

# Structure Quasi-Po-Ternary Ideals in Po-Ternary Semirings

Dr. D. Madhusudhana Rao, P. Siva Prasad, G. Srinivasa Rao

**Abstract**— In this paper we introduce the definitions related to quasi-PO-ternary ideals and Bi-PO-ternary ideals in PO-ternary semirings and we study the relation between quasi-PO-ternary ideals and Bi-PO-ternary ideals in PO-ternary semirings.

**Mathematics Subject Classification:** 16YIII0, 16Y99.

**Index Terms** — *Quasi PO-k-ternary ideal, quasi-PO-ternary ideal of T generated by A, quasi simple, 0-quasi simple, quasi k-PO-ternary ideal.*

## I. INTRODUCTION

The notion of semiring was introduced by Vandiver [III] in 19III4. In fact semiring is a generalization of ring. In 1971 Lister [2] characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. MadusudhanaRao. D, Siva Prasad. P and SrinivasaRao. G [4, 5, 6, 7, 8] studied and investigated some results on partially ordered ternary semiring.

## II. PRELIMINARIES

**Definition II.1[6]** : A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by  $[ ]$  is said to be a **ternary semiring** if T is an additive commutative semigroup satisfying the following conditions :

- i)  $[[abc]de] = [a[bcd]e] = [ab[cde]]$ ,
- ii)  $[(a+b)cd] = [acd] + [bcd]$ ,
- iii)  $[a(b+c)d] = [abd] + [acd]$ ,
- iv)  $[ab(c+d)] = [abc] + [abd]$  for all  $a; b; c; d; e \in T$ .

**Note II.2[6]** : For the convenience we write  $x_1x_2x_3$  instead of  $[x_1x_2x_3]$

**Note II.3[6]**: Let T be a ternary semiring. If A, B and C are three subsets of T, we shall denote the set  $ABC = \{\sum abc : a \in A, b \in B, c \in C\}$ .

**Dr. D. Madhusudhana Rao**, Head, Department of Mathematics, V.S.R. & N.V.R. College, Tenali, A. P. India.

**P. Siva Prasad**, Asst. Prof of Mathematics, Universal College of Engineering & Technology, Perecherla, Guntur, A. P. India.

**G. Srinivasa Rao**, Asst. Prof of Mathematics, Tirumala Engineering College, Narasaraopet, A. P. India.

**Note II.4[6]** : Let T be a ternary semiring. If A, B are two subsets of T, we shall denote the set  $A + B = \{a+b : a \in A, b \in B\}$  and  $2A = \{a+a : a \in A\}$ .

**Note II.5[6]** : Any semiring can be reduced to a ternary semiring.

**Definition II.6 [6]**: A ternary semiring T is said to be a **partially ordered ternary semiring** or simply **PO Ternary Semiring** or **Ordered Ternary Semiring** provided T is partially ordered set such that  $a \leq b$  then

- (1)  $a + c \leq b + c$  and  $c + a \leq c + b$ ,
  - (2)  $acd \leq bcd, cad \leq cbd$  and  $cda \leq cdb$  for all  $a, b, c, d \in T$ .
- Throughout T will denote as PO-ternary semiring unless otherwise stated.

**Theorem II.7 [6]**: Let T be a po-ternary semiring and  $A \subseteq T, B \subseteq T$  and  $C \subseteq T$ . Then (i)  $A \subseteq [A]$ , (ii)  $[A] = [A]$ , (iii)  $[A][B][C] \subseteq [ABC]$  and (iv)  $A \subseteq B \Rightarrow A \subseteq [B]$  and (v)  $A \subseteq B \Rightarrow [A] \subseteq [B]$ , (vi)  $(A \cap B) = [A] \cap [B]$ , (vii)  $(A \cup B) = [A] \cup [B]$ .

**Definition II.8 [6]**: A nonempty subset A of a PO-ternary semiring T is a **PO-ternary ideal** of T provided A is additive subsemigroup of T,  $ATT \subseteq A, TTA \subseteq A, TAT \subseteq A$  and  $[A] \subseteq A$ .

**Theorem II.9[8]** : Let T be a PO-ternary semiring and A, B be two PO-ternary ideals of T, then the sum of A, B denoted by  $A + B$  is a PO-ternary ideal of T where  $A + B = \{x = a + b / a \in A, b \in B\}$ .

**Theorem II.10[8]**: Let A be a PO-ternary ideal of T. Then  $[A]$  is an ordered ternary ideal of T generated by A.

**Theorem II.11[8]** : The left PO-ternary ideal of a PO-ternary semiring T generated by a non-empty subset A is the intersection of all left PO-ideals of T containing A.

**Theorem II.12[8]** : The lateral ideal of a ternary semiring T generated by a non-empty subset A is the intersection of all lateral ideals of T containing A.

**Theorem II.13[8]** : The right PO-ternary ideal of a PO-ternary semiring T generated by a nonempty subset A is the intersection of all right PO-ternary ideals of T containing A.

### III. QUASI-PO-TERNARY IDEALS

We now introduce the notion of quasi-PO-ternary ideals in PO-ternary semirings.

**Definition III.1:** A non-empty subset  $Q$  of a PO-ternary semiring  $T$  is said to be *quasi-PO-ternary ideal* provided  $Q$  is a subsemigroups of  $(T, +)$  satisfying

- (1)  $TTQ \cap TQT \cap QTT \subseteq Q$
- (2)  $TTQ \cap TTQTT \cap QTT \subseteq Q$  and
- (III)  $(Q) \subseteq Q$ .

**Example III.2:** Let  $T = M_2(Z_0^-)$  be the PO-ternary semiring of the set of all  $2 \times 2$  square matrices over  $Z_0^-$ , the set of all non positive integers. Let  $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^- \right\}$ . Then we can easily verify that  $Q$  is a quasi-PO-ternary ideal of  $T$ .

**Theorem III.3:** Every left PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Assume that  $Q$  is a left PO-ternary ideal of  $T$ . Then  $(TTQ) \subseteq Q$ , but  $(TTQ) \cap (TQT \cup TTQTT) \cap (QTT) \subseteq (TTQ) \subseteq Q$ . Hence  $Q$  is a quasi-PO-ternary ideal of  $T$ .

**Theorem III.4:** Every lateral PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Similar to III.III.

**Theorem III.5:** Every right PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Similar to III.III.

**Theorem III.6:** Every two sided PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Any two sided PO-ternary ideal of  $T$  is a left PO-ternary ideal and right PO-ternary ideal and any left PO-ternary or any right PO-ternary ideal of  $T$  is a quasi-PO-ternary ideal of  $T$ . Therefore any two sided PO-ternary ideal of  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Theorem III.7:** Every PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Similar to III.6.

**Note III.8:** In general a quasi-PO-ternary ideal need not be a left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of  $T$ .

**Example III.9:** In example III.2,  $Q$  is a quasi-PO-ternary ideal of  $T$ , but  $Q$  is not left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of  $T$ .

**Theorem III.10:** Let  $T$  be a commutative PO-ternary semiring, then every quasi-PO-ternary ideal of  $T$  is a three sided PO-ternary ideal of  $T$ .

**Proof:** Assume that  $T$  be a commutative PO-ternary semiring. Let  $Q$  be a quasi-PO-ternary ideal of  $T$ . Then  $(TTQ) \cap (TQT \cup TTQTT) \cap (QTT) \subseteq Q$ . Since  $T$  be a commutative and  $Q \subseteq T$ , then  $TTQ = TQT = QTT = TTQTT$ . Now  $TTQ \cap (TQT \cup TTQTT) \cap QTT = (TTQ \cap TQT \cap QTT) \cup (TTQ \cap TTQTT \cap QTT) \subseteq TTQ \cup TTQ = (TTQ) + (TTQ) = (TTQ) \subseteq Q$ . Hence  $Q$  is a left PO-ternary ideal of  $T$ . Similarly  $Q$  is lateral PO-ternary ideal and right PO-ternary ideal of  $T$ . Therefore, every quasi-PO-ternary ideal of  $T$  is a three sided ideal of  $T$ .

**Theorem III.11:** The intersection of any system of quasi-PO-ternary ideals is a quasi-PO-ternary ideal of  $T$  or empty.

**Proof:** Let  $\{Q_\alpha\}_{\alpha \in \Delta}$  be a family of PO-ternary ideals of  $T$

and let  $Q = \bigcap_{\alpha \in \Delta} Q_\alpha$

Assume that  $Q$  is not empty. Since  $Q_\alpha$  is a quasi-PO-ternary ideal for each  $\alpha \in \Delta$ . Then  $(Q_\alpha TT) \cap (TQ_\alpha T + TTQ_\alpha TT) \cap (TTQ_\alpha) \subseteq Q_\alpha$  for each  $\alpha \in \Delta$ .

Now for each  $\alpha \in \Delta$   $TTQ = TT(\bigcap_{\alpha \in \Delta} Q_\alpha) = \bigcap_{\alpha \in \Delta} TTQ_\alpha \subseteq$

$TTQ_\alpha \subseteq (TTQ_\alpha)$ ,  $TQT = T(\bigcap_{\alpha \in \Delta} Q_\alpha)T = \bigcap_{\alpha \in \Delta} TQ_\alpha T \subseteq$

$TQ_\alpha T \subseteq (TQ_\alpha T)$ ,  $TTQTT = TT(\bigcap_{\alpha \in \Delta} Q_\alpha)TT =$

$\bigcap_{\alpha \in \Delta} TTQ_\alpha TT \subseteq TTQ_\alpha TT \subseteq (TTQ_\alpha TT)$ , and  $TTQ =$

$TT(\bigcap_{\alpha \in \Delta} Q_\alpha) = \bigcap_{\alpha \in \Delta} Q_\alpha TT \subseteq Q_\alpha TT \subseteq (Q_\alpha TT)$ . Then  $(TTQ) \cap$

$(TQT \cup TTQTT) \cap (QTT) \subseteq (TTQ_\alpha) \cap (TQ_\alpha T \cup TTQ_\alpha TT) \cap (Q_\alpha TT) \subseteq Q_\alpha$  for each  $\alpha \in \Delta$ . Therefore

$(TTQ) \cap (TQT + TTQTT) \cap (QTT) \subseteq \bigcap_{\alpha \in \Delta} Q_\alpha = Q$ .

Let  $x \in \bigcap_{\alpha \in \Delta} Q_\alpha$  and  $y \in T$  be such that  $y \leq x$ . Let for each

$\alpha \in \Delta$ . Since  $y \leq x$  and  $x \in Q_\alpha$ ,  $y \in Q_\alpha$ . Thus  $y \in \bigcap_{\alpha \in \Delta} Q_\alpha$

(i.e.  $(Q) = \left[ \bigcap_{\alpha \in \Delta} Q_\alpha \right] \subseteq \bigcap_{\alpha \in \Delta} (Q_\alpha) = \bigcap_{\alpha \in \Delta} Q_\alpha = Q$ ). Therefore

$Q = \bigcap_{\alpha \in \Delta} Q_\alpha$ , is a quasi-PO-ternary ideal of  $T$ .

**Theorem III.12:** Every quasi-PO-ternary ideal of a PO-ternary semiring  $T$  is a PO-ternary subsemiring of  $T$ .

**Proof:** Let  $Q$  be a quasi-PO-ternary ideal of  $T$ . By definition III.1,  $Q$  is a subsemigroup of  $(T, +)$ . Let  $a, b, c \in Q \subseteq T$ . Then  $abc \in (TTQ)$ ,  $abc \in (TQT)$ , and  $abc \in (QTT)$ . Therefore  $abc \in (TTQ) \cap (TQT \cup TTQTT) \cap (QTT) \subseteq Q$ , since  $Q$  is a quasi-PO-ternary ideal of  $T$  and hence  $abc \in Q$ . Therefore  $Q$  is a PO-ternary subsemiring of  $T$ .

**Lemma III.13:** If  $Q$  is a quasi-PO-ternary ideal of a PO-ternary semiring  $T$  and  $S$  is a PO-ternary subsemiring of  $T$ , then  $Q \cap S$  is a quasi-ideal of  $S$ .

**Proof:** Assume that  $Q_1 = Q \cap S \neq \emptyset$ . Since  $Q_1 \subseteq Q$ , it follows that  $SSQ_1 \cap SQ_1S \cap Q_1SS \subseteq TTQ \cap TQT \cap QTT \subseteq Q$ . Since  $Q_1 \subseteq S$  and  $S$  is a PO-ternary subsmigroup of  $T$ . We have  $SSQ_1 \cap SQ_1S \cap Q_1SS \subseteq S$ . Then  $SSQ_1 \cap SQ_1S \cap Q_1SS \subseteq Q_1$ . Let  $x \in Q_1$  and  $y \in S$  such that  $y \leq x$ . Since  $x \in Q$ ,  $y \in Q$ . So  $y \in Q_1$ . Therefore  $Q_1$  is quasi-PO-ternary ideal of  $S$ .

**Theorem III.14:** The intersection of left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal is a quasi-PO-ternary ideal of T.

**Proof :** Let L, M, and R be the left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T respectively. Let  $Q = L \cap M \cap R$ . Choose  $l \in L, m \in M, r \in R$ . Since  $lmr \in L \cap M \cap R$ , Q is not empty. Since  $TTQ \subseteq L, TQT \subseteq M$  and  $QTT \subseteq R$ , then we have  $TTQ \cap TQT \cap QTT \subseteq L \cap M \cap R = Q$ . Similarly,  $TTQ \cap TTQTT \cap QTT \subseteq Q$ . Let  $x \in L \cap M \cap R$  and  $y \in T$  such that  $y \leq x$ . Since  $x \in L \cap M \cap R, y \in L \cap M \cap R$ . Therefore  $Q = L \cap M \cap R$  is a quasi-PO-ternary ideal of T.

**Theorem III.15:** An additive subsemigroup Q of PO-ternary semiring T is a quasi-PO-ternary ideal of T, if Q is the intersection of a left PO-ternary ideal, a lateral PO-ternary ideal and a right PO-ternary ideal of T.

**Proof :** Assume that Q is a quasi-PO-ternary ideal of T and  $L = (TTQ \cup Q), M = (TQT \cup TTQTT \cup Q), R = (QTT \cup Q)$ , then by theorems 2.10, III.4, III.5, III.6, we have L is left PO-ternary ideal, M is lateral PO-ternary ideal and R is right PO-ternary ideal of T containing Q respectively. Thus  $Q \subseteq L \cap M \cap R$ . Since Q is quasi-PO-ternary ideal of T. We have  $L \cap M \cap R = (TTQ \cup Q) \cap (TQT \cup TTQTT \cup Q) \cap (QTT \cup Q) = ((TTQ) \cap (TQT \cup TTQTT) \cap (QTT)) \cup (Q) \subseteq Q \cup (Q) = Q$ . Therefore  $Q = L \cap M \cap R$  and hence Q is the intersection of left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T.

**Definition III.16:** Let A be a nonempty subset of an ordered ternary semi-ring T. The intersection of all quasi-PO-ternary ideals of T containing A is called the *quasi-PO-ternary ideal of T generated by A* and is denoted by  $Q(A)$ . Moreover,  $Q(A)$  is the smallest quasi-PO-ternary ideal of T containing A. If  $A = \{a\}$ , we also write  $Q(\{a\})$  as  $Q(a)$  or  $\langle a \rangle_q$ .

**Theorem III.17:** Let A be a nonempty subset of an ordered ternary semi-ring T. Then

$$Q(A) = (A) \cup [(TTA) \cap (TATUTTATT) \cap (ATT)].$$

In particular,  $Q(a) = \langle a \rangle_q = (a) \cup [(TTa) \cap (TaTUTTaTT) \cap (aTT)]$  for all  $a \in T$ .

**Proof:** By the theorem II.11, II.12, and II.13, we have  $(A \cup TTA), (A \cup TAT \cup TTATT)$  and  $(A \cup ATT)$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T containing A, respectively. By theorem III.15, we have  $(TTA \cup A) \cap (TAT \cup TTATT \cup A) \cap (ATT \cup A)$  is a quasi-PO-ternary ideal of T containing A. Thus  $Q(A) \subseteq (TTA \cup A) \cap (TAT \cup TTATT \cup A) \cap (ATT \cup A)$

$= (A) \cup [(TTA) \cup (TAT \cup TTATT) \cup (ATT)]$ . By the theorem III.15, we have

$$\begin{aligned} & (A) \cup [(TTA) \cup (TAT \cup TTATT) \cup (ATT)] \\ &= (TTA \cup A) \cap (TAT \cup TTATT \cup A) \cap (ATT \cup A) \\ &\subseteq (TTQ(a) \cup Q(A)) \cap (TQ(A) \cup TTQ(A)TT) \cap (TTQ(A) \cup Q(A)) \subseteq Q(A). \end{aligned}$$

Hence  $Q(A) = (A) \cup [(TTA) \cup (TAT \cup TTATT) \cup (ATT)]$ .

Now we characterize the relationship between the minimality of the quasi-PO-ternary ideals and a quasi-simple and 0-quasi simple-PO-ternary semirings.

**Definition III.18 :** Let T be a PO-ternary semiring with a zero element. Then T is called *quasi simple* if T has no proper quasi-PO-ternary ideals of T.

**Theorem III.19:** Let T be a PO-ternary semiring without a zero element. Then the following are equivalent.

(1) T is a quasi-simple.

(2)  $(TTa) \cap (TaTUTTaTT) \cap (aTT) = T$  for all  $a \in T$ .

(III)  $Q(a) = T$  for all  $a \in T$ .

**Proof:** (1)  $\Rightarrow$  (2) : Suppose that T is quasi-simple and let  $a \in T$ . By the theorem 2.11, 2.12, and 2.1III, we have  $(aUTT a), (aUTaTUTT aTT)$  and  $(aUaTT)$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T containing A, respectively. By theorem III.15, we have  $(TTaUa) \cap (TaTUTT aTTUa) \cap (aTTUa)$  is a quasi-PO-ternary ideal of T containing a. Since T is quasi-simple, thus  $(TTaUa) \cap (TaTUTT aTTUa) \cap (aTTUa) = T$ .

(2)  $\Rightarrow$  (III) : Assume that  $(TTa) \cap (TaTUTT aTT) \cap (aTT) = T$  for all  $a \in T$ . By theorem III.18, we have  $T = (TTa) \cap (TaTUTT aTT) \cap (aTT) \subseteq (a) \cup ((TTa) \cap (TaTUTT aTT) \cap (aTT)) = Q(a)$ . Therefore  $Q(a) = T$ .

(III)  $\Rightarrow$  (1) : Assume that  $Q(a) = T$  for all  $a \in T$ . Let Q be a quasi-PO-ternary ideal of T and let  $a \in Q$ . Then  $Q(a) = T$ , and so  $Q(a) \subseteq Q \subseteq T$ . Therefore  $T = Q$  and hence T is quasi-simple.

**Definition III.20:** Let T be a PO-ternary semiring with zero element,  $T^{III} \neq \{0\}$  and  $|T| > 1$ . Then T is called *0-quasi-simple* if T has no non zero proper quasi-PO-ternary ideals.

**Theorem III.21:** Let T be a PO-ternary semiring with zero element,  $T^{III} \neq \{0\}$  and  $|T| > 1$ . Then T is a 0-quasi-simple if and only if  $Q(a) = T$  for  $a \in T \setminus \{0\}$ .

**Proof:** Suppose that T is a 0-quasi-simple and  $a \in T \setminus \{0\}$ . Then  $Q(a) \neq \{0\}$ . Since T is 0-quasi simple, therefore  $Q(a) = T$ .

Conversely, suppose that  $Q(a) = T$  for all  $a \in T \setminus \{0\}$ . Let Q be a non-zero quasi-PO-ternary ideal of T and  $a \in Q \setminus \{0\}$ . Then  $Q(a) = T$  and  $Q(a) \subseteq Q \subseteq T$  implies that  $T = Q$ . Therefore, T is a 0-quasi-simple.

**Definition III.22:** A left quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a *minimal left quasi-PO-ternary ideal* of T if there is no left quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any left quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.23:** A lateral quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a *minimal lateral quasi-PO-ternary ideal* of T if there is no lateral quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any lateral quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.24:** A right quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a *minimal right quasi-PO-ternary ideal* of T if there is no right quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any right quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.25:** A two sided quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a *minimal two sided quasi-PO-ternary ideal* of T if there is no two sided quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ .

Equivalently, if for any two sided quasi-PO-ternary ideal  $A$  of  $T$  such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.26:** A quasi-PO-ternary ideal  $Q$  of an ordered ternary semiring  $T$  without a zero element is called a *minimal quasi-PO-ternary ideal* of  $T$  if there is no quasi-PO-ternary ideal  $A$  of  $T$  such that  $A \subseteq Q$ . Equivalently, if for any quasi-PO-ternary ideal  $A$  of  $T$  such that  $A \subseteq Q$ , we have  $A = Q$ .

**Theorem III.27:** Let  $Q$  be a quasi-PO-ternary ideal of an ordered ternary semi-ring  $T$  without a zero element. Then  $Q$  is a minimal quasi-PO-ternary ideal of  $T$  if and only if it is the intersection of a minimal ordered left, a minimal ordered right and a minimal ordered lateral PO-ternary ideal of  $T$ .

**Proof:** Suppose that  $Q$  is a minimal quasi-PO-ternary ideal of  $T$ . Then  $(TTQ] \cap (TQT \cup TTQTT] \cap (QTT]) \subseteq Q$ . By the theorems 2.11, 2.12, and 2.1III, we have  $(TTQ]$ ,  $(TQT \cup TTQTT]$  and  $(QTT])$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of  $T$  and by theorem III.15, we have  $(TTQ] \cap (TQT \cup TTQTT] \cap (QTT])$  is a quasi-PO-ternary ideal of  $T$ .

Since  $Q$  is minimal quasi-PO-ternary ideal of  $T$ .

We have  $Q = (TTQ] \cap (TQT \cup TTQTT] \cap (QTT])$ .

We claim that  $(TTQ]$  is a minimal left PO-ternary ideal of  $T$ .

Let  $L$  be a left PO-ternary ideal of  $T$  such that  $L \subseteq (TTQ]$ .

Then  $(TTL] \subseteq (L) = L \subseteq (TTQ]$ .

Therefore,  $(TTL] \cap (TQT \cup TTQTT] \cap (QTT]) \subseteq (TTQ] \cap (TQT \cup TTQTT] \cap (QTT]) = Q$ .

Since  $(TTL] \cap (TQT \cup TTQTT] \cap (QTT])$  is a quasi-PO-ternary ideal of  $T$  and  $Q$  is a minimal quasi-PO-ternary ideal of  $T$ , we have  $(TTL] \cap (TQT \cup TTQTT] \cap (QTT]) = Q$ .

Thus  $Q \subseteq (TTL]$  and so  $(TTQ] \subseteq (TT(TTL]) \subseteq (TT(L)) = (TTL] \subseteq L$ .

Hence,  $L = (TTQ]$ . Therefore,  $(TTQ]$  is a minimal left PO-ternary ideal of  $T$ .

Similarly, we can show that  $(QTT]$  and  $(TQT \cup TTQTT]$  are minimal right PO-ternary ideal and minimal lateral PO-ternary ideal of  $T$ , respectively.

Conversely, let  $Q = L \cap M \cap R$  where  $L$ ,  $M$  and  $R$  are a minimal left PO-ternary ideal, a minimal lateral PO-ternary ideal and a minimal right PO-ternary ideal of a PO-ternary semiring  $T$ , respectively. By theorem III.15,  $Q$  is a quasi-PO-ternary ideal of  $T$ . Let  $A$  be a quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . By theorems 2.11, 2.12, and 2.1III, we have  $(TTA]$ ,  $(TAT \cup TTATT]$  and  $(ATT]$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of  $T$ . Now  $(TTA] \subseteq (TTQ] \subseteq (TTL) \subseteq (L) = L$ . Since  $L$  is a minimal left PO-ternary ideal of  $T$ , we have  $(TTA] = L$ . Similarly,  $(TAT \cup TTATT] = M$  and  $(ATT] = R$ . Since  $A$  is a quasi-PO-ternary ideal of  $T$  and hence  $Q = L \cap M \cap R = (TTA] \cap (TAT \cup TTATT] \cap (ATT]) \subseteq A$ . Therefore  $A = Q$ . Hence,  $Q$  is a minimal quasi-PO-ternary ideal of PO-ternary semiring  $T$ .

**Theorem III.28:** Let  $Q$  be a quasi-PO-ternary ideal of an ordered ternary semi-ring  $T$  without a zero element. If  $Q$  is quasi-simple, then  $Q$  is a minimal quasi-PO-ternary ideal of  $T$ .

**Proof :** Suppose that  $Q$  is quasi-simple and let  $A$  be a quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . Therefore  $(QQA] \cap (QAQ \cup QQAQQ] \cap (AQQ] \subseteq (TTA] \cap (TAT \cup TTATT] \cap (ATT]) \subseteq A$  and  $(A) \cap Q \subseteq (A) = A$ . Therefore  $A$

is a quasi-PO-ternary ideal of  $Q$ . Since  $Q$  is quasi-simple and hence  $Q = A$ . Hence  $Q$  is a minimal quasi-PO-ternary ideal of  $T$ .

**Theorem III.29:** Let  $T$  be an ordered ternary semiring without a zero element having proper quasi-PO-ternary ideals. Then every proper quasi-PO-ternary ideal of  $T$  is minimal if and only if the intersection of any two distinct proper quasi-PO-ternary ideals is empty.

**Proof:** Let  $Q_1$  and  $Q_2$  be two distinct proper quasi-PO-ternary ideals of  $T$ . By assumption, we have that  $Q_1$  and  $Q_2$  are minimal. If  $Q_1 \cap Q_2 \neq \emptyset$ , then by Theorem III.11,  $Q_1 \cap Q_2$  is a quasi-PO-ternary ideal of  $T$ . Since  $Q_1 \cap Q_2 \subseteq Q_1$  and  $Q_1$  is minimal, we have  $Q_1 \cap Q_2 = Q_1$ . Since  $Q_1 \cap Q_2 \subseteq Q_2$  and  $Q_2$  is minimal, we have  $Q_1 \cap Q_2 = Q_2$ . Therefore  $Q_1 = Q_1 \cap Q_2 = Q_2$ . This is a contradiction and hence  $Q_1 \cap Q_2 = \emptyset$ .

Conversely, suppose that  $Q$  be a proper quasi-PO-ternary ideal of  $T$  and let  $A$  be a quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . Then  $A$  is a proper quasi-PO-ternary ideal of  $T$ . If  $A \neq Q$ , then by assumption,  $A = A \cap Q = \emptyset$ . That is a contradiction. Hence,  $A = Q$ . Therefore,  $Q$  is a minimal quasi-PO-ternary ideal of  $T$ .

**Definition III.30:** A nonzero quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  with a zero element is called a *0-minimal quasi-PO-ternary ideal* of  $T$  if there is no a nonzero quasi-PO-ternary ideal  $A$  of  $T$  such that  $A \subseteq Q$ . Equivalently, if for any nonzero quasi-PO-ternary ideal  $A$  of  $T$  such that  $A \subseteq Q$ , we have  $A = Q$ .

**Note III.31:** We also define a 0-minimal left PO-ternary ideal, a 0-minimal lateral PO-ternary and a 0-minimal right PO-ternary ideal of an ordered ternary semiring  $T$  with a zero element in the same way of a 0-minimal quasi-PO-ternary ideal.

**Theorem III.32:** Let  $T$  be a PO-ternary semiring with a zero element. Then the intersection of a 0-minimal left PO-ternary ideal, a 0-minimal right PO-ternary ideal and a 0-minimal lateral PO-ternary ideal of  $T$  is either  $\{0\}$  or a 0-minimal quasi-PO-ternary ideal of  $T$ .

**Proof :** Let  $Q = L \cap M \cap R \neq \{0\}$  where  $L$ ,  $M$ , and  $R$  are a 0-minimal left PO-ternary ideal, a 0-minimal lateral PO-ternary ideal and a 0-minimal right PO-ternary ideal of  $T$ , respectively. By theorem III.14,  $Q$  is a quasi-PO-ternary ideal of  $T$ . Let  $A$  be a non zero quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . By theorems II.11, II.12, and II.13,  $(TTA]$ ,  $(TAT \cup TTATT]$ ,  $(ATT]$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideals of  $T$  respectively. Then we get the following two cases:

**Case-1:**  $(TTA] = \{0\}$ ,  $(TAT \cup TTATT] = \{0\}$ ,  $(ATT] = \{0\}$ . If  $(TTA] = \{0\}$ , then  $(TTA] = \{0\} \subseteq A$ . Thus  $A$  is a nonzero left PO-ternary ideal of  $T$ . Since  $A \subseteq Q \subseteq L$  and  $L$  is a 0-minimal left PO-ternary ideal of  $T$ . Then we have  $A = L$ . Therefore  $A = Q$ . Similarly, if  $(ATT] = \{0\}$  or  $(TAT \cup TTATT] = \{0\}$ , we get  $A = Q$ .

**Case-2:**  $(TTA] \neq \{0\}$ ,  $(TAT \cup TTATT] \neq \{0\}$ ,  $(ATT] \neq \{0\}$ . Now  $(TTA] \subseteq (TTQ] \subseteq (TTL) \subseteq (L) = L$ . Since  $L$  is a 0-minimal left PO-ternary ideal of  $T$ , we have  $(TTA] = L$ . Similarly,  $(TAT \cup TTATT] = M$  and  $(ATT] = R$ .

Since  $A$  is a quasi-PO-ideal of  $T$ , we have  $Q = L \cap M \cap R = (TTA] \cap (TAT \cup TTATT] \cap (ATT]) \subseteq A \Rightarrow A = Q$ . Hence,  $Q$  is a 0-minimal ordered quasi-ideal of  $T$ .



**Theorem III.33:** Let  $Q$  be anon-zero quasi-PO-ternary ideal of an ordered ternary semi-ring  $T$  with a zero element. If  $Q$  is 0-quasi-simple, then  $Q$  is a 0-minimal quasi-PO-ternary ideal of  $T$ .

**Proof:** Suppose that  $Q$  is 0-quasi-simple and let  $A$  be anon-zero quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . Therefore  $(QQA) \cap (QAQ \cup QQAQQ) \cap (AQQ) \subseteq (TTA) \cap (TAT \cup TTATT) \cap (ATT) \subseteq A$  and  $(A) \cap Q \subseteq (A) = A$ . Therefore  $A$  is a non-zero quasi-PO-ternary ideal of  $Q$ . Since  $Q$  is 0-quasi-simple and hence  $Q = A$ . Hence  $Q$  is a 0-minimal quasi-PO-ternary ideal of  $T$ .

**Theorem III.34:** Let  $T$  be an ordered ternary semiring with a zero element having non-zero proper quasi-PO-ternary ideals. Then every non-zero proper quasi-PO-ternary ideal of  $T$  is 0-minimal if and only if the intersection of any two distinct non-zero proper quasi-PO-ternary ideals is  $\{0\}$ .

**Proof:** Let  $Q_1$  and  $Q_2$  be two distinct non-zero proper quasi-PO-ternary ideals of  $T$ . By assumption, we have that  $Q_1$  and  $Q_2$  are 0-minimal. If  $Q_1 \cap Q_2 \neq \{0\}$ , then by Theorem III.11,  $Q_1 \cap Q_2$  is anon-zero quasi-PO-ternary ideal of  $T$ . Since  $Q_1 \cap Q_2 \subseteq Q_1$  and  $Q_1$  is 0-minimal, we have  $Q_1 \cap Q_2 = Q_1$ . Since  $Q_1 \cap Q_2 \subseteq Q_2$  and  $Q_2$  is 0-minimal, we have  $Q_1 \cap Q_2 = Q_2$ . Therefore  $Q_1 = Q_1 \cap Q_2 = Q_2$ . This is a contradiction and hence  $Q_1 \cap Q_2 = \{0\}$ .

Conversely, suppose that  $Q$  be anon-zero proper quasi-PO-ternary ideal of  $T$  and let  $A$  be anon-zero quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . Then  $A$  is a non-zero proper quasi-PO-ternary ideal of  $T$ . If  $A \neq Q$ , then by assumption,  $A = A \cap Q = \{0\}$ . That is a contradiction. Hence,  $A = Q$ . Therefore,  $Q$  is a 0-minimal quasi-PO-ternary ideal of  $T$ .

**Theorem III.35:** Let  $x$  be an idempotent element of a PO-ternary semiring  $T$ , that is,  $x^{III} (= xxx) \geq x$ . If  $R$  is a right PO-ternary ideal,  $M$  a lateral PO-ternary ideal, and  $L$  a left PO-ternary ideal of  $T$ , then  $(Rxx]$ ,  $(xxMxx]$ , and  $(xxL]$  are quasi-PO-ternary ideals of  $T$ .

**Proof:** To show  $(Rxx]$ ,  $(xxMxx]$ , and  $(xxL]$  are quasi-ideals of  $S$ , it is sufficient to show that

$$(Rxx] = (R] \cap (TxT + TTxTT] \cap (TTx], (xxMxx] = (xTT] \cap (M] \cap (SSx], \text{ and } (xxL] = (xTT] \cap (TxT \cup TTxTT] \cap (L].$$

For the first case, it is clear that  $(Rxx] \subseteq R \cap TTx = (R \cap TTx] = (R] \cap (TTx]$ .

Let  $a \in (R] \cap (TTx] \Rightarrow a \in (R]$  and  $a \in (TTx]$ .

Now,  $a \in (TTx] \Rightarrow a \leq \sum_{i=1}^n s_i t_i x$  for some  $s_i, t_i \in T$ .

$$\text{Therefore } axx \leq \left( \sum_{i=1}^n s_i t_i x \right) xx = \sum_{i=1}^n s_i t_i (xxx) \geq \sum_{i=1}^n s_i t_i x \geq a.$$

It follows that  $a \in (Rxx]$  and hence  $(Rxx] = (R] \cap (TTx]$ .

Again  $a \leq axx \Rightarrow a \in (TxT]$ . Therefore we have  $a \in (TxT \cup TTxTT]$ .

Thus  $(R] \cap (TTx] \subseteq (TxT \cup TTxTT]$ . Therefore,  $(Rxx] = (R] \cap (TxT + TTxTT] \cap (TTx]$ .

For the second case, we see that  $(xxMxx] \subseteq (xTT] \cap (M] \cap (TTx]$ . Let  $a \in (xTT] \cap (M] \cap (TTx]$ . Then  $a \in (xTT]$ ,  $a \in (M]$  and  $a \in (TTx]$ . Now  $a \in (xTT]$  and  $a \in (TTx]$

$$\Rightarrow a \leq \sum_{i=1}^n s_i t_i x = \sum_{j=1}^m xu_j v_j \text{ for some } s_i, t_i, u_j, v_j \in T.$$

$$\begin{aligned} \text{Therefore } xxaxx &\leq xx \left( \sum_{i=1}^n s_i t_i x \right) xx \\ &= xx \sum_{i=1}^n s_i t_i (xxx) \geq xx \sum_{i=1}^n s_i t_i x = xx \sum_{j=1}^m xu_j v_j = \\ &\sum_{j=1}^m (xxx) u_j v_j \geq \sum_{j=1}^m xu_j v_j \geq a \end{aligned}$$

$$\Rightarrow a \in (xxMxx] \text{ and hence } (xxMxx] = (xTT] \cap (M] \cap (SSx].$$

For the third case it is similar to first case.

#### IV. PRIME QUASI-PO-TERNARY IDEALS

In this section, we introduce the notions of prime and semiprime quasi-PO-ternary ideals in PO-ternary semirings and some relevant counter examples are also indicated.

**Definition IV.1:** A proper quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is said to be **prime quasi-PO-ternary ideal** provided  $ABC \subseteq Q$  implies that  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$  for some quasi-PO-ternary ideals  $A, B, C$  of  $T$ .

**Definition IV.2:** A proper quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is said to be **semiprime quasi-PO-ternary ideal** provided  $A^3 \subseteq Q$  implies that  $A \subseteq Q$  for some quasi-PO-ternary ideal  $A$  of  $T$ .

**Definition IV.3:** A proper quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is said to be **weakly prime quasi-PO-ternary ideal** provided  $Q \subseteq A, B \subseteq Q, C \subseteq Q$  and  $ABC \subseteq Q$  implies that  $A = Q$  or  $B = Q$  or  $C = Q$  for some quasi-PO-ternary ideals of  $T$ .

**Theorem IV.4:** Every prime quasi-PO-ternary ideal of  $T$  is a Semiprime quasi-PO-ternary ideal of  $T$ .

**Proof:** Suppose that  $Q$  is a prime quasi-PO-ternary ideal of  $T$  and  $A$  be any quasi-PO-ternary ideal of  $T$  such that  $A^3 = AAA \subseteq Q$ . Since  $Q$  is prime. Therefore  $A \subseteq Q$  and hence  $Q$  is a Semiprime quasi-PO-ternary ideal of  $T$ .

**Note IV.5:** The converse of the theorem 4.4. need not be true. i.e., every Semiprime quasi-PO-ternary ideal of  $T$  need not be a prime quasi-PO-ternary ideal of  $T$ .

**Example IV.6:** Let  $T = M_2(Z_0^-)$  is a PO-ternary semiring of  $2 \times 2$  square matrices over  $Z_0^-$ . Let

$$Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^- \right\} \text{ and } a \leq b \text{ for } a, b \in Z_0^-.$$

Then  $Q$  is a Semiprime quasi-PO-ternary ideal of  $T$ . But  $Q$  is not a prime quasi-PO-ternary ideal of  $T$ .

$$\text{Since } A = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in Z_0^- \right\},$$

$$B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in Z_0^- \right\} \text{ and}$$

$$C = \left\{ \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} : d \in Z_0^- \right\} \text{ are quasi-PO-ternary ideal of } T$$

such that  $ABC \subseteq Q$ . But  $A \not\subseteq Q$ ,  $B \not\subseteq Q$  and  $C \not\subseteq Q$ .

**Theorem IV.7:** Every prime quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is a weakly prime quasi-PO-ternary ideal of  $T$ .

**Proof:** Suppose that  $Q$  is a prime quasi-PO-ternary ideal of  $T$ . Then there exist quasi-PO-ternary ideals  $A, B, C$  of  $T$  such that  $ABC \subseteq Q$ . If  $Q \subseteq A$ ,  $Q \subseteq B$ ,  $C \subseteq Q$  and  $ABC \subseteq Q$ ,  $Q$  is a prime quasi-PO-ternary ideal of  $T$  implies that  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$ . Therefore  $A = Q$  or  $B = Q$  or  $C = Q$  and hence  $Q$  is a weakly prime quasi-PO-ternary ideal of  $T$ .

**Note IV.8:** The converse of the theorem 4.7. need not be true. i.e., every weakly prime quasi-PO-ternary ideal of  $T$  is not prime quasi-PO-ternary ideal of  $T$ .

**Example IV.9:** Let  $T = M_2(Z_0^-)$  is a PO-ternary semiring of  $2 \times 2$  square matrices over  $Z_0^-$ . Let  $Q =$

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 3Z_0^- \right\} \text{ and } a \leq b \text{ for } a, b \in Z_0^- . \text{ Then } Q$$

is a weakly prime quasi-PO-ternary ideal of  $T$ . But  $Q$  is not a prime quasi-PO-ternary ideal of  $T$ . Since  $A =$

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 2Z_0^- \right\}, B = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 3Z_0^- \right\} \text{ and}$$

$$C = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 5Z_0^- \right\} \text{ are quasi-PO-ternary ideal of } T$$

such that  $ABC \subseteq Q$ . But  $Q \not\subseteq A$ ,  $Q \not\subseteq B$  and  $Q \not\subseteq C$ .

**Theorem IV.10:** Let  $T$  be a PO-ternary semiring and  $Q$  be a quasi-PO-ternary ideal of  $T$ . If  $Q$  is prime, then  $Q$  is left or lateral or right PO-ternary ideal of  $T$ .

**Proof:** Let  $Q$  be a prime quasi-PO-ternary ideal of  $T$ . Then  $(TTQ)(TQT \cup TTQTT)(QTT) \subseteq TTQ \cap (TQT \cup TTQTT) \cap QTT \subseteq Q$ . Since  $Q$  is prime, we have  $TTQ \subseteq Q$  or  $TQT \cap TTQTT \subseteq Q$  or  $QTT \subseteq Q$ . therefore  $Q$  is left or lateral or right PO-ternary ideal of  $T$ .

**Theorem IV.11:** Let  $T$  be a commutative PO-ternary semiring and  $Q$  be a quasi-PO-ternary ideal of  $T$ . Then  $Q$  is prime if and only if  $abc \in Q$  implies  $a \in Q$  or  $b \in Q$  or  $c \in Q$ .

**Proof:** Suppose that  $Q$  is a prime quasi-PO-ternary ideal of  $T$ . Let  $abc \in Q$ . Then by theorem 4.10,  $Q$  is a PO-ternary ideal of  $T$ . Let  $x \in \langle a \rangle_q \langle b \rangle_q \langle c \rangle_q$ . Then  $x = ((a] \cap (TTa] \cap (TaT \cup TTTaTT] \cap (aTT)).((b] \cap (TTb] \cap (TbT \cup TTTbTT] \cap (bTT)).((c] \cap (TTc] \cap (TcT \cup TTTcTT] \cap (cTT))$ . Since  $abc \in Q$  and  $Q$  is a PO-ternary ideal of  $T$ . Therefore  $x \in Q$ . Thus  $\langle a \rangle_q \langle b \rangle_q \langle c \rangle_q \subseteq Q$ . Since  $Q$  is prime quasi-PO-ternary ideal of  $T$ . Hence  $a \in Q$  or  $b \in Q$  or  $c \in Q$ .

Converse is obvious.

**Theorem IV.12:** Let  $T$  be a PO-ternary semiring and  $Q$  be a quasi-PO-ternary ideal of  $T$ . Then  $Q$  is prime if and only if  $((TTa] \cap (TaT \cup TTTaTT] \cap (aTT)).((TTb] \cap (TbT \cup TTTbTT] \cap (bTT)).((TTc] \cap (TcT \cup TTTcTT] \cap (cTT)) \subseteq Q$  implies  $a \in Q$  or  $b \in Q$  or  $c \in Q$ .

**Proof:** Suppose that  $Q$  is a prime quasi-PO-ternary ideal of  $T$  and let  $((TTa] \cap (TaT \cup TTTaTT] \cap (aTT)).((TTb] \cap (TbT \cup TTTbTT] \cap (bTT)).((TTc] \cap (TcT \cup TTTcTT] \cap (cTT)) \subseteq Q$  for some  $a, b, c \in T$ . Clearly,  $((TTa] \cap (TaT \cup TTTaTT] \cap (aTT))$ ,  $((TTb] \cap (TbT \cup TTTbTT] \cap (bTT))$ ,  $((TTc] \cap (TcT \cup TTTcTT] \cap (cTT))$  are quasi-PO-ternary ideals of  $T$ . Since  $Q$  is prime, therefore  $((TTa] \cap (TaT \cup TTTaTT] \cap (aTT)) \subseteq Q$  or  $((TTb] \cap (TbT \cup TTTbTT] \cap (bTT)) \subseteq Q$  or  $((TTc] \cap (TcT \cup TTTcTT] \cap (cTT)) \subseteq Q$ . If  $((TTa] \cap (TaT \cup TTTaTT] \cap (aTT)) \subseteq Q$ , then  $\langle a \rangle_q \subseteq Q$  implies that  $a \in Q$ . Similarly,  $b \in Q$  or  $c \in Q$ .

Converse is obvious.

**Theorem IV.13:** Let  $T$  be a PO-ternary semiring. If the quasi-PO-ternary ideal of  $T$  with respect to inclusion relation form a chain, then every weakly prime quasi-PO-ternary ideal is a prime quasi-PO-ternary ideal of  $T$ .

**Proof:** Let  $Q$  be a weakly prime quasi-PO-ternary ideal of  $T$ . Let  $A, B, C$  are quasi-PO-ternary ideal of  $T$  such that  $ABC \subseteq Q$ . Suppose that  $A \not\subseteq Q$ ,  $B \not\subseteq Q$  and  $C \not\subseteq Q$ . By the statement since  $Q \subseteq A$ ,  $Q \subseteq B$  and  $Q \subseteq C$ . Since  $Q$  is weakly prime quasi-PO-ternary ideal of  $T$ . Therefore  $A = Q$  or  $B = Q$  or  $C = Q$ . This is a contradiction. Hence  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$ . Therefore  $Q$  is a prime quasi-PO-ternary ideal of  $T$ .

**Theorem IV.14:** Let  $T$  be a PO-ternary semiring. Then the following are equivalent

- (1) The quasi-PO-ternary ideal of  $T$  is idempotent.
- (2) If  $A, B, C$  are three quasi-PO-ternary ideals of  $T$  such that  $A \cap B \cap C \neq \emptyset$ , then  $A \cap B \cap C = ABC$ .
- (3)  $\langle a \rangle_q = [\langle a \rangle_q]^3$  for all  $a \in T$ .

**Proof:** (1)  $\Rightarrow$  (2): Suppose that  $A, B, C$  are quasi-PO-ternary ideals of  $T$  such that  $A \cap B \cap C \neq \emptyset$ . Then by theorem III.11,  $A \cap B \cap C$  is a quasi-PO-ternary ideal of  $T$ . Since every quasi-PO-ternary ideal is an idempotent. Therefore  $(A \cap B \cap C) = (A \cap B \cap C)^3 = (A \cap B \cap C)(A \cap B \cap C)(A \cap B \cap C) \subseteq ABC$ .

(2)  $\Rightarrow$  (3): It is straight forward and (3)  $\Rightarrow$  (1) is obvious.

**Definition IV.15:** A non-empty subset  $A$  of a PO-ternary semiring  $T$  is said to be  $m_q$ -system provided for any  $a, b, c \in A$ , there exist  $x \in \langle a \rangle_q$ ,  $y \in \langle b \rangle_q$ ,  $z \in \langle c \rangle_q$  and  $d \in A$  such that  $xyz \leq d$ .

**Note IV.16:** A non-empty subset  $A$  of a PO-ternary semiring  $T$  is called an  $m_q$ -system if  $a, b, c \in A$ , there exist  $x \in \langle a \rangle_q$ ,  $y \in \langle b \rangle_q$ ,  $z \in \langle c \rangle_q$  such that  $xyz \in (A]$ .

**Definition IV.17:** A non-empty subset  $A$  of a PO-ternary semiring  $T$  is said to be  $n_q$ -system provided for any  $a \in A$ , there exist  $x, y, z \in \langle a \rangle_q$  and  $d \in A$  such that  $xyz \leq d$ .

**Note IV.18:** A non-empty subset  $A$  of a PO-ternary semiring  $T$  is said to be  $n_q$ -system provided for any  $a \in A$ , there exist  $x, y, z \in \langle a \rangle_q$  such that  $xyz \in (A]$ .

**Theorem IV.19:** Every  $m_q$ -system is an  $n_q$ -system of PO-ternary semiring  $T$ .

**Proof:** Suppose that the non-empty subset  $A$  of a PO-ternary semiring  $T$  is an  $m_q$ -system. Let for any  $a \in A$ , there exist  $x, y, z \in \langle a \rangle_q$ . Since  $A$  is an  $m_q$ -system and hence  $xyz \in (A]$ . therefore  $A$  is a  $n_q$ -system of  $T$ .

**Note IV.20:** The converse of the theorem 4.19, need not be true. i.e., every  $n_q$ -system of a PO-ternary semiring T need not be  $am_q$ -system of T.

**Example IV.21:** Let  $T = Z_6^-$  is a PO-ternary semiring under usual addition, multiplication modulo 6 and natural ordering. Let  $A = \{-2, -3\}$ . Then A is  $an_q$ -system but not  $anm_q$ -system.

**Theorem IV.22:** Let T be a PO-ternary semiring and Q is a quasi-PO-ternary ideal of T. Then Q is prime quasi-PO-ternary ideal of T if and only if  $T \setminus Q$  is  $anm_q$ -system of T.

**Proof:** Suppose that Q is a prime quasi-PO-ternary ideal of T. Let  $a, b, c \in T \setminus Q$ .

Suppose that  $xyz \not\leq d$  for all  $d \in T \setminus Q$  and for all  $x \in \langle a \rangle_q, y \in \langle b \rangle_q$  and  $z \in \langle c \rangle_q$ .

Then  $\langle a \rangle_q \langle b \rangle_q \langle c \rangle_q \subseteq Q$ . Since Q is a prime quasi-PO-ternary ideal of T.

Therefore  $a \in Q$  or  $b \in Q$  or  $c \in Q$ . This is a contradiction.

Therefore  $xyz \leq d$  for some  $x \in \langle a \rangle_q, y \in \langle b \rangle_q$  and  $z \in \langle c \rangle_q$ .

Hence  $T \setminus Q$  is  $anm_q$ -system of T.

Conversely suppose that, A, B, C are quasi-PO-ternary ideals of T such that  $ABC \subseteq Q$ . Assume that  $A \not\subseteq Q, B \not\subseteq Q$  and  $C \not\subseteq Q$ . Let  $a \in A \setminus Q, b \in B \setminus Q$  and  $c \in C \setminus Q$ . Then  $a, b, c \in T \setminus Q$ . Since  $T \setminus Q$  is an  $m_q$ -system. Therefore there exist an element  $d \in T \setminus Q$  such that  $xyz \leq d$  for some  $x \in \langle a \rangle_q, y \in \langle b \rangle_q$  and  $z \in \langle c \rangle_q$ . But  $xyz \in \langle a \rangle_q \langle b \rangle_q \langle c \rangle_q \subseteq Q$ . This is a contradiction. Hence  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$ . Therefore Q is a prime quasi-PO-ternary ideal of T.

**Theorem IV.23:** Let T be a PO-ternary semiring and Q is a quasi-PO-ternary ideal of T. Then Q is semiprime quasi-PO-ternary ideal of T if and only if  $T \setminus Q$  is an  $n_q$ -system of T.

**Proof:** Similar to the proof of the theorem IV.22.

**Definition IV.24:** A quasi-PO-ternary ideal of a PO-ternary semiring T is said to be **T-prime quasi-PO-ternary ideal** of T provided  $xTyTz \subseteq Q$  implies  $x \in Q$  or  $y \in Q$  or  $z \in Q$ .

**Definition IV.25:** A quasi-PO-ternary ideal of a PO-ternary semiring T is said to be **T-semiprime quasi-PO-ternary ideal** of T provided  $xTxTx \subseteq Q$  implies  $x \in Q$ .

**Theorem IV.26:** A quasi-PO-ternary ideal Q of a PO-ternary semiring T is T-prime if and only if  $RML \subseteq Q$  implies  $R \subseteq Q$  or  $M \subseteq Q$  or  $L \subseteq Q$  for any right PO-ternary ideal R, lateral PO-ternary ideal M and left PO-ternary ideal L of T.

**Proof:** Let Q be a T-prime quasi-PO-ternary ideal of T and  $RML \subseteq Q$ . Suppose  $R \not\subseteq Q$  and  $M \not\subseteq Q$ . Then there exist  $x \in R \setminus Q$  and  $y \in M \setminus Q$ . Let  $z \in L$ . Then  $xTyTz \subseteq RTMTL \subseteq RML \subseteq Q$ . Since Q is T-prime. Therefore,  $x \in Q$  or  $y \in Q$  or  $z \in Q$ . But  $x \notin Q$  and  $y \notin Q$ . Hence  $z \in Q$  and hence  $L \subseteq Q$ .

Conversely, suppose that  $xTyTz \subseteq Q$ . Then  $(xTT)(TyT)(TTz) \subseteq xTyTz \subseteq Q$ . Since  $xTT$  is a right PO-ternary ideal of T,  $TyT$  is a lateral PO-ternary ideal of T and  $TTz$  is a left PO-ternary ideal of T. Therefore, by hypothesis  $xTT \subseteq Q$  or  $TyT \subseteq Q$  or  $TTz \subseteq Q$ . If  $xTT \subseteq Q$ , then  $x^3 \in xTT \subseteq Q$ .

Now  $\langle x \rangle_r \langle x \rangle_m \langle x \rangle_l = (x \cup xTT)(x \cup TxTUTT xTT)(x \cup TT x) \subseteq (x)^3 \cup (xTT) \subseteq Q$ . By hypothesis  $\langle x \rangle_r \subseteq Q$  or  $\langle x \rangle_m \subseteq Q$  or

$\langle x \rangle_l \subseteq Q$ . Therefore  $x \in Q$ . Similarly, if  $TyT \subseteq Q \Rightarrow y \in Q$  and if  $TTz \subseteq Q \Rightarrow z \in Q$ . Hence Q is T-prime PO-ternary ideal of T.

**Notation IV.27:** we use the following set defined as

$$\begin{aligned} L(Q) &= \{x \in Q : (TTx) \subseteq Q\}, \\ M(Q) &= \{x \in Q : (TxTUTT xTT) \subseteq Q\} \\ R(Q) &= \{x \in Q : (xTT) \subseteq Q\} \\ I_L &= \{x \in L(Q) : (TTx) \subseteq L(Q)\} \\ M^I_M &= \{x \in M(Q) : (TxTUTT xTT) \subseteq M(Q)\} \\ I_R &= \{x \in R(Q) : (xTT) \subseteq R(Q)\}. \end{aligned}$$

**Theorem IV.28:** Let Q be a quasi-PO-ternary ideal of T. Then  $L(Q)$  is a left PO-ternary ideal of T contained in Q if Q is non-empty.

**Proof:** Let  $x, y \in L(Q)$  and  $s, t \in T$ . Then  $x, y \in L(Q) \Rightarrow stx \in (TTx) \subseteq Q, sty \in (TTy) \subseteq Q$ .

$\Rightarrow stx, sty \in Q \Rightarrow stx + sty = st(x + y) \in (TT(x + y)) \subseteq Q \Rightarrow x + y \in L(Q)$

Now  $TTstx \subseteq TTx \Rightarrow (TTstx) \subseteq (TTx) \subseteq Q$ . Therefore  $stx \in L(Q)$ .

Consequently,  $TTL(Q) \subseteq Q$ . Hence  $L(Q)$  is a left PO-ternary ideal of T.

**Theorem IV.29:** Let Q be a quasi-PO-ternary ideal of T. Then  $M(Q)$  is a lateral PO-ternary ideal of T contained in Q if Q is non-empty.

**Proof:** Let  $x, y \in M(Q)$  and  $s, t \in T$ . Then  $x, y \in M(Q) \Rightarrow sxt \in (TxT) \subseteq Q, syt \in (TyT) \subseteq Q$ .

$\Rightarrow sxt, syt \in Q \Rightarrow sxt + syt = s(x + y)t \in (T(x + y)T) \subseteq Q$

$\Rightarrow x + y \in M(Q)$

Now  $TsxtT \subseteq TxTUTT xTT \Rightarrow (TsxtT) \subseteq (TxTUTT xTT) \subseteq Q$ . Therefore  $sxt \in M(Q)$ .

Consequently,  $TM(Q)TU TTM(Q)TT \subseteq Q$ . Hence  $M(Q)$  is a lateral PO-ternary ideal of T.

**Theorem IV.30:** Let Q be a quasi-PO-ternary ideal of T. Then  $R(Q)$  is a right PO-ternary ideal of T contained in Q if Q is non-empty.

**Proof:** Let  $x, y \in R(Q)$  and  $s, t \in T$ . Then  $x, y \in R(Q) \Rightarrow xst \in (xTx) \subseteq Q, yst \in (yTT) \subseteq Q$ .

$\Rightarrow xst, yst \in Q \Rightarrow xst + yst = (x + y)st \in ((x + y)TT) \subseteq Q$

$\Rightarrow x + y \in R(Q)$

Now  $xstTT \subseteq xTT \Rightarrow (xstTT) \subseteq (xTT) \subseteq Q$ .

Therefore  $xst \in R(Q)$ .

Consequently,  $R(Q)TT \subseteq Q$ .

Hence  $R(Q)$  is a right PO-ternary ideal of T.

**Theorem IV.31:** Let Q is a T-prime quasi-PO-ternary ideal of a PO-ternary semiring T. Then  $I_Q$  is a prime quasi-PO-ternary ideal of T.

**Proof:** Let Q be a T-prime quasi-PO-ternary ideal of a PO-ternary semiring T.

Suppose  $RML \subseteq I_Q$  for any PO-ternary ideals R, M and L of T. Now  $I_Q \subseteq L(Q) \subseteq Q$  implies  $RML \subseteq Q$ . Since Q is T-prime,

therefore, by theorem 4.26, we have  $R \subseteq Q$  or  $M \subseteq Q$  or  $L \subseteq Q$ . Also  $I_Q$  is the largest PO-ternary ideal contained in Q, therefore,  $R \subseteq I_Q$  or  $M \subseteq I_Q$  or  $L \subseteq I_Q$ . Hence  $I_Q$  is a prime PO-ternary ideal of T.

**Corollary IV.32:** Let Q is a Semiprime quasi-PO-ternary ideal of a PO-ternary semiring T. Then  $I_Q$  is a Semiprime PO-ternary ideal of T.

**Theorem IV.33:** If a PO-ternary semiring T is a regular, then every quasi-PO-ternary ideal of T is T-semiprime.



**Proof:** Suppose that  $T$  is regular and  $Q$  be a quasi-PO-ternary ideal of  $T$ . Let  $aTaTa \subseteq Q$  for  $a \in T$ . Since  $T$  is regular, therefore, for  $a \in T$ , there exist  $x, y \in T$  such that  $a \leq axaya$ . Thus  $a \leq axaya \Rightarrow a \in (aTaTa) \subseteq aTaTa \subseteq Q$ . Therefore  $a \in Q$ . Hence  $Q$  is a  $T$ -semiprime.

## V. QUASI-K-PO-TERNARY IDEALS

**Definition V.1:** An additive subsemigroup  $Q$  of a PO-ternary semiring  $T$  is said to be *quasi-k-PO-ternary ideal* of  $T$  provided  $\overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ} \subseteq Q$  and  $(Q) \subseteq Q$ .

**Theorem V.2:** Let  $T$  be a PO-ternary semiring and  $A, B, C \subseteq T$ . Then  $\overline{ABC} = \overline{\overline{ABC}}$ .

**Proof:** Since  $A \subseteq \overline{A}$ ,  $B \subseteq \overline{B}$  and  $C \subseteq \overline{C}$ , therefore,  $ABC \subseteq \overline{ABC}$ . Hence  $\overline{ABC} \subseteq \overline{\overline{ABC}}$ . Again, let  $x \in \overline{A}$ ,  $y \in \overline{B}$  and  $z \in \overline{C}$ . Then there exist  $a_1, a_2 \in A$ ,  $b_1, b_2 \in B$  and  $c_1, c_2 \in C$  such that  $x + a_1 = a_2$ ,  $y + b_1 = b_2$  and  $z + c_1 = c_2$ . Now

$$\begin{aligned} &xyz + a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ &= xyz + (x + a_1)(y + b_1)c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ &= xyz + xyc_1 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ &= xyc_2 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ &= xyc_2 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + (x + a_1)b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ &= x(y + b_1)c_2 + xb_1c_1 + a_1(y + b_1)c_1 + a_1b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ &= xb_2c_2 + (x + a_1)b_1c_1 + a_1b_2c_1 + a_1b_1c_2 + a_1b_2c_2 \\ &= (x + a_1)b_2c_2 + a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2 = a_2b_2c_2 + a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2. \end{aligned}$$

As  $a_i b_i c_i \in ABC$ , where  $i = 1, 2$ . Therefore we can prove that  $xyz \in \overline{ABC}$  for  $x \in \overline{A}$ ,  $y \in \overline{B}$  and  $z \in \overline{C}$ . Suppose that

$$t \in \overline{ABC}. \text{ Then } t = \sum_{i=1}^n a_i b_i c_i \text{ for some } a_i \in \overline{A}, b_i \in \overline{B},$$

$c_i \in \overline{C}$ . Thus  $t \in \overline{ABC}$ . Therefore  $\overline{ABC} \subseteq \overline{\overline{ABC}}$ . Hence  $\overline{ABC} \subseteq \overline{\overline{ABC}} = \overline{ABC}$ . Therefore  $\overline{ABC} = \overline{\overline{ABC}}$ .

**Definition IV.3:** A PO-ternary semiring  $T$  is said to be *k-regular* provided for each  $a \in T$  there exist  $x, y \in T$  such that  $a + axa = aya$ .

**Theorem IV.4:** If a PO-ternary semiring  $T$  is *k-regular*. Then every quasi-k-PO-ternary ideal  $Q$  of  $T$  is of the form  $Q = \overline{QTQTQ} = \overline{TTQ} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{QTT}$ .

**Proof:** Let  $Q$  be a quasi-k-PO-ternary ideal of  $T$ . Then  $\overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ} \subseteq Q$  and  $(Q) \subseteq Q$ . Let  $a \in Q$  and  $T$  is *k-regular*, then there exist  $x, y \in T$  such that  $a + axa = aya \Rightarrow axa + axaxa = ayaxa$ . Since  $axaxa, ayaxa \in QTQTQ$ . Therefore  $axa \in \overline{QTQTQ}$ . Similarly,  $aya \in \overline{QTQTQ}$ . Since  $\overline{QTQTQ}$  is *k-closed* and hence  $a \in \overline{QTQTQ} = \overline{QTQTQ}$ . Therefore  $Q \subseteq \overline{QTQTQ}$ . Again  $QTQTQ \subseteq Q(TTT)T \subseteq QTT$  and  $QTQTQ \subseteq TTQ$  and  $QTQTQ \subseteq TTQTT$  and hence  $\overline{QTQTQ} \subseteq \overline{TTQ}$ ,  $\overline{QTQTQ} \subseteq \overline{QTT}$  and  $\overline{QTQTQ} \subseteq \overline{TQT \cup TTQTT}$  as  $0 \in \overline{TQT}$ .

Thus we have  $Q$

$$\subseteq \overline{QTQTQ} \subseteq \overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ} \subseteq Q \text{ as } Q \text{ is quasi-}k\text{-PO-ternary ideal of } T. \text{ Hence } Q = \overline{QTQTQ} = \overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ}.$$

## VI. CONCLUSION

In this paper mainly we studied about quasi po-k-ternary ideals and full quasi po-k-ternary ideals in PO-ternary semiring.

## ACKNOWLEDGMENT

Our thanks to the experts who have contributed towards preparation and development of the paper.

## REFERENCES

- [1] Dubey.M. K., A Note on Quasi k-Ideals and Bi k-Ideals in Ternary Semirings-italian journal of pure and applied mathematics - n. 28;2011 (143;150)
- [2] Lister.W.G., Ternary Rings, Amer.Math.Soc.154, 37-55 (1971).
- [3] Vandiver, Note on Simple type of algebra in which the cancellation law of addition does not hold, Bull.Am.Math.Soc.40, 920 (1934).
- [3] B. Smith, "An approach to graphs of linear forms (Unpublished work style)," unpublished.
- [4] Siva Prasad. P, MadhusudhanaRao. D., SrinivasaRao. G., Concepts on Ordered Ternary Semirings- International Journal of Innovative Science, Engineering & Technology, Volume 2, Issue 4, April 2015, pp. 435-438.
- [5] Siva Prasad. P, MadhusudhanaRao. D., SrinivasaRao. G., Concepts on PO-Ternary Semirings-International Organization of Scientific Research Journal of Mathematics, Volume 11, Issue 3, May-Jun 2015, pp 01-06.
- [6] Siva Prasad. P, MadhusudhanaRao. D., SrinivasaRao. G., A Study on Structure of PO-ternary semirings-Journal of Advances in Mathematics, Volume 10, No 8, pp 3717-3724.
- [7] Siva Prasad. P, MadhusudhanaRao. D., SrinivasaRao. G., A Note on One Sided and Two Sided PO-Ternary Ideals in PO-Ternary Semiring-Journal of Progressive Research in Mathematics, volume- 4, issue-3, pp: 339-347.
- [8] Siva Prasad. P, MadhusudhanaRao. D., SrinivasaRao. G., Theory of PO-ternary Ideals in PO-ternary Semirings-Global journal of Mathematics, Volume 3, issue 2, july 2015, pp 297-309.



**Dr. D. MadhusudhanaRao:** He completed his M.Sc. from Osmania University, Hyderabad, Telangana, India. M. Phil. from M. K. University, Madurai, Tamil Nadu, India. Ph. D. from AcharyaNagarjuna University, Andhra Pradesh, India. He joined as Lecturer in Mathematics, in the department of Mathematics, VSR & NVR College,

Tenali, A. P. India in the year 1997, after that he promoted as Head, Department of Mathematics, VSR & NVR College, Tenali. He helped more than 5 Ph. D's. At present he is guiding 7 Ph. D. Scholars and 3 M. Phil., Scholars in the department of Mathematics, AcharyaNagarjuna University, Nagarjuna Nagar, Guntur, A. P.

A major part of his research work has been devoted to the use of semigroups, Gamma semigroups, duo gamma semigroups, partially ordered gamma semigroups and ternary semigroups, Gamma semirings and ternary semirings, Near rings ect. He acting as peer review member to (1) "British Journal of Mathematics & Computer Science", (2) "International Journal of Mathematics and Computer Applications Research", (3) "Journal of Advances in Mathematics" and Editorial Board Member of (4) "International Journal of New Technology and Research". He is life member of (1) Andhra Pradesh Society for Mathematical Sciences, (2) Heath Awareness Research Institution Technology Association, (3) Asian Council of Science Editors, Membership No: 91.7347, (4) Council for Innovative Research for Journal of Advances in Mathematics". He published more than 64 research papers in different International Journals to his credit in the last four academic years.





**P. Siva Prasad:** He is working as Assistant Professor in the department of mathematics, Universal College of Engineering & Technology, perecharla, Guntur(Dt), Andhra Pradesh, India. He is pursuing Ph.D. under the guidance of Dr.D.MadhusudanaRao in AcharyaNagarjuna University. He published more than 7 research papers in popular international Journals to his credit. His area of interests are ternary semirings, ordered ternary semirings, semirings. Presently he is working on Partially Ordered Ternary semirings.

**<sup>3</sup>G. SrinivasaRao:** He is working as an Assistant Professor in the Department of Applied Sciences & Humanities, Tirumala Engineering College. He completed his M.Phil. inMadhuraiKamaraj University. He is pursuing Ph.D. under the guidance of Dr.D.MadhusudanaRao in AcharyaNagarjuna University. He published more than 20 research papers in popular international Journals to his credit. His area of interests are ternary semirings, ordered ternary semirings, semirings and topology. Presentlyhe is working on Ternary semirings.

