On application of Laplace transform to Exponential Distribution

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Abstract— **Exponential distribution has a probabilistic terms concerning the time we need to wait before a given event occurs. If this waiting time is unknown, it is often appropriate to think of it as a random variable having an exponential distribution. In this paper, emphasis is placed on obtaining the mean and variance of exponential distribution by method of Laplace transform. The aim of establishing this new method is to compare the efficiency (i.e. how best the results) with that of an existing method of exponential distribution in solving real life problems. EasyFit professional software is used to analyze the entire work. Based on the results obtained, we conclude that the exponential distribution by Laplace transform is the most appropriate model to approximate the time until the next phone call arrives than an existing exponential distribution.**

Index Terms— **Exponential distribution, Laplace transforms Mean, Variance, and Probability.**

I. Introduction

In statistics, the exponential distribution is the probability distribution that describes the time between events in a Poisson process, i.e. a process in which events occur continuously and independently at a constant average rate. It is a particular case of the gamma distribution. It is the continuous analogue of the geometric distribution, and it has the key property of being memoryless [2]. The exponential distribution is not the same as the class of exponential families of distributions, which is a large class of probability distributions that includes the exponential distribution as one of its members, but also includes the normal distribution, binomial distribution, gamma distribution, Poisson, and many others [7].The characteristics of the exponential distribution and the related distribution functions including gamma, weibull and lognormal then relates some of their properties to the application of Markov models. One of the major properties is forgetfulness, the consequence of this is that Markov and stationarity assumptions imply that the times between events must be negative-exponentially distributed.

To make a decision on the application of Markov model to any process in real life situation, it is advised that it should be fitted to the form of the negative exponential density functions which implies that the most likely times are close to zero, and very long times are increasingly unlikely. That is, the most likely values are considered to be clustered about the mean, and large deviations from the mean are viewed as increasingly unlike. If this characteristic of the negative exponential distribution seems incompatible with the application one has in mind then a Markov model may not be appropriate [8]. A random variable having an exponential distribution is also called an exponential random variable.In real-world scenarios, the assumption of a constant rate (or probability per unit time) is rarely satisfied. For example, the rate of incoming phone calls differs according to the time of day. But if we focus on a time interval during which the rate is roughly constant, such as from 2 to 4 p.m. during work days, the exponential distribution can be used as a good approximate model for the time until the next phone call arrives. Similar caveats apply to the following examples which yield approximately exponentially distributed variables: the time until a radioactive particle decays, or the time between beeps of a geiger counter; the time it takes before your next telephone call; the time until default (on payment to company debt holders) in reduced form credit risk modeling [5]. Exponential variables can also be used to model situations where certain events occur with a constant probability per unit distance: monthly and annual maximum values of daily rainfall and river discharge volumes [4].

The exponential distribution is the only continuous memoryless random distribution. It is a continuous analog of the geometric distribution [1]. This distribution is properly normalized since;

$$\int_0^\infty f(x)\,dx = \frac{1}{\beta}\int_0^\infty \lambda^{-x/\beta}\,dx = 1 \qquad (1)$$

Given a Poisson distribution with rate of change λ, the distribution of waiting times between successive changes (k = 0) is

$$D(x) \equiv P(X \le x) = 1 - P(X > x) = 1 - \lambda^{-\lambda x} \quad [9] \qquad (2)$$

The probability density function (pdf) of an exponential distribution has the form

$$f(x) = \begin{cases} \lambda\lambda^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases} \qquad (3)$$

where $\lambda > 0$ is a parameter of the distribution, often called the rate parameter. The distribution is supported on the interval $[0,\infty)$. If a random variable X has this distribution, we write X ~ exponential (λ).

A commonly used alternate parameterization is to define the probability density function (pdf) of an exponential distribution with parameter $\frac{1}{\beta}$ as

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$$f(x) = \begin{cases} \frac{1}{\beta}\lambda^{-x/\beta}, & x \ge 0 \\ 0, & x < 0 \end{cases} \qquad (4)$$

Where β is the scale parameter (the scale parameter is often referred to as λ which equals $1/\beta$). The case where $\beta = 1$ is called the standard exponential distribution [6]. The equation for the standard exponential distribution is

$$f(x) = \lambda^{-x/\beta},\ x \ge 0,\ \beta > 0 \qquad (5)$$

The exponential distribution is primarily used in reliability applications. The exponential distribution is used to model data with a constant failure rate (indicated by the hazard plot which is simply equal to a constant). The formula for the hazard function of the exponential distribution is

$$h(x) = \frac{1}{\beta},\ \beta > 0 \qquad (6)$$

The Laplace transform is a frequency-domain approach for continuous time signals irrespective of whether the system is stable or unstable. The Laplace transform of a function $f(x)$, defined for all real numbers $x \ge 0$, is the function $F(s)$, which is a unilateral transform defined by

$$F(s) = \int_0^\infty f(x)\lambda^{-sx}dx \qquad (7)$$

Where s is a complex number frequency parameter.

In pure and applied probability, the Laplace transform is defined as an expected value. If X is a random variable with probability density function f, then the Laplace transform of f is given by the expectation.

$$L\{f\}(s) = E[\lambda^{-sx}] \qquad (8)$$

By abuse of language, this is referred to as the Laplace transform of the random variable X itself. Replacing s by $-x$ gives the moment generating function of X. The Laplace transform has applications throughout probability theory, including first passage times of stochastic processes such as Markov chains, and renewal theory [3].

Of particular use is the ability to recover the cumulative distribution function of a continuous random variable X by means of the Laplace transform as follows;

$$f(x) = L^{-1}\left\{\frac{1}{s}E[\lambda^{-sx}]\right\}(x)$$

$$= L^{-1}\left\{\frac{1}{s}L\{f\}(s)\right\}(x) \qquad (9)$$

II. Methodology

The method used in this paper is known as Laplace transform to exponential distribution with a unit value of s. In this context, the exponential distribution is defined as

$$f(x) = \frac{1}{\beta}\lambda^{-x/\beta},\ \beta > 0 \qquad (10)$$

and the Laplace transform is defined as

$$L\{f(x)\} = \lambda^{-sx},\ x \ge 0 \qquad (11)$$

Therefore, the Laplace transform of exponential distribution is defined as

$$L\{f(x)\} = \int_0^\infty \lambda^{-sx} f(x)\,dx$$

$$= \int_0^\infty \lambda^{-sx}\frac{1}{\beta}\lambda^{-x/\beta}\,dx \qquad (12)$$

Solving (12), we have

$$L\{f(x)\} = \frac{1}{\beta}\int_0^\infty \lambda^{-sx}\lambda^{-x/\beta}\,dx = \frac{1}{\beta}\int_0^\infty \lambda^{-(s+1/\beta)x}\,dx$$

$$= \frac{1}{\beta}\left[-\frac{\lambda^{-(s-1/\beta)x}}{(s-1/\beta)}\right]_0^\infty = \frac{1}{\beta}\left[-\frac{1}{(s-1/\beta)}\right]$$

$$= \frac{1}{(1+\beta s)} = \frac{1}{(1+\beta)},\ \beta > -1 \text{ [for unity of s]} \qquad (13)$$

III. Mean of Exponential Distribution by Laplace Transform

$$E[x] = \int_0^\infty x\lambda^{-sx} f(x)\,dx = \frac{1}{\beta}\int_0^\infty x\lambda^{-sx}\lambda^{-x/\beta}\,dx$$

$$E[x] = \frac{1}{\beta}\int_0^\infty x\lambda^{-(s+1/\beta)x}\,dx$$

$$= \frac{1}{\beta}\left[-\frac{x\lambda^{-(s+1/\beta)x}}{(s+1/\beta)}\right]_0^\infty +$$

$$\frac{1}{\beta(s+1/\beta)}\int_0^\infty \lambda^{-(s+1/\beta)x}\,dx$$

$$= \frac{1}{\beta\left(s + \frac{1}{\beta}\right)} \int_0^\infty \lambda^{-\left(s + \frac{1}{\beta}\right)x} \, dx$$

$$= \frac{1}{\beta\left(s + \frac{1}{\beta}\right)} \left[-\frac{\lambda^{-\left(s + \frac{1}{\beta}\right)x}}{\left(s + \frac{1}{\beta}\right)} \right]_0^\infty$$

$$= \frac{1}{\beta\left(s + \frac{1}{\beta}\right)^2} = \frac{\beta}{(1+\beta)^2} \text{ [for unity of s]} \qquad (14)$$

IV. Variance of Exponential Distribution by Laplace Transform

$$E[x^2] = \int_0^\infty x^2 \lambda^{-sx} f(x)\,dx = \frac{1}{\beta}\int_0^\infty x^2 \lambda^{-sx}\lambda^{-x/\beta}\,dx$$

$$E[x^2] = \frac{1}{\beta}\int_0^\infty x^2 \lambda^{-\left(s + \frac{1}{\beta}\right)x}\,dx$$

$$= \frac{1}{\beta}\left[-\frac{x^2\lambda^{-\left(s + \frac{1}{\beta}\right)x}}{\left(s + \frac{1}{\beta}\right)} \right]_0^\infty + \frac{2}{\beta\left(s + \frac{1}{\beta}\right)}\int_0^\infty x\lambda^{-\left(s + \frac{1}{\beta}\right)x}\,dx$$

$$= \frac{2}{\beta\left(s + \frac{1}{\beta}\right)}\int_0^\infty x\lambda^{-\left(s + \frac{1}{\beta}\right)x}\,dx$$

$$= \frac{2}{\beta\left(s + \frac{1}{\beta}\right)}\left[\left[-\frac{x\lambda^{-\left(s + \frac{1}{\beta}\right)x}}{\left(s + \frac{1}{\beta}\right)} \right]_0^\infty + \frac{1}{\left(s + \frac{1}{\beta}\right)}\int_0^\infty \lambda^{-\left(s + \frac{1}{\beta}\right)x}\,dx \right]$$

$$= \frac{2}{\beta\left(s + \frac{1}{\beta}\right)^2}\int_0^\infty \lambda^{-\left(s + \frac{1}{\beta}\right)x}\,dx$$

$$= \frac{2}{\beta\left(s + \frac{1}{\beta}\right)^2}\left[-\frac{\lambda^{-\left(s + \frac{1}{\beta}\right)x}}{\left(s + \frac{1}{\beta}\right)} \right]_0^\infty$$

$$= \frac{2}{\beta\left(s + \frac{1}{\beta}\right)^3} = \frac{2\beta^2}{(1+\beta s)^3}$$

$$= \frac{2\beta^2}{(1+\beta)^3} \qquad (15)$$

$$Var(x) = E[x^2] - [E(x)]^2 \qquad (16)$$

Substituting (13) and (14) into (16), we have

$$Var(x) = \frac{2\beta^2}{(1+\beta)^3} - \left[\frac{\beta}{(1+\beta)^2}\right]^2$$

$$= \frac{2\beta^2}{(1+\beta)^3} - \frac{\beta^2}{(1+\beta)^4}$$

$$= \frac{2\beta^2(1+\beta) - \beta^2}{(1+\beta)^4} = \frac{\beta^2 + 2\beta^3}{(1+\beta)^4}$$

$$= \frac{\beta^2(1+2\beta)}{(1+\beta)^4} \qquad (17)$$

V. Mean of Existing Exponential Distribution

The exponential distribution is defined as

$$f(x) = \int_0^\infty \frac{1}{\beta}\lambda^{-x/\beta}\,dx$$

$$E[x] = \int_0^\infty \frac{1}{\beta}x\lambda^{-x/\beta}\,dx = \frac{1}{\beta}\int_0^\infty x\lambda^{-x/\beta}\,dx$$

$$= \frac{1}{\beta}\left[-\frac{x\lambda^{-x/\beta}}{\frac{1}{\beta}} \right]_0^\infty + \int_0^\infty \lambda^{-x/\beta}\,dx$$

$$= \beta\left[-\lambda^{-x/\beta} \right]_0^\infty = \beta \qquad (18)$$

VI. Variance of Existing Exponential Distribution

$$E[x^2] = \int_0^\infty \frac{1}{\beta}x^2\lambda^{-x/\beta}\,dx = \frac{1}{\beta}\int_0^\infty x^2\lambda^{-x/\beta}\,dx$$

$$= \frac{1}{\beta}\left[-\frac{x^2\lambda^{-x/\beta}}{\frac{1}{\beta}} \right]_0^\infty + 2\int_0^\infty x\lambda^{-x/\beta}\,dx$$

$$= 2\int_0^\infty x\lambda^{-x/\beta}\,dx = 2\left[-x\lambda^{-x/\beta} \right]_0^\infty + 2\beta\int_0^\infty \lambda^{-x/\beta}\,dx$$

$$\Rightarrow 2\beta\int_0^\infty \lambda^{-x/\beta}\,dx = 2\beta^2 \qquad (19)$$

$$Var(x) = E[x^2] - [E(x)]^2 = 2\beta^2 - \beta^2 = \beta^2 \qquad (20)$$

VII. Data Presentation/Analysis

Primary source of data collection is obtainable in this research. Data for this study were collected by observation method from individual caller; the waiting time interval of incoming phone calls per day is between 10:00 – 12 noon during working hours administered by me. Data were

collected for a week (i.e. the five working days) and the summary of the data is shown below
Incoming calls per day in a week

Date	Time	Arrival freq (rate)
27/07/2017	10:00 – 12 noon	21
28/07/2017	10:00 – 12 noon	27
29/07/2017	10:00 – 12 noon	23
30/07/2017	10:00 – 12 noon	24
31/07/2017	10:00 – 12 noon	26
Total		121

VIII. RESULTS

Descriptive Statistics

Statistic	Value
Sample Size	6
Range	100
Mean	40.33
Variance	1566.0
Std. Deviation	39.58
Coef. of Variation	0.9812
Std. Error	16.16
Skewness	2.433
Excess Kurtosis	5.939

Percentile	Value
Min	21
5%	21
10%	21
25% (Q1)	22.5
50% (Median)	25
75% (Q3)	50.5
90%	121
95%	121
Max	121

From (13) and (16), Mean and variance of exponential distribution by Laplace transform is

$$E[x] = 0.024 \text{ and } V[x] = 0.046$$

From (17) and (19), mean and Variance of existing exponential distribution is

$$E[x] = 40.33 \text{ and } V[x] = 1566.0$$

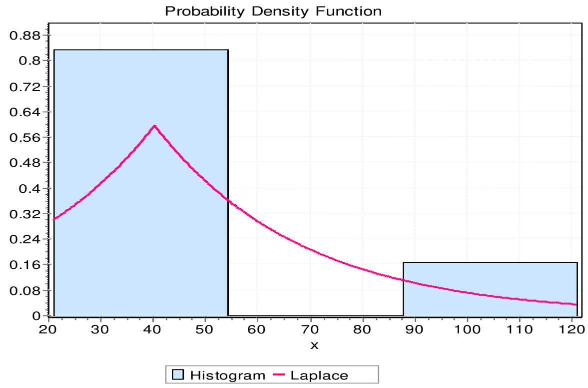

Figure 1: Graph of Probability density function showing Laplace exponential distribution of waiting time for incoming calls per day

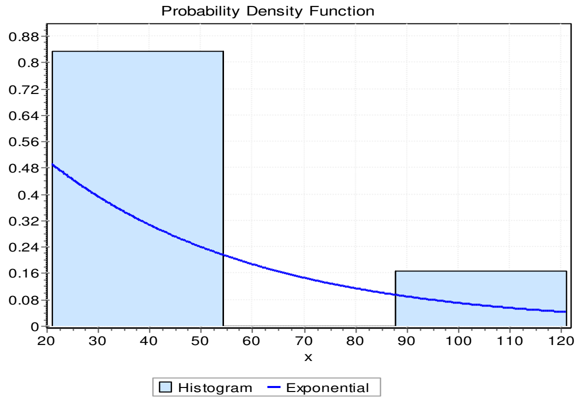

Figure 2: Graph of Probability density function showing exponential distribution of waiting time for incoming calls per day

IX. CONCLUSION

Based on the results obtained, it shows that expected value of an exponential distribution by Laplace transform is less than that of an existing exponential distribution and the same with the variance. By means of efficiency where the smallest variance is more efficiency than the highest variance, we conclude that exponential distribution by Laplace transform is the most suitable model to approximate the waited time of an incoming phone calls, an individual make per day in a given time interval. The graphs show that the higher the scale parameter, it decreases the mean rate of an existing exponential distribution than exponential distribution by Laplace transform giving the peak form of a bell shape.

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