|\nabla^{-m} u_{tt}\|^2 \le \frac{t^2+1}{t^2} c e^{-a_2 t} + c, \quad t > 0 \qquad (3.8)$$

(3) The following Lipschitz continuity holds:

$$\|(z, z_t)\|_X^2 \le c(\|(z_0, z_1)\|_X^2) e^{-a_6 t}, \quad t \ge 0 \qquad (3.9)$$

for some $a_6 > 0$, where $z = u^1 - u^2$, u^1, u^2 are the solution of problem (1.1)-(1.3) corresponding to initial data (u_0^i, u_1^i) $i = 1,2$, respectively.

Proof: (1) We use $u_t + \varepsilon u$ multiply both sides of equation (1.1) and obtain,

$$\frac{d}{dt}\Phi(\xi_u(t)) + K(\xi_u(t)) = 0 \qquad (3.10)$$

where $\xi_u(t) = (u(t), u_t(t))$

$$\Phi(\xi_u(t)) = \frac{1}{2}\left(\|u_t\|^2 + \int_0^{\|\nabla^m u\|^2} \phi(s)ds + 2\tilde{G}(u) - 2(f,u)\right) + \varepsilon\left(\int_0^{\|\nabla^m u\|^2} \sigma(s)ds + (u_t, u)\right) \qquad (3.11)$$

$$K(\xi_u(t)) = \sigma\left(\|\nabla^m u\|^2\right)\|\nabla^m u_t\|^2 - \varepsilon\|u_t\|^2 + \phi\left(\|\nabla^m u\|^2\right)\|\nabla^m u\|^2 + \varepsilon(g(\cdot,u),u) - \varepsilon(f,u) \qquad (3.12)$$

Obviously, $\Phi\colon X \to R$ is continuous function.

From hypothesis, we get

$$\Phi(\xi_u(t)) \ge a\left(\|u_t\|^2 + \|\nabla^m u\|^2\right) - c(\|f\|), \qquad (3.13)$$

$$K(\xi_u(t)) \ge b\|\nabla^m u_t\| + a\left(\|u_t\|^2 + \|\nabla^m u\|^2\right) - c(\|f\|) \qquad (3.14)$$

where $a, b > 0$ and a, b is very small.

Substituting (3.13), (3.14) into (3.10), we have

$$\frac{d}{dt}\Phi(\xi_u(t)) + a\|\xi_u(t)\|_X^2 \le c(\|f\|). \qquad (3.15)$$

According to Lemma 3.1, we get

$$\Phi(\xi_u(t)) = \sup_{\varsigma \in X}\left\{\Phi(\varsigma) \middle| \|\varsigma\|_X^2 \le \frac{c(\|f\|)+1}{a}\right\}, \quad t \ge t_0 = c(R_0, \|f\|) \qquad (3.16)$$

Hence,

$$\|u_t\|^2 + \|\nabla^m u\|^2 \le c(R_0, \|f\|), \quad \forall t \ge t_0. \qquad (3.17)$$

Integrating (3.15) over the interval $[0, \ t]$, and $t \le t_0$, we can get

$$\|u_t\|^2 + \|\nabla^m u\|^2 \le c(R_0, \|f\|), \quad \forall t \in [0, t_0] \qquad (3.18)$$

Combining (3.17) and (3.18), when $\varepsilon = 0$, integrating (3.10) over the interval $[0, \ t]$, we obtain

$$\int_0^t \|\nabla^m u_t\|^2 d\tau \le c(\|R_0\|, \|f\|) \qquad (3.19)$$

According to (3.17), (3.18) and (3.19), we have

$$\|u_t\|^2 + \|\nabla^m u\|^2 + \int_0^t \|\nabla^m u_t\|^2 d\tau \le c(R_0, \|f\|) \qquad (3.20)$$

(2) Let $v = u_t$,

$$v_{tt} + \sigma\left(\|\nabla^m u\|^2\right)(-\Delta)^m v_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m v + \frac{\partial g}{\partial u}\cdot v + H(u, u_t; t) = 0 \qquad (3.21)$$

where

$$H(u, u_t; t) = 2\left(\sigma'\left(\|\nabla^m u\|^2\right)(-\Delta)^m v + \phi'\left(\|\nabla^m u\|^2\right)(-\Delta)^m u\right)(\nabla^m u, \nabla^m v)$$

We use $(-\Delta)^{-m} v_t + \varepsilon v$ multiply both sides of equation (3.21), and we get

$$\left(v_{tt}, (-\Delta)^{-m} v_t + \varepsilon v\right) = \frac{1}{2}\frac{d}{dt}\|\nabla^{-m} v_t\|^2 + \varepsilon\frac{d}{dt}(v_t, v) - \varepsilon\|v_t\|^2 \qquad (3.22)$$

$$\begin{aligned}&\left(\sigma\left(\|\nabla^m u\|^2\right)(-\Delta)^m v_t,(-\Delta)^{-m} v_t+\varepsilon v\right)\\&=\sigma\left(\|\nabla^m u\|^2\right)\|v_t\|^2+\frac{\varepsilon}{2}\sigma\left(\|\nabla^m u\|^2\right)\frac{d}{dt}\|\nabla^m v\|^2\\&=\sigma\left(\|\nabla^m u\|^2\right)\|v_t\|^2+\frac{1}{2}\frac{d}{dt}(\varepsilon\sigma\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2)\\&-\varepsilon\|\nabla^m v\|^2\sigma'\left(\|\nabla^m u\|^2\right)(\nabla^m u,\nabla^m v)\end{aligned} \tag{3.23}$$

$$\left(\phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m v,(-\Delta)^{-m} v_t+\varepsilon v\right)=\phi\left(\|\nabla^m u\|^2\right)(v,v_t)+\varepsilon\phi\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2 \tag{3.24}$$

$$\begin{aligned}&(H(u,u_t;t),(-\Delta)^{-m} v_t+\varepsilon v)\\&=2\left(\sigma'\left(\|\nabla^m u\|^2\right)(v,v_t)(\nabla^m u,\nabla^m v)+\phi'\left(\|\nabla^m u\|^2\right)(u,v_t)(\nabla^m u,\nabla^m v)\right)\\&+\varepsilon\cdot 2\left(\sigma'\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2(\nabla^m u,\nabla^m v)+\phi'\left(\|\nabla^m u\|^2\right)\|\nabla^m u\|^2(\nabla^m u,\nabla^m v)\right)\\&\le C_R\|\nabla^m v\|\|v_t\|+\varepsilon C_R\left((\nabla^m u,\nabla^m v)^2+(\nabla^m u,\nabla^m v)\|\nabla^m v\|^2\right)\\&\le C_R\|\nabla^m v\|\|v_t\|+\varepsilon C_R(1+\|\nabla^m v\|)\|\nabla^m v\|^2\end{aligned} \tag{3.25}$$

$$\left(\frac{\partial g}{\partial u}\cdot v,(-\Delta)^{-m} v_t+\varepsilon v\right)=\left(\frac{\partial g}{\partial u}\cdot v,(-\Delta)^{-m} v_t\right)+\varepsilon\int_\Omega\frac{\partial g}{\partial u}\cdot v^2dx \tag{3.26}$$

From assumption, we have

$$\begin{aligned}&\left(\frac{\partial g}{\partial u}\cdot v,(-\Delta)^{-m} v_t\right)=\int_\Omega\frac{\partial g}{\partial u}\cdot v\cdot(-\Delta)^{-m} v_t dx\\&\le C\left(\|v\|\cdot\|(-\Delta)^{-m} v_t\|+\|u\|_{p+1}^{p-1}\|v\|_{p+1}\|(-\Delta)^{-m} v_t\|_{p+1}\right)\\&\le\varepsilon'\|\nabla^{-m} v_t\|^2+C\|\nabla^m v\|^2\end{aligned} \tag{3.27}$$

$$\varepsilon\int_\Omega\frac{\partial g}{\partial u}\cdot v^2dx\le C\left(\varepsilon\|v\|^2+\varepsilon\|u\|_{p+1}^{p-1}\|v\|_{p+1}^2\right)\le C\|\nabla^m v\|^2 \tag{3.28}$$

Combining (3.22)-(3.28), we receive,

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\nabla^{-m} v_t\|^2+\varepsilon(v_t,v)+\frac{1}{2}\varepsilon\sigma\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2\right)\\&+\sigma\left(\|\nabla^m u\|^2\right)\|v_t\|^2+\varepsilon\phi\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2\\&\le C_R\|\nabla^m v\|\cdot\|v_t\|+\varepsilon C_R(1+\|\nabla^m v\|)\|\nabla^m v\|^2+\varepsilon\|v_t\|^2\\&+\varepsilon\|\nabla^m v\|^2\sigma'\left(\|\nabla^m u\|^2\right)(\nabla^m u,\nabla^m v)\\&-\phi\left(\|\nabla^m u\|^2\right)(v,v_t)+\varepsilon'\|\nabla^{-m} v_t\|^2+C\|\nabla^m v\|^2\\&\le\varepsilon C_R(1+\|\nabla^m v\|)\|\nabla^m v\|^2+2\varepsilon\|v_t\|^2+\varepsilon'\|\nabla^{-m} v_t\|^2\end{aligned} \tag{3.29}$$

Let

$$\Psi(t)=\frac{1}{2}\|\nabla^{-m} v_t\|^2+\varepsilon(v_t,v)+\frac{1}{2}\varepsilon\sigma\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2$$

Obviously,

$$a_R\left(\|\nabla^{-m} v_t\|^2+\|\nabla^m v\|^2\right)\le\Psi(t)\le b_R\left(\|\nabla^{-m} v_t\|^2+\|\nabla^m v\|^2\right) \tag{3.30}$$

According to (3.29), we obtain,

$$\frac{d}{dt}\Psi+a_2\Psi+a_3\|v_t\|^2\le C\|\nabla^m v\|\Psi+C\|\nabla^m v\|^2 \tag{3.31}$$

When $t\in(0,1]$, we use $t^\alpha(\alpha\ge 2)$ multiply both sides of equation (3.31) and obtain,

$$\begin{aligned}&\frac{d}{dt}(t^\alpha\Psi)+a_2t^\alpha\Psi+a_3t^\alpha\|v_t\|^2\\&\le C\|\nabla^m v\|t^\alpha\Psi+Ct^\alpha\|\nabla^m v\|^2\\&+\alpha t^{\alpha-1}b_R\left(\|\nabla^{-m} v_t\|^2+\|\nabla^m v\|^2\right)\end{aligned} \tag{3.32}$$

Due to

$$\|\nabla^{-m} v_t\|^2\le C\|v_t\|^\delta\|(-\Delta)^{-m} u_{tt}\|^{2-\delta},\quad\delta\in[1,2) \tag{3.33}$$

We can get

$$t^{\alpha-1}\|\nabla^{-m} v_t\|^2\le C_\delta t^{\alpha-1}\|v_t\|^\delta\le\varepsilon t^\alpha\|v_t\|^2+C_{\delta,\varepsilon} \tag{3.34}$$

Because (1.1), we have

$$(-\Delta)^{-m} u_{tt}=-(\sigma\left(\|\nabla^m u\|^2\right)u_t+\phi\left(\|\nabla^m u\|^2\right)u)+(-\Delta)^{-m}(f-g(\cdot,u)) \tag{3.35}$$

$$\|(-\Delta)^m u_{tt}\|\le C_R+\int_\Omega g(\cdot,u)dx\le\tilde{C}_R \tag{3.36}$$

Therefore, by the above inequality

$$\frac{d}{dt}(t^\alpha\Psi)+a_2t^\alpha\Psi+a_3t^\alpha\|v_t\|^2\le C\|\nabla^m v\|t^\alpha\Psi+\alpha b_R\|\nabla^m v\|^2 \tag{3.37}$$

Owing to

$$c\int_s^t\|\nabla^m v\|d\tau\le c\left(\int_s^t\|\nabla^m u_t\|d\tau\right)^{\frac{1}{2}}(t-s)^{\frac{1}{2}}\le\frac{k}{2}(t-s)+m \tag{3.38}$$

and by Lemma 3.2, we have

$$t^2\Psi\le C_1, \tag{3.39}$$

that is

$$\|\nabla^{-m} u_{tt}\|^2+\|\nabla^m u_t\|^2\le\frac{C_1}{t^2},\quad 0<t\le 1. \tag{3.40}$$

When $t \ge 1$, and according to (3.31) and Lemma 3.2, we will obtain

$$\left\|\nabla^{-m}u_{tt}\right\|^2 + \left\|\nabla^m u_t\right\|^2 \le c \cdot e^{-a_2 t} + c < c \quad (3.41)$$

Combining (3.40) and (3.41), we get

$$\left\|\nabla u_t\right\|^2 + \left\|\nabla^{-m}u_{tt}\right\|^2 \le \frac{t^2+1}{t^2} c e^{-a_2 t} + c, \quad t > 0 \quad (3.42)$$

(3) Let $z = u^1 - u^2$, $u^i(i=1,2)$ is the solution of the problem (1.1)-(1.3), and $\left(u_0^i, u_1^i\right) \in X$. Then the two equations subtract and obtain

$$z_{tt} + \frac{1}{2}\sigma_{12}(t)(-\Delta)^m z_t + \frac{1}{2}\phi_{12}(t)(-\Delta)^m z + G_{12}\left(u^1, u^2; t\right) = 0 \quad (3.43)$$

where $\sigma_{12} = \sigma_1 + \sigma_2$, $\phi_{12} = \phi_1 + \phi_2$, $\sigma_i = \sigma\left(\left\|\nabla^m u^i\right\|^2\right)$, $\phi_i = \phi\left(\left\|\nabla^m u^i\right\|^2\right)(i=1,2)$

$$G_{12}\left(u^1, u^2; t\right) = \frac{1}{2}\left\{(\sigma_1 - \sigma_2)(-\Delta)^m\left(u_t^1 + u_t^2\right) + (\phi_1 - \phi_2)(-\Delta)^m\left(u^1 + u^2\right)\right\} + g\left(\cdot, u^1\right) - g\left(\cdot, u^2\right)$$

By multiplying (3.43) by z_t, we get

$$\frac{1}{2}\frac{d}{dt}\|z_t\|^2 + \frac{1}{2}\sigma_{12}\left\|\nabla^m z_t\right\|^2 + \frac{1}{4}\phi_{12}\frac{d}{dt}\left\|\nabla^m z\right\|^2\left(\frac{1}{2}\phi_{12}\left(\nabla^m z, \nabla^m z_t\right)\right) + (G_{12}, z_t) = 0, \quad (3.44)$$

By multiplying (3.43) by z, we get

$$\frac{d}{dt}(z_t, z) + \frac{1}{4}\sigma_{12}\frac{d}{dt}\left\|\nabla^m z\right\|^2 + \frac{1}{2}\phi_{12}\left\|\nabla^m z\right\|^2 + (G_{12}, z) = \|z_t\|^2 \quad (3.45)$$

About (3.45), we have

$$\frac{d}{dt}\left((z_t, z) + \frac{1}{4}\sigma_{12}\left\|\nabla^m z\right\|^2\right) + \frac{1}{2}\phi_{12}\left\|\nabla^m z\right\|^2 + \left(g(\cdot, u^1) - g(\cdot, u^2), z\right)$$
$$+ \frac{1}{2}\left((\phi_1 - \phi_2)(-\Delta)^m\left(u^1 + u^2\right), z\right) + \frac{1}{2}\left((\sigma_1 - \sigma_2)(-\Delta)^m\left(u_t^1 + u_t^2\right), z\right)$$
$$= \|z_t\|^2 + \frac{1}{4}\left\|\nabla^m z\right\|^2\left(\sigma_1'\left(\nabla^m u^1, \nabla^m u_t^1\right) + \sigma_2'\left(\nabla^m u^2, \nabla^m u_t^2\right)\right) \quad (3.46)$$

Handle some items of (3.46), as follow

$$\frac{1}{2}\left((\phi_1 - \phi_2)(-\Delta)^m\left(u^1 + u^2\right), z\right) = \tilde{\phi}_{12}\left(\nabla\left(u^1 + u^2\right), \nabla z\right)^2, \quad (3.47)$$

where

$$\tilde{\phi}_{12} = \frac{1}{2}\int_0^1 \phi'\left(\lambda\left\|\nabla^m u^1\right\|^2 + (1-\lambda)\left\|\nabla^m u^2\right\|^2\right)d\lambda \ge 0.$$

$$\left|\left((\sigma_1 - \sigma_2)(-\Delta)^m\left(u_t^1 + u_t^2\right), z\right)\right|$$
$$= \left|(\sigma_1 - \sigma_2)\left(\nabla^m u_t^1 + \nabla^m u_t^2, \nabla^m z\right)\right|$$
$$\le C_R\left(\left\|\nabla^m u_t^1\right\| + \left\|\nabla^m u_t^2\right\|\right)\left\|\nabla^m z\right\|^2 \quad (3.48)$$

$$\left\|\nabla^m z\right\|^2\left(\sigma_1'\left(\nabla^m u^1, \nabla^m u_t^1\right) + \sigma_2'\left(\nabla^m u^2, \nabla^m u_t^2\right)\right)$$
$$\le C_R\left(\left\|\nabla^m u_t^1\right\| + \left\|\nabla^m u_t^2\right\|\right)\left\|\nabla^m z\right\|^2 \quad (3.49)$$

$$\left(g(\cdot, u^1) - g(\cdot, u^2), z\right)$$
$$\le c\int\left(\left|u^1\right|^{p-1} + \left|u^2\right|^{p-1} + 1\right)|z|^2 dx$$
$$\le c\left(\left(\left\|u^1\right\|_{p+1}^{p-1} + \left\|u^2\right\|_{p+1}^{p-1}\right)\|z\|_{p+1}^2 + \|z\|^2\right)$$
$$\le c\left\|\nabla^m z\right\|^2 \quad (3.50)$$

From the above, we have

$$\frac{d}{dt}\left((z_t, z) + \frac{1}{4}\sigma_{12}\left\|\nabla^m z\right\|^2\right) + \frac{1}{2}\phi_{12}\left\|\nabla^m z\right\|^2 \le \|z_t\|^2 + C_R\left(1 + \left\|\nabla^m u_t^1\right\| + \left\|\nabla^m u_t^2\right\|\right)\left\|\nabla^m z\right\|^2 \quad (3.51)$$

Now, we deal with (3.44)

$$(G_{12}, z_t) = G_1(t) + G_2(t) + G_3(t) \quad (3.52)$$

where $G_1(t) = \frac{1}{2}(\sigma_1 - \sigma_2)\left((-\Delta)^m\left(u_t^1 + u_t^2\right), z_t\right)$

$$G_2(t) = \tilde{\phi}_{12}\left(\nabla^m u^1 + \nabla^m u^2, \nabla^m z\right)\left(\nabla^m(u^1 + u^2), \nabla^m z_t\right)$$

$$G_3(t) = \left(g\left(x, u^1\right) - g\left(x, u^2\right), z_t\right)$$

So

$$\left|G_1 + G_2\right| \le C_k\left\|\nabla^m z_t\right\|\left\|\nabla^m z\right\| \quad (3.53)$$

$$\left|G_3\right| \le C\int_\Omega\left(\left|u^1\right|^{p-1} + \left|u^2\right|^{p-1} + 1\right)|z||z_t|dx$$
$$\le C\left(\|z\|\|z_t\| + \left(\left|u^1\right|_{p+1}^{p-1} + \left|u^2\right|_{p+1}^{p-1}\right)\|z\|_{p+1}\|z_t\|_{p+1}\right)$$
$$\le a_3\left\|\nabla^m z_t\right\|^2 + C\left\|\nabla^m z\right\|^2$$

$$\phi_{12}\left(\nabla^m z,\nabla^m z_t\right)\le a_4\left\|\nabla^m z_t\right\|^2+C\left\|\nabla^m z\right\|^2 \quad (3.55)$$

From the above, we get

$$\frac{1}{2}\frac{d}{dt}\left\|z_t\right\|^2+a_5\left\|\nabla^m z_t\right\|^2\le C\left\|\nabla^m z\right\|^2 \quad (3.56)$$

Let

$$\Phi^*=\left\|z_t\right\|^2+\eta\left(\left(z,z_t\right)+\frac{1}{4}\sigma_{12}\left\|\nabla^m z\right\|^2\right)\sim\left\|z_t\right\|^2+\left\|\nabla^m z\right\|^2 \quad (3.57)$$

Combining (3.51) and (3.56), we obtain

$$\frac{d}{dt}\Phi^*+a_5\left\|\nabla^m z_t\right\|^2+a_6\Phi^*\le C_k\left(1+\left\|\nabla^m u_t^1\right\|+\left\|\nabla^m u_t^2\right\|\right)\Phi^* \quad (3.58)$$

therefore

$$\left\|\left(z,z_t\right)\right\|_X^2\le C\left(\left\|\left(z_0,z_1\right)\right\|_X^2\right)e^{-a_6 t}. \quad (3.59)$$

IV. The global attractor

Definition mapping $S(t):X\to X$, such that $S(t)(u_0,u_1)=(u,u_t)$, where u is the solution of the problem (1.1). According to theorem 3.1, $S(t)$ constitute the continuous operator semigroup in X.

Theorem 4.1 Under the assume Theorem 3.1, the continuous operator semigroup $S(t)$ exists the bounded absorbing set.

Lemma 4.1 Let the assumption of Theorem 3.1 be in force, and u^1,u^2 be two solution of the problem (1.1)-(1.3) corresponding to initial data $\left(u_0^i,u_1^i\right)\in X$, $i=1,2$ $z=u^1-u^2$. Then

$$\left\|\left(z,z_t\right)\right\|_X^2\le C\left(\left\|\left(z_0,z_1\right)\right\|_X^2\right)e^{-a_6 t}+c\int_0^t e^{-k(t-\tau)}\left\|\nabla^{m-1}z\right\|^2 d\tau \quad (4.1)$$

Proof The proof process of this part is similar to Theorem 3.1(3).

Now, we deal with (3.44) again.

$$\begin{aligned}\left|G_3\right|&\le C\int_\Omega\left(\left|u^1\right|^{p-1}+\left|u^2\right|^{p-1}+1\right)|z|\left|z_t\right|dx\\&\le C\left(\|z\|\left\|z_t\right\|+\left(\left|u^1\right|_{p+1}^{p-1}+\left|u^2\right|_{p+1}^{p-1}\right)\|z\|_{p+1}\left\|z_t\right\|_{p+1}\right)\\&\le a_3\left\|\nabla^m z_t\right\|^2+c\left\|\nabla^{m-1}z\right\|^2\end{aligned} \quad (4.2)$$

By (3.52), (3.53), (3.55) and (4.2), we have

$$\frac{1}{2}\frac{d}{dt}\left\|z_t\right\|^2+a_5\left\|\nabla^m z_t\right\|^2\le c\left\|\nabla^{m-1}z\right\|^2 \quad (4.3)$$

Combining (3.51) and (4.3), we obtain

$$\frac{d}{dt}\Phi^*+a_5\left\|\nabla^m z_t\right\|^2+a_6\Phi^*\le c\left\|\nabla^{m-1}z\right\|^2+C_k\left(1+\left\|\nabla^m u_t^1\right\|+\left\|\nabla^m u_t^2\right\|\right)\Phi^* \quad (4.4)$$

Therefore

$$\left\|\left(z,z_t\right)\right\|_X^2\le C\left(\left\|\left(z_0,z_1\right)\right\|_X^2\right)e^{-a_6 t}+c\int_0^t e^{-k(t-\tau)}\left\|\nabla^{m-1}z\right\|^2 d\tau \quad (4.5)$$

Theorem 4.2 Under the assume Theorem 3.1, the semigroup $S(t)$ exists the global attractor in X.

Proof.
Set $w^{m,n}=u^n(t+t_n-T)-u^m(t+t_m-T),\ t_n>t_m>T>0$, we have

$$\left\|\left(w^{m,n},w_t^{m,n}\right)\right\|_X^2\le ce^{-kt}+c\sup_{0\le s\le t}\left\|u^n(t_n-T+s)-u^m(t_m-T+s)\right\|^2 \quad (4.6)$$

According to (3.59), we get

$$\left\|\left(w,w_t\right)\right\|_X^2\le C\left(\left\|\left(w_0,w_1\right)\right\|_X^2\right)e^{-a_6 t}+c\int_0^t e^{-a_6(t-\tau)}\left\|\nabla^{m-1}w\right\|^2 d\tau \quad (4.7)$$

Take $\left\{\left(u_0^n,u_1^n\right)\right\}\subset B_1$, $\left\|\left(u_0^n,u_1^n\right)\right\|_X\le R$, then $\left\{u^n(t)\right\}$ is the solution of the problem (1.1), $u^n\in C\left(0,+\infty;H_0^m\right)\cap C^1\left(0,+\infty;L^2\right)$.

We take $t_n>t_m>T>0$, then

$$w^{m,n}=u^n(t+t_n-T)-u^m(t+t_m-T),\quad t\ge 0 \quad (4.8)$$

By (4.7), we have

$$\left\|\left(w^{m,n},w_t^{m,n}\right)\right\|_X^2\le ce^{-a_6 t}+c\sup_{0\le s\le t}\left\|\nabla^{m-1}u^n(t_n-T+s)-\nabla^{m-1}u^m(t_m-T+s)\right\|^2 \quad (4.9)$$

When $t=T$, we receive

$$\begin{aligned}&\left\|\left(u^n(t_n)-u^m(t_m),u_t^n(t_n)-u_t^m(t_m)\right)\right\|_X^2\\&\le ce^{-a_6 T}+c\sup_{0\le s\le T}\left\|\nabla^{m-1}u^n(t_n+s)-\nabla^{m-1}u^m(t_m+s)\right\|^2\end{aligned} \quad (4.10)$$

Because $C\left(0,c;H_0^m\right)\cap C^1\left(0,c;L^2\right)\subset C\left(0,c;H_0^{m-1}\right)$, for any $c>0$, we can pull out a subsequence $\left\{u^{n'}\right\}$ in

$C\left(0,c;H_0^{m-1}\right)$, and it is convergence. For any $\varepsilon>0$, we set $T>0$, firstly, such that

$$ce^{-a_6 T}<\frac{\varepsilon}{2}\ . \tag{4.11}$$

Then, when m',n' is largh enough, we have

$$\sup_{0\le s\le T}\left\|\nabla^{m-1}u^{n'}(t_{n'}+s)-\nabla^{m-1}u^{m'}(t_{m'}+s)\right\|^2\le\frac{\varepsilon}{2}. \tag{4.12}$$

Therefore

$$\left\|\left(u^{n'}(t_{n'})-u^{m'}(t_{m'}),u_t^{n'}(t_{n'})-u_t^{m'}(t_{m'})\right)\right\|_X^2\le\varepsilon \tag{4.13}$$

Thus, continuous operator semigroup $S(t)$ is asymptotically compact. So, the semigroup $S(t)$ in X exists the global attractor.

REFERENCES

[1] Kirchhoff, G. Vorlesungen fiber Mechanik. Tenbner, Stuttgarty, 1883.

[2] Igor, Chueshov. Longtime Dynamics of Kirchhoff Wave Models with Strong Nonlinear Damping. *Journal of Differential Equations*,2012, **252**, 1229-1262.

[3] Igor, Chueshov. Global attractors for a class of Kirchhoff wave models with a structural nonlinear damping, J. Abstr. Differ. Equ. Appl. 2010, 1,86-106

[4] Zhijian Yang, Pengyan Ding, Zhiming Liu. Global attractor for the Kirchhoff type equations with strong nonlinear damping and supercritical nonlinearity, Appl. Math. Lett. 2014,33,12-17.

[5] Zhijian,Yang, Pengyan Ding. Longtime Dynamics of the Kirchhoff Equations with Strong Damping and Critical Nonlinearity on R^N. Journal of Mathematical Analysis Application, 2016,434, 1826-1851.

[6]Zhijian,Yang, Pengyan Ding, Lei Li. Longtime Dynamics of the Kirchhoff Equations with Fractional Damping and Supercritical Nonlinearity. Journal of Mathematical Analysis Application, 2016,442, 485-510.

[7] Ling Chen, Weiwang, Guoguang Lin. The global attractor and their Hausdorff and fractal dimensions estimation for the higher-order nonlinear Kirchhoff-type equation. Journal of Advances in Mathematics, 2016, 10,1-15.

[8] Yuting Sun, Yunlong Gao, Guoguang Lin, The global attractors for the Higher-order Kirchhoff-type equation with nonlinear strongly damped term. International Journal of Modern Nonlinear Theory and Application, 2016, 5, 203-217