

# Structure Quasi-Po-Ternary Ideals in Po-Ternary Semirings

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**Abstract**— In this paper we introduce the definitions related to quasi-PO-ternary ideals and Bi-PO-ternary ideals in PO-ternary semirings and we study the relation between quasi-PO-ternary ideals and Bi-PO-ternary ideals in PO-ternary semirings.

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**Index Terms** — *Quasi PO- $k$ -ternary ideal, quasi-PO-ternary ideal of  $T$  generated by  $A$ , quasi simple, 0-quasi simple, quasi  $k$ -PO-ternary ideal.*

## I. INTRODUCTION

The notion of semiring was introduced by Vandiver [III] in 19III4. In fact semiring is a generalization of ring. In 1971 Lister [2] characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. MadusudhanaRao, D, Siva Prasad, P and SrinivasaRao, G [4, 5, 6, 7, 8] studied and investigated some results on partially ordered ternary semiring.

## II. PRELIMINARIES

**Definition II.1[ 6]** : A nonempty set  $T$  together with a binary operation called addition and a ternary multiplication denoted by  $[ ]$  is said to be a *ternary semiring* if  $T$  is an additive commutative semigroup satisfying the following conditions :  
 i)  $[[abc]de] = [a[bcd]e] = [ab[cde]]$ ,  
 ii)  $[(a + b)cd] = [acd] + [bcd]$ ,  
 iii)  $[a(b + c)d] = [abd] + [acd]$ ,  
 iv)  $[ab(c + d)] = [abc] + [abd]$  for all  $a; b; c; d; e \in T$ .

**Note II.2[6]** : For the convenience we write  $x_1x_2x_3$  instead of  $[x_1x_2x_3]$

**Note II.3[6]**: Let  $T$  be a ternary semiring. If  $A$ ,  $B$  and  $C$  are three subsets of  $T$ , we shall denote the set  $ABC = \{\sum abc : a \in A, b \in B, c \in C\}$ .

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**Note II.4[6]** : Let  $T$  be a ternary semiring. If  $A$ ,  $B$  are two subsets of  $T$ , we shall denote the set  $A + B = \{a + b : a \in A, b \in B\}$  and  $2A = \{a + a : a \in A\}$ .

**Note II.5[6]** : Any semiring can be reduced to a ternary semiring.

**Definition II.6 [6]**: A ternary semiring  $T$  is said to be a *partially ordered ternary semiring* or simply *PO Ternary Semiring* or *Ordered Ternary Semiring* provided  $T$  is partially ordered set such that  $a \leq b$  then

(1)  $a + c \leq b + c$  and  $c + a \leq c + b$ ,  
 (2)  $acd \leq bcd$ ,  $cad \leq cbd$  and  $cda \leq cdb$  for all  $a, b, c, d \in T$ .  
 Throughout  $T$  will denote as PO-ternary semiring unless otherwise stated.

**Theorem II.7 [6]**: Let  $T$  be a po-ternary semiring and  $A \subseteq T$ ,  $B \subseteq T$  and  $C \subseteq T$ . Then (i)  $A \subseteq (A)$ , (ii)  $((A)) = (A)$ , (iii)  $(A)(B)(C) \subseteq (ABC)$  and (iv)  $A \subseteq B \Rightarrow A \subseteq (B)$  and (v)  $A \subseteq B \Rightarrow (A) \subseteq (B)$ , (vi)  $(A \cap B) = (A) \cap (B)$ , (vii)  $(A \cup B) = (A) \cup (B)$ .

**Definition II.8 [6]**: A nonempty subset  $A$  of a PO-ternary semiring  $T$  is a *PO-ternary ideal* of  $T$  provided  $A$  is additive subsemigroup of  $T$ ,  $ATT \subseteq A$ ,  $TTA \subseteq A$ ,  $TAT \subseteq A$  and  $(A) \subseteq A$ .

**Theorem II.9[8]** : Let  $T$  be a PO-ternary semiring and  $A$ ,  $B$  be two PO-ternary ideals of  $T$ , then the sum of  $A$ ,  $B$  denoted by  $A + B$  is a PO-ternary ideal of  $T$  where  $A + B = \{x = a + b / a \in A, b \in B\}$ .

**Theorem II.10[8]**: Let  $A$  be a PO-ternary ideal of  $T$ . Then  $(A)$  is an ordered ternary ideal of  $T$  generated by  $A$ .

**Theorem II.11[8]** : The left PO-ternary ideal of a PO-ternary semiring  $T$  generated by a non-empty subset  $A$  is the intersection of all left PO-ideals of  $T$  containing  $A$ .

**Theorem II.12[8]** : The lateral ideal of a ternary semiring  $T$  generated by a non-empty subset  $A$  is the intersection of all lateral ideals of  $T$  containing  $A$ .

**Theorem II.13[8]** : The right PO-ternary ideal of a PO-ternary semiring  $T$  generated by a nonempty subset  $A$  is the intersection of all right PO-ternary ideals of  $T$  containing  $A$ .

### III. QUASI-PO-TERNARY IDEALS

We now introduce the notion of quasi-PO-ternary ideals in PO-ternary semirings.

**Definition III.1:** A non-empty subset  $Q$  of a PO-ternary semiring  $T$  is said to be **quasi-PO-ternary ideal** provided  $Q$  is a subsemigroups of  $(T, +)$  satisfying

- (1)  $TTQ \cap TQT \cap QTT \subseteq Q$
- (2)  $TTQ \cap TTQTT \cap QTT \subseteq Q$  and
- (III)  $(Q] \subseteq Q$ .

**Example III.2:** Let  $T = M_2(Z_0^-)$  be the PO-ternary semiring of the set of all  $2 \times 2$  square matrices over  $Z_0^-$ , the set of all non positive integers. Let  $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^- \right\}$ . Then we can easily verify that  $Q$  is a quasi-PO-ternary ideal of  $T$ .

**Theorem III.3:** Every left PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Assume that  $Q$  is a left PO-ternary ideal of  $T$ . Then  $(TTQ] \subseteq Q$ , but  $(TTQ] \cap (TQT \cup TTQTT) \cap (QTT) \subseteq (TTQ] \subseteq Q$ . Hence  $Q$  is a quasi-PO-ternary ideal of  $T$ .

**Theorem III.4:** Every lateral PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Similar to III.3.

**Theorem III.5:** Every right PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof:** Similar to III.3.

**Theorem III.6:** Every two sided PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof :** Any two sided PO-ternary ideal of  $T$  is a left PO-ternary ideal and right PO-ternary ideal and any left PO-ternary or any right PO-ternary ideal of  $T$  is a quasi-PO-ternary ideal of  $T$ . Therefore any two sided PO-ternary ideal of  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Theorem III.7:** Every PO-ternary ideal of a PO-ternary semiring  $T$  is a quasi-PO-ternary ideal of  $T$ .

**Proof :** Similar to III.6.

**Note III.8:** In general a quasi-PO-ternary ideal need not be a left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of  $T$ .

**Example III.9:** In example III.2,  $Q$  is a quasi-PO-ternary ideal of  $T$ , but  $Q$  is not left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of  $T$ .

**Theorem III.10:** Let  $T$  be a commutative PO-ternary semiring, then every quasi-PO-ternary ideal of  $T$  is a three sided PO-ternary ideal of  $T$ .

**Proof:** Assume that  $T$  be a commutative PO-ternary semiring. Let  $Q$  be a quasi-PO-ternary ideal of  $T$ . Then  $(TTQ] \cap (TQT \cup TTQTT) \cap (QTT) \subseteq Q$ . Since  $T$  be a commutative and  $Q \subseteq T$ , then  $TTQ = TQT = QTT = TTQTT$ . Now  $TTQ \cap (TQT \cup TTQTT) \cap (QTT) = (TTQ \cap TQT \cap QTT) \cup (TTQ \cap TTQTT \cap QTT) \subseteq TTQ \cup TTQ = (TTQ] + (TTQ] = (TTQ] \subseteq Q$ . Hence  $Q$  is a left PO-ternary ideal of  $T$ . Similarly  $Q$  is lateral PO-ternary ideal and right PO-ternary ideal of  $T$ . Therefore, every quasi-PO-ternary ideal of  $T$  is a three sided ideal of  $T$ .

**Theorem III.11:** The intersection of any system of quasi-PO-ternary ideals is a quasi-PO-ternary ideal of  $T$  or empty.

**Proof:** Let  $\{Q_\alpha\}_{\alpha \in \Delta}$  be a family of PO-ternary ideals of  $T$

$$\text{and let } Q = \bigcap_{\alpha \in \Delta} Q_\alpha$$

Assume that  $Q$  is not empty. Since  $Q_\alpha$  is a quasi-PO-ternary ideal for each  $\alpha \in \Delta$ . Then  $(Q_\alpha TT] \cap (TQ_\alpha T + TTQ_\alpha TT] \cap (TTQ_\alpha] \subseteq Q_\alpha$  for each  $\alpha \in \Delta$ .

$$\text{Now for each } \alpha \in \Delta \text{ } TTQ = TT(\bigcap_{\alpha \in \Delta} Q_\alpha) = \bigcap_{\alpha \in \Delta} TTQ_\alpha \subseteq$$

$$TTQ_\alpha \subseteq (TTQ_\alpha], \quad TQT = T(\bigcap_{\alpha \in \Delta} Q_\alpha)T = \bigcap_{\alpha \in \Delta} TQ_\alpha T \subseteq$$

$$TQ_\alpha T \subseteq (TQ_\alpha T], \quad TTQTT = TT(\bigcap_{\alpha \in \Delta} Q_\alpha)TT = \bigcap_{\alpha \in \Delta} TTQ_\alpha TT \subseteq TTQ_\alpha TT \subseteq (TTQ_\alpha TT], \text{ and } TTQ =$$

$$TT(\bigcap_{\alpha \in \Delta} Q_\alpha) = \bigcap_{\alpha \in \Delta} Q_\alpha TT \subseteq Q_\alpha TT \subseteq (Q_\alpha TT]. \text{ Then } (TTQ] \cap (TQT \cup TTQTT) \cap (QTT) \subseteq (TTQ_\alpha] \cap (TQ_\alpha T + TTQ_\alpha TT) \cap (Q_\alpha TT] \subseteq Q_\alpha \text{ for each } \alpha \in \Delta. \text{ Therefore } (TTQ] \cap (TQT + TTQTT) \cap (QTT) \subseteq \bigcap_{\alpha \in \Delta} Q_\alpha = Q.$$

Let  $x \in \bigcap_{\alpha \in \Delta} Q_\alpha$  and  $y \in T$  be such that  $y \leq x$ . Let for each

$\alpha \in \Delta$ . Since  $y \leq x$  and  $x \in Q_\alpha$ ,  $y \in Q_\alpha$ . Thus  $y \in \bigcap_{\alpha \in \Delta} Q_\alpha$

$$(\text{i.e. } (Q] = \left[ \bigcap_{\alpha \in \Delta} Q_\alpha \right] \subseteq \bigcap_{\alpha \in \Delta} (Q_\alpha] = \bigcap_{\alpha \in \Delta} Q_\alpha = Q). \text{ Therefore } Q = \bigcap_{\alpha \in \Delta} Q_\alpha, \text{ is a quasi-PO-ternary ideal of } T.$$

**Theorem III.12:** Every quasi-PO-ternary ideal of a PO-ternary semiring  $T$  is a PO-ternary subsemiring of  $T$ .

**Proof :** Let  $Q$  be a quasi-PO-ternary ideal of  $T$ . By definition III.1,  $Q$  is a subsemigroup of  $(T, +)$ . Let  $a, b, c \in Q \subseteq T$ . Then  $abc \in (TTQ]$ ,  $abc \in (TQT]$ , and  $abc \in (QTT)$ . Therefore  $abc \in (TTQ] \cap (TQT \cup TTQTT) \cap (QTT) \subseteq Q$ , since  $Q$  is a quasi-PO-ternary ideal of  $T$  and hence  $abc \in Q$ . Therefore  $Q$  is a PO-ternary subsemiring of  $T$ .

**Lemma III.13:** If  $Q$  is a quasi-PO-ternary ideal of a PO-ternary semiring  $T$  and  $S$  is a PO-ternary subsemiring of  $T$ , then  $Q \cap S$  is a quasi-ideal of  $S$ .

**Proof:** Assume that  $Q_1 = Q \cap S \neq \emptyset$ . Since  $Q_1 \subseteq Q$ , it follows that  $SSQ_1 \cap SQ_1S \cap Q_1SS \subseteq TTQ \cap TQT \cap QTT \subseteq Q$ . Since  $Q_1 \subseteq S$  and  $S$  is a PO-ternary subsemiring of  $T$ . We have  $SSQ_1 \cap SQ_1S \cap Q_1SS \subseteq S$ . Then  $SSQ_1 \cap SQ_1S \cap Q_1SS \subseteq Q_1$ . Let  $x \in Q_1$  and  $y \in S$  such that  $y \leq x$ . Since  $x \in Q$ ,  $y \in Q$ . So  $y \in Q_1$ . Therefore  $Q_1$  is quasi-PO-ternary ideal of  $S$ .

**Theorem III.14:** The intersection of left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal is a quasi-PO-ternary ideal of T.

**Proof :** Let L, M, and R be the left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T respectively. Let  $Q = L \cap M \cap R$ . Choose  $l \in L, m \in M, r \in R$ . Since  $lmr \in L \cap M \cap R$ , Q is not empty. Since  $TTQ \subseteq L$ ,  $TQT \subseteq M$  and  $QTT \subseteq R$ , then we have  $TTQ \cap TQT \cap QTT \subseteq L \cap M \cap R = Q$ . Similarly,  $TTQ \cap TTQTT \cap QTT \subseteq Q$ . Let  $x \in L \cap M \cap R$  and  $y \in T$  such that  $y \leq x$ . Since  $x \in L \cap M \cap R$ ,  $y \in L \cap M \cap R$ . Therefore  $Q = L \cap M \cap R$  is a quasi-PO-ternary ideal of T.

**Theorem III.15:** An additive subsemigroup Q of PO-ternary semiring T is a quasi-PO-ternary ideal of T, if Q is the intersection of a left PO-ternary ideal, a lateral PO-ternary ideal and a right PO-ternary ideal of T.

**Proof :** Assume that Q is a quasi-PO-ternary ideal of T and  $L = (TTQ \cup Q)$ ,  $M = (TQT \cup TTQTT \cup Q)$ ,  $R = (QTT \cup Q)$ , then by theorems 2.10, III.4, III.5, III.6, we have L is left PO-ternary ideal, M is lateral PO-ternary ideal and R is right PO-ternary ideal of T containing Q respectively. Thus  $Q \subseteq L \cap M \cap R$ . Since Q is quasi-PO-ternary ideal of T. We have

$L \cap M \cap R = (TTQ \cup Q) \cap (TQT \cup TTQTT \cup Q) \cap (QTT \cup Q) = ((TTQ) \cap (TQT \cup TTQTT) \cap (QTT)) \cup (Q) \subseteq Q \cup (Q) = Q$ . Therefore  $Q = L \cap M \cap R$  and hence Q is the intersection of left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T.

**Definition III.16:** Let A be a nonempty subset of an ordered ternary semi-ring T. The intersection of all quasi-PO-ternary ideals of T containing A is called the **quasi-PO-ternary ideal of T generated by A** and is denoted by  $Q(A)$ . Moreover,  $Q(A)$  is the smallest quasi-PO-ternary ideal of T containing A. If  $A = \{a\}$ , we also write  $Q(\{a\})$  as  $Q(a)$  or  $\langle a \rangle_q$ .

**Theorem III.17:** Let A be a nonempty subset of an ordered ternary semi-ring T. Then

$$Q(A) = [A] \cup [((TTA) \cap (TaTUTT) \cap (ATT))].$$

In particular,  $Q(a) = \langle a \rangle_q = [a] \cup [((TTA) \cap (TaTUTT) \cap (ATT))]$  for all  $a \in T$ .

**Proof:** By the theorem II.11, II.12, and II.13, we have  $(A \cup TTA)$ ,  $(A \cup TAT \cup TTATT)$  and  $(A \cup ATT)$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T containing A, respectively. By theorem III.15, we have  $(TTA \cup A) \cap (TAT \cup TTATT \cup A) \cap (ATT \cup A)$  is a quasi-PO-ternary ideal of T containing A. Thus  $Q(A) \subseteq (TTA \cup A) \cap (TAT \cup TTATT \cup A) \cap (ATT \cup A)$

$= [A] \cup [((TTA) \cup (TAT \cup TTATT) \cup (ATT))]$ . By the theorem III.15, we have

$$\begin{aligned} & [A] \cup [((TTA) \cup (TAT \cup TTATT) \cup (ATT))] \\ &= [TTA \cup A] \cap [TAT \cup TTATT \cup A] \cap [ATT \cup A] \\ &\subseteq [(TTQ(a) \cup Q(A))] \cap [(TQ(A)T \cup TTQ(A)TT) \cap (TTQ(A) \cup Q(A))] \subseteq Q(A). \end{aligned}$$

Hence  $Q(A) = [A] \cup [((TTA) \cup (TAT \cup TTATT) \cup (ATT))]$ .

Now we characterize the relationship between the minimality of the quasi-PO-ternary ideals and a quasi-simple and 0-quasi simple-PO-ternary semirings.

**Definition III.18 :** Let T be a PO-ternary semiring with a zero element. Then T is called **quasi simple** if T has no proper quasi-PO-ternary ideals of T.

**Theorem III.19:** Let T be a PO-ternary semiring without a zero element. Then the following are equivalent.

- (1) T is a quasi-simple.
- (2)  $(TTa) \cap (TaTUTT) \cap (aTT) = T$  for all  $a \in T$ .
- (III)  $Q(a) = T$  for all  $a \in T$ .

**Proof:** (1)  $\Rightarrow$  (2) : Suppose that T is quasi-simple and let  $a \in T$ . By the theorem 2.11, 2.12, and 2.1III, we have  $(aUTTa)$ ,  $(aUTaTUTT)$  and  $(aUaTT)$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T containing A, respectively. By theorem III.15, we have  $(TTaUa) \cap (TaTUTT) \cap (aTTUa)$  is a quasi-PO-ternary ideal of T containing a. Since T is quasi-simple, thus  $(TTaUa) \cap (TaTUTT) \cap (aTTUa) = T$ .

(2)  $\Rightarrow$  (III) : Assume that  $(TTa) \cap (TaTUTT) \cap (aTT) = T$  for all  $a \in T$ . By theorem III.18, we have  $T = (TTa) \cap (TaTUTT) \cap (aTT) \subseteq [a] \cup [((TTa) \cap (TaTUTT) \cap (aTT))] = Q(a)$ . Therefore  $Q(a) = T$ .

(III)  $\Rightarrow$  (1) : Assume that  $Q(a) = T$  for all  $a \in T$ . Let Q be a quasi-PO-ternary ideal of T and let  $a \in Q$ . Then  $Q(a) = T$ , and so  $Q(a) \subseteq Q \subseteq T$ . Therefore  $T = Q$  and hence T is quasi-simple.

**Definition III.20:** Let T be a PO-ternary semiring with zero element,  $T^{\text{III}} \neq \{0\}$  and  $|T| > 1$ . Then T is called **0-quasi-simple** if T has no non zero proper quasi-PO-ternary ideals.

**Theorem III.21:** Let T be a PO-ternary semiring with zero element,  $T^{\text{III}} \neq \{0\}$  and  $|T| > 1$ . Then T is a 0-quasi-simple if and only if  $Q(a) = T$  for  $a \in T \setminus \{0\}$ .

**Proof:** Suppose that T is a 0-quasi-simple and  $a \in T \setminus \{0\}$ . Then  $Q(a) \neq \{0\}$ . Since T is 0-quasi simple, therefore  $Q(a) = T$ .

Conversely, suppose that  $Q(a) = T$  for all  $a \in T \setminus \{0\}$ . Let Q be a non-zero quasi-PO-ternary ideal of T and  $a \in Q \setminus \{0\}$ . Then  $Q(a) = T$  and  $Q(a) \subseteq Q \subseteq T$  implies that  $T = Q$ . Therefore, T is a 0-quasi-simple.

**Definition III.22:** A left quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a **minimal left quasi-PO-ternary ideal** of T if there is no a left quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any left quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.23:** A lateral quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a **minimal lateral quasi-PO-ternary ideal** of T if there is no a lateral quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any lateral quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.24:** A right quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a **minimal right quasi-PO-ternary ideal** of T if there is no a right quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any right quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.25:** A two sided quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a **minimal two sided quasi-PO-ternary ideal** of T if there is no a two sided quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ .

Equivalently, if for any two sided quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Definition III.26:** A quasi-PO-ternary ideal Q of an ordered ternary semiring T without a zero element is called a *minimal quasi-PO-ternary ideal* of T if there is no a quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Theorem III.27:** Let Q be a quasi-PO-ternary ideal of an ordered ternary semi-ring T without a zero element. Then Q is a minimal quasi-PO-ternary ideal of T if and only if it is the intersection of a minimal ordered left, a minimal ordered right and a minimal ordered lateral PO-ternary ideal of T.

**Proof:** Suppose that Q is a minimal quasi-PO-ternary ideal of T. Then  $(TTQ] \cap (TQT \cup TTQTT] \cap (QTT)] \subseteq Q$ . By the theorems 2.11, 2.12, and 2.1III, we have  $(TTQ]$ ,  $(TQT \cup TTQTT]$  and  $(QTT)]$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T and by theorem III.15, we have  $(TTQ] \cap (TQT \cup TTQTT] \cap (QTT)]$  is a quasi-PO-ternary ideal of T.

Since Q is minimal quasi-PO-ternary ideal of T.

We have  $Q = (TTQ] \cap (TQT \cup TTQTT] \cap (QTT)]$ .

We claim that  $(TTQ]$  is a minimal left PO-ternary ideal of T. Let L be a left PO-ternary ideal of T such that  $L \subseteq (TTQ]$ . Then  $(TTL] \subseteq (L] = L \subseteq (TTQ]$ .

Therefore,  $(TTL] \cap (TQT \cup TTQTT] \cap (QTT)] \subseteq (TTQ] \cap (TQT \cup TTQTT] \cap (QTT)] = Q$ .

Since  $(TTL] \cap (TQT \cup TTQTT] \cap (QTT)]$  is a quasi-PO-ternary ideal of T and Q is a minimal quasi-PO-ternary ideal of T, we have  $(TTL] \cap (TQT \cup TTQTT] \cap (QTT)] = Q$ .

Thus  $Q \subseteq (TTL]$  and so  $(TTQ] \subseteq (TT(TTL]) \subseteq (TT(L]) = (TTL] \subseteq L$ .

Hence,  $L = (TTQ]$ . Therefore,  $(TTQ]$  is a minimal left PO-ternary ideal of T.

Similarly, we can show that  $(QTT]$  and  $(TQT \cup TTQTT]$  are minimal right PO-ternary ideal and minimal lateral PO-ternary ideal of T, respectively.

Conversely, let  $Q = L \cap M \cap R$  where L, M and R are a minimal left PO-ternary ideal, a minimal lateral PO-ternary ideal and a minimal right PO-ternary ideal of a PO-ternary semiring T, respectively. By theorem III.15, Q is a quasi-PO-ternary ideal of T. Let A be a quasi-PO-ternary ideal of T such that  $A \subseteq Q$ . By theorems 2.11, 2.12, and 2.1III, we have  $(TTA]$ ,  $(TAT \cup TTATT]$  and  $(ATT]$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideal of T. Now  $(TTA] \subseteq (TTQ] \subseteq (TTL] \subseteq (L] = L$ . Since L is a minimal left PO-ternary ideal of T, we have  $(TTA] = L$ . Similarly,  $(TAT \cup TTATT] = M$  and  $(ATT] = R$ . Since A is a quasi-PO-ternary ideal of T and hence  $Q = L \cap M \cap R = (TTA] \cap (TAT \cup TTATT] \cap (ATT] \subseteq A$ . Therefore  $A = Q$ . Hence, Q is a minimal quasi-PO-ternary ideal of PO-ternary semiring T.

**Theorem III.28:** Let Q be a quasi-PO-ternary ideal of an ordered ternary semi-ring T without a zero element. If Q is quasi-simple, then Q is a minimal quasi-PO-ternary ideal of T.

**Proof :** Suppose that Q is quasi-simple and let A be a quasi-PO-ternary ideal of T such that  $A \subseteq Q$ . Therefore  $(QQA] \cap (QAQ \cup QQAQQ] \cap (AQQ] \subseteq (TTA] \cap (TAT \cup TTATT] \cap (ATT] \subseteq A$  and  $(A] \cap Q \subseteq (A] = A$ . Therefore A

is a quasi-PO-ternary ideal of Q. Since Q is quasi-simple and hence  $Q = A$ . Hence Q is a minimal quasi-PO-ternary ideal of T.

**Theorem III.29:** Let T be an ordered ternary semiring without a zero element having proper quasi-PO-ternary ideals. Then every proper quasi-PO-ternary ideal of T is minimal if and only if the intersection of any two distinct proper quasi-PO-ternary ideals is empty.

**Proof:** Let  $Q_1$  and  $Q_2$  be two distinct proper quasi-PO-ternary ideals of T. By assumption, we have that  $Q_1$  and  $Q_2$  are minimal. If  $Q_1 \cap Q_2 \neq \emptyset$ , then by Theorem III.11,  $Q_1 \cap Q_2$  is a quasi-PO-ternary ideal of T. Since  $Q_1 \cap Q_2 \subseteq Q_1$  and  $Q_1$  is minimal, we have  $Q_1 \cap Q_2 = Q_1$ . Since  $Q_1 \cap Q_2 \subseteq Q_2$  and  $Q_2$  is minimal, we have  $Q_1 \cap Q_2 = Q_2$ . Therefore  $Q_1 = Q_1 \cap Q_2 = Q_2$ . This is a contradiction and hence  $Q_1 \cap Q_2 = \emptyset$ .

Conversely, suppose that Q be a proper quasi-PO-ternary ideal of T and let A be a quasi-PO-ternary ideal of T such that  $A \subseteq Q$ . Then A is a proper quasi-PO-ternary ideal of T. If  $A \neq Q$ , then by assumption,  $A = A \cap Q = \emptyset$ . That is a contradiction. Hence,  $A = Q$ . Therefore, Q is a minimal quasi-PO-ternary ideal of T.

**Definition III.30:** A nonzero quasi-PO-ternary ideal Q of a PO-ternary semiring T with a zero element is called a *0-minimal quasi-PO-ternary ideal* of T if there is no a nonzero quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ . Equivalently, if for any nonzero quasi-PO-ternary ideal A of T such that  $A \subseteq Q$ , we have  $A = Q$ .

**Note III.31:** We also define a 0-minimal left PO-ternary ideal, a 0-minimal lateral PO-ternary and a 0-minimal right PO-ternary ideal of an ordered ternary semiring T with a zero element in the same way of a 0-minimal quasi-PO-ternary ideal.

**Theorem III.32:** Let T be a PO-ternary semiring with a zero element. Then the intersection of a 0-minimal left PO-ternary ideal, a 0-minimal right PO-ternary ideal and a 0-minimal lateral PO-ternary ideal of T is either {0} or a 0-minimal quasi-PO-ternary ideal of T.

**Proof :** Let  $Q = L \cap M \cap R \neq \{0\}$  where L, M, and R are a 0-minimal left PO-ternary ideal, a 0-minimal lateral PO-ternary ideal and a 0-minimal right PO-ternary ideal of T, respectively. By theorem III.14, Q is a quasi-PO-ternary ideal of T. Let A be a non zero quasi-PO-ternary ideal of T such that  $A \subseteq Q$ . By theorems II.11, II.12, and II.13,  $(TTA]$ ,  $(TAT \cup TTATT]$ ,  $(ATT]$  are left PO-ternary ideal, lateral PO-ternary ideal and right PO-ternary ideals of T respectively. Then we get the following two cases:

**Case-1:**  $(TTA] = \{0\}$ ,  $(TAT \cup TTATT] = \{0\}$ ,  $(ATT] = \{0\}$ . If  $(TTA] = \{0\}$ , then  $(TTA] = \{0\} \subseteq A$ . Thus A is a nonzero left PO-ternary ideal of T. Since  $A \subseteq Q \subseteq L$  and L is a 0-minimal left PO-ternary ideal of T. Then we have  $A = L$ . Therefore  $A = Q$ . Similarly, if  $(ATT] = \{0\}$  or  $(TAT \cup TTATT] = \{0\}$ , we get  $A = Q$ .

**Case-2:**  $(TTA] \neq \{0\}$ ,  $(TAT \cup TTATT] \neq \{0\}$ ,  $(ATT] \neq \{0\}$ . Now  $(TTA] \subseteq (TTQ] \subseteq (TTL) \subseteq (L] = L$ . Since L is a 0-minimal left PO-ternary ideal of T, we have  $(TTA] = L$ . Similarly,  $(TAT \cup TTATT] = M$  and  $(ATT] = R$ . Since A is a quasi-PO-ideal of T, we have  $Q = L \cap M \cap R = (TTA] \cap (TAT \cup TTATT] \cap (ATT] \subseteq A \Rightarrow A = Q$ . Hence, Q is a 0-minimal ordered quasi-ideal of T.

**Theorem III.33:** Let  $Q$  be a non-zero quasi-PO-ternary ideal of an ordered ternary semi-ring  $T$  with a zero element. If  $Q$  is 0-quasi-simple, then  $Q$  is a 0-minimal quasi-PO-ternary ideal of  $T$ .

**Proof:** Suppose that  $Q$  is 0-quasi-simple and let  $A$  be a non-zero quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . Therefore  $(QQA) \cap (QAQ \cup QQAQQ) \cap (AQQ) \subseteq (TTA) \cap (TAT \cup TTATT) \cap (ATT) \subseteq A$  and  $(A) \cap Q \subseteq (A) = A$ . Therefore  $A$  is a non-zero quasi-PO-ternary ideal of  $Q$ . Since  $Q$  is 0-quasi-simple and hence  $Q = A$ . Hence  $Q$  is a 0-minimal quasi-PO-ternary ideal of  $T$ .

**Theorem III.34:** Let  $T$  be an ordered ternary semiring with a zero element having non-zero proper quasi-PO-ternary ideals. Then every non-zero proper quasi-PO-ternary ideal of  $T$  is 0-minimal if and only if the intersection of any two distinct non-zero proper quasi-PO-ternary ideals is  $\{0\}$ .

**Proof:** Let  $Q_1$  and  $Q_2$  be two distinct non-zero proper quasi-PO-ternary ideals of  $T$ . By assumption, we have that  $Q_1$  and  $Q_2$  are 0-minimal. If  $Q_1 \cap Q_2 \neq \{0\}$ , then by Theorem III.11,  $Q_1 \cap Q_2$  is a non-zero quasi-PO-ternary ideal of  $T$ . Since  $Q_1 \cap Q_2 \subseteq Q_1$  and  $Q_1$  is 0-minimal, we have  $Q_1 \cap Q_2 = Q_1$ . Since  $Q_1 \cap Q_2 \subseteq Q_2$  and  $Q_2$  is 0-minimal, we have  $Q_1 \cap Q_2 = Q_2$ . Therefore  $Q_1 = Q_1 \cap Q_2 = Q_2$ . This is a contradiction and hence  $Q_1 \cap Q_2 = \{0\}$ .

Conversely, suppose that  $Q$  be a non-zero proper quasi-PO-ternary ideal of  $T$  and let  $A$  be a non-zero quasi-PO-ternary ideal of  $T$  such that  $A \subseteq Q$ . Then  $A$  is a non-zero proper quasi-PO-ternary ideal of  $T$ . If  $A \neq Q$ , then by assumption,  $A = A \cap Q = \{0\}$ . That is a contradiction. Hence,  $A = Q$ . Therefore,  $Q$  is a 0-minimal quasi-PO-ternary ideal of  $T$ .

**Theorem III.35:** Let  $x$  be an idempotent element of a PO-ternary semiring  $T$ , that is,  $x^{\text{III}}(=xxx) \geq x$ . If  $R$  is a right PO-ternary ideal,  $M$  a lateral PO-ternary ideal, and  $L$  a left PO-ternary ideal of  $T$ , then  $(Rxx]$ ,  $(xxMxx]$ , and  $(xxL]$  are quasi-PO-ternary ideals of  $T$ .

**Proof:** To show  $(Rxx]$ ,  $(xxMxx]$ , and  $(xxL]$  are quasi-ideals of  $S$ , it is sufficient to show that

$$(Rxx] = (R) \cap (TxT + TTxTT) \cap (TTx], (xxMxx] = (xTT) \cap (M) \cap (SSx], \text{ and } (xxL] = (xTT) \cap (TxT \cup TTxTT) \cap (L).$$

For the first case, it is clear that  $(Rxx] \subseteq R \cap TTx = (R \cap TTx) = (R) \cap (TTx)$ .

Let  $a \in (R) \cap (TTx) \Rightarrow a \in (R)$  and  $a \in (TTx)$ .

Now,  $a \in (TTx) \Rightarrow a \leq \sum_{i=1}^n s_i t_i x$  for some  $s_i, t_i \in T$ .

Therefore  $axx \leq (\sum_{i=1}^n s_i t_i x)xx = \sum_{i=1}^n s_i t_i (xxx) \geq \sum_{i=1}^n s_i t_i x \geq a$ .

It follows that  $a \in (Rxx]$  and hence  $(Rxx] = (R) \cap (TTx)$ .

Again  $a \leq axx \Rightarrow a \in (TxT)$ . Therefore we have  $a \in (TxT \cup TTxTT)$ .

Thus  $(R) \cap (TTx) \subseteq (TxT \cup TTxTT)$ . Therefore,  $(Rxx] = (R) \cap (TxT + TTxTT) \cap (TTx)$ .

For the second case, we see that  $(xxMxx] \subseteq (xTT) \cap (M) \cap (TTx)$ . Let  $a \in (xTT) \cap (M) \cap (TTx)$ . Then  $a \in (xTT)$ ,  $a \in (M)$  and  $a \in (TTx)$ . Now  $a \in (xTT)$  and  $a \in (TTx)$

$$\Rightarrow a \leq \sum_{i=1}^n s_i t_i x = \sum_{j=1}^m x u_j v_j \text{ for some } s_i, t_i, u_j, v_j \in T.$$

$$\begin{aligned} \text{Therefore } xxaxx &\leq xx(\sum_{i=1}^n s_i t_i x)xx \\ &= xx \sum_{i=1}^n s_i t_i (xxx) \geq xx \sum_{i=1}^n s_i t_i x = xx \sum_{j=1}^m x u_j v_j = \\ &\sum_{j=1}^m (xxx) u_j v_j \geq \sum_{j=1}^m x u_j v_j \geq a \\ \Rightarrow a &\in (xxMxx] \text{ and hence } (xxMxx] = (xTT) \cap (M) \cap (SSx]. \end{aligned}$$

For the third case it is similar to first case.

#### IV. PRIME QUASI-PO-TERNARY IDEALS

In this section, we introduce the notions of prime and semiprime quasi-PO-ternary ideals in PO-ternary semirings and some relevant counter examples are also indicated.

**Definition IV.1:** A proper quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is said to be **prime quasi-PO-ternary ideal** provided  $ABC \subseteq Q$  implies that  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$  for some quasi-PO-ternary ideals  $A, B, C$  of  $T$ .

**Definition IV.2:** A proper quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is said to be **semiprime quasi-PO-ternary ideal** provided  $A^3 \subseteq Q$  implies that  $A \subseteq Q$  for some quasi-PO-ternary ideal  $A$  of  $T$ .

**Definition IV.3:** A proper quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is said to be **weakly prime quasi-PO-ternary ideal** provided  $Q \subseteq A, B \subseteq Q, C \subseteq Q$  and  $ABC \subseteq Q$  implies that  $A = Q$  or  $B = Q$  or  $C = Q$  for some quasi-PO-ternary ideals of  $T$ .

**Theorem IV.4:** Every prime quasi-PO-ternary ideal of  $T$  is a Semiprime quasi-PO-ternary ideal of  $T$ .

**Proof:** Suppose that  $Q$  is a prime quasi-PO-ternary ideal of  $T$  and  $A$  be any quasi-PO-ternary ideal of  $T$  such that  $A^3 = AAA \subseteq Q$ . Since  $Q$  is prime. Therefore  $A \subseteq Q$  and hence  $Q$  is a Semiprime quasi-PO-ternary ideal of  $T$ .

**Note IV.5:** The converse of the theorem 4.4. need not be true. i.e., every Semiprime quasi-PO-ternary ideal of  $T$  need not be a prime quasi-PO-ternary ideal of  $T$ .

**Example IV.6:** Let  $T = M_2(\mathbb{Z}_0^-)$  is a PO-ternary semiring of  $2 \times 2$  square matrices over  $\mathbb{Z}_0^-$ . Let  $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_0^- \right\}$  and  $a \leq b$  for  $a, b \in \mathbb{Z}_0^-$ . Then  $Q$  is a Semiprime quasi-PO-ternary ideal of  $T$ . But  $Q$  is not a prime quasi-PO-ternary ideal of  $T$ .

Since  $A = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z}_0^- \right\}$ ,

$B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in \mathbb{Z}_0^- \right\}$  and

$C = \left\{ \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} : d \in Z_0^- \right\}$  are quasi-PO-ternary ideal of T such that  $ABC \subseteq Q$ . But  $A \not\subseteq Q$ ,  $B \not\subseteq Q$  and  $C \not\subseteq Q$ .

**Theorem IV.7:** Every prime quasi-PO-ternary ideal Q of a PO-ternary semiring T is a weakly prime quasi-PO-ternary ideal of T.

**Proof:** Suppose that Q is a prime quasi-PO-ternary ideal of T. Then there exist quasi-PO-ternary ideals A, B, C of T such that  $ABC \subseteq Q$ . If  $Q \subseteq A$ ,  $Q \subseteq B$ ,  $C \subseteq Q$  and  $ABC \subseteq Q$ , Q is a prime quasi-PO-ternary ideal of T implies that  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$ . Therefore  $A = Q$  or  $B = Q$  or  $C = Q$  and hence Q is a weakly prime quasi-PO-ternary ideal of T.

**Note IV.8:** The converse of the theorem 4.7. need not be true. i.e., every weakly prime quasi-PO-ternary ideal of T is not prime quasi-PO-ternary ideal of T.

**Example IV.9:** Let  $T = M_2(Z_0^-)$  is a PO-ternary semiring of  $2 \times 2$  square matrices over  $Z_0^-$ . Let  $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 30Z_0^- \right\}$  and  $a \leq b$  for  $a, b \in Z_0^-$ . Then Q is a weakly prime quasi-PO-ternary ideal of T. But Q is not a prime quasi-PO-ternary ideal of T. Since  $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 2Z_0^- \right\}$ ,  $B = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 3Z_0^- \right\}$  and  $C = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 5Z_0^- \right\}$  are quasi-PO-ternary ideal of T such that  $ABC \subseteq Q$ . But  $Q \not\subseteq A$ ,  $Q \not\subseteq B$  and  $Q \not\subseteq C$ .

**Theorem IV.10:** Let T be a PO-ternary semiring and Q be a quasi-PO-ternary ideal of T. If Q is prime, then Q is left or lateral or right PO-ternary ideal of T.

**Proof:** Let Q be a prime quasi-PO-ternary ideal of T. Then  $(TTQ)(TQT \cup TTQTT)(QTT) \subseteq TTQ \cap (TQT \cup TTQTT) \cap QTT \subseteq Q$ . Since Q is prime, we have  $TTQ \subseteq Q$  or  $TQT \cap TTQTT \subseteq Q$  or  $QTT \subseteq Q$ . therefore Q is left or lateral or right PO-ternary ideal of T.

**Theorem IV.11:** Let T be a commutative PO-ternary semiring and Q be a quasi-PO-ternary ideal of T. Then Q is prime if and only if  $abc \in Q$  implies  $a \in Q$  or  $b \in Q$  or  $c \in Q$ .

**Proof:** Suppose that Q is a prime quasi-PO-ternary ideal of T. Let  $abc \in Q$ . Then by theorem 4.10, Q is a PO-ternary ideal of T. Let  $x \in \langle a \rangle_q \langle b \rangle_q \langle c \rangle_q$ . Then  $x = ((a) \cap (TTa) \cap (TaTUTTaaTT) \cap (aTT)) \cup ((b) \cap (TTb) \cap (TbTUTTbTT) \cap (bTT)) \cup ((c) \cap (TTc) \cap (TcTUTTcTT) \cap (cTT))$ . Since  $abc \in Q$  and Q is a PO-ternary ideal of T. Therefore  $x \in Q$ . Thus  $\langle a \rangle_q \langle b \rangle_q \langle c \rangle_q \subseteq Q$ . Since Q is prime quasi-PO-ternary ideal of T. Hence  $a \in Q$  or  $b \in Q$  or  $c \in Q$ . Converse is obvious.

**Theorem IV.12:** Let T be a PO-ternary semiring and Q be a quasi-PO-ternary ideal of T. Then Q is prime if and only if  $((TTa) \cap (TaTUTTaaTT) \cap (aTT)) \cup ((TTb) \cap (TbTUTTbTT) \cap (bTT)) \cup ((TTc) \cap (TcTUTTcTT) \cap (cTT)) \subseteq Q$  implies  $a \in Q$  or  $b \in Q$  or  $c \in Q$ .

**Proof:** Suppose that Q is a prime quasi-PO-ternary ideal of T and let  $((TTa) \cap (TaTUTTaaTT) \cap (aTT)) \cup ((TTb) \cap (TbTUTTbTT) \cap (bTT)) \cup ((TTc) \cap (TcTUTTcTT) \cap (cTT)) \subseteq Q$  for some  $a, b, c \in T$ . Clearly,  $((TTa) \cap (TaTUTTaaTT) \cap (aTT))$ ,  $((TTb) \cap (TbTUTTbTT) \cap (bTT))$ ,  $((TTc) \cap (TcTUTTcTT) \cap (cTT))$  are quasi-PO-ternary ideals of T. Since Q is prime, therefore  $((TTa) \cap (TaTUTTaaTT) \cap (aTT)) \subseteq Q$  or  $((TTb) \cap (TbTUTTbTT) \cap (bTT)) \subseteq Q$  or  $((TTc) \cap (TcTUTTcTT) \cap (cTT)) \subseteq Q$ . If  $((TTa) \cap (TaTUTTaaTT) \cap (aTT)) \subseteq Q$ , then  $\langle a \rangle_q \subseteq Q$  implies that  $a \in Q$ . Similarly,  $b \in Q$  or  $c \in Q$ . Converse is obvious.

**Theorem IV.13:** Let T be a PO-ternary semiring. If the quasi-PO-ternary ideal of T with respect to inclusion relation form a chain, then every weakly prime quasi-PO-ternary ideal is a prime quasi-PO-ternary ideal of T.

**Proof:** Let Q be a weakly prime quasi-PO-ternary ideal of T. Let A, B, C are quasi-PO-ternary ideal of T such that  $ABC \subseteq Q$ . Suppose that  $A \not\subseteq Q$ ,  $B \not\subseteq Q$  and  $C \not\subseteq Q$ . By the statement since  $Q \subseteq A$ ,  $Q \subseteq B$  and  $Q \subseteq C$ . Since Q is weakly prime quasi-PO-ternary ideal of T. Therefore  $A = Q$  or  $B = Q$  or  $C = Q$ . This is a contradiction. Hence  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$ . Therefore Q is a prime quasi-PO-ternary ideal of T.

**Theorem IV.14:** Let T be a PO-ternary semiring. Then the following are equivalent

- (1) The quasi-PO-ternary ideal of T is idempotent.
- (2) If A, B, C are three quasi-PO-ternary ideals of T such that  $A \cap B \cap C \neq \emptyset$ , then  $A \cap B \cap C = ABC$ .
- (3)  $\langle a \rangle_q = [\langle a \rangle_q]^3$  for all  $a \in T$ .

**Proof:** (1)  $\Rightarrow$  (2): Suppose that A, B, C are quasi-PO-ternary ideals of T such that  $A \cap B \cap C \neq \emptyset$ . Then by theorem III.11,  $A \cap B \cap C$  is a quasi-PO-ternary ideal of T. Since every quasi-PO-ternary ideal is an idempotent. Therefore  $(A \cap B \cap C) = (A \cap B \cap C)^3 = (A \cap B \cap C)(A \cap B \cap C)(A \cap B \cap C) \subseteq ABC$ .

(2)  $\Rightarrow$  (3): It is straight forward and (3)  $\Rightarrow$  (1) is obvious.

**Definition IV.15:** A non-empty subset A of a PO-ternary semiring T is said to be  **$m_q$ -system** provided for any  $a, b, c \in A$ , there exist  $x \in \langle a \rangle_q$ ,  $y \in \langle b \rangle_q$ ,  $z \in \langle c \rangle_q$  and  $d \in A$  such that  $xyz \leq d$ .

**Note IV.16:** A non-empty subset A of a PO-ternary semiring T is called an  **$m_q$ -system** if  $a, b, c \in A$ , there exist  $x \in \langle a \rangle_q$ ,  $y \in \langle b \rangle_q$ ,  $z \in \langle c \rangle_q$  such that  $xyz \in (A)$ .

**Definition IV.17:** A non-empty subset A of a PO-ternary semiring T is said to be  **$n_q$ -system** provided for any  $a \in A$ , there exist  $x, y, z \in \langle a \rangle_q$  and  $d \in A$  such that  $xyz \leq d$ .

**Note IV.18:** A non-empty subset A of a PO-ternary semiring T is said to be  **$n_q$ -system** provided for any  $a \in A$ , there exist  $x, y, z \in \langle a \rangle_q$  such that  $xyz \in (A)$ .

**Theorem IV.19:** Every  $m_q$ -system is an  $n_q$ -system of PO-ternary semiring T.

**Proof:** Suppose that the non-empty subset A of a PO-ternary semiring T is an  $m_q$ -system. Let for any  $a \in A$ , there exist  $x, y, z \in \langle a \rangle_q$ . Since A is an  $m_q$ -system and hence  $xyz \in (A)$ . therefore A is a  $n_q$ -system of T.

**Note IV.20:** The converse of the theorem 4.19, need not be true. i.e., every  $n_q$ -system of a PO-ternary semiring  $T$  need not be  $an_q$ -system of  $T$ .

**Example IV.21:** Let  $T = Z_6^-$  is a PO-ternary semiring under usual addition, multiplication modulo 6 and natural ordering. Let  $A = \{-2, -3\}$ . Then  $A$  is an  $n_q$ -system but not an  $an_q$ -system.

**Theorem IV.22:** Let  $T$  be a PO-ternary semiring and  $Q$  is a quasi-PO-ternary ideal of  $T$ . Then  $Q$  is prime quasi-PO-ternary ideal of  $T$  if and only if  $T \setminus Q$  is an  $m_q$ -system of  $T$ .

**Proof:** Suppose that  $Q$  is a prime quasi-PO-ternary ideal of  $T$ . Let  $a, b, c \in T \setminus Q$ .

Suppose that  $xyz \leq d$  for all  $d \in T \setminus Q$  and for all  $x \in \langle a \rangle_q, y \in \langle b \rangle_q$  and  $z \in \langle c \rangle_q$ .

Then  $\langle a \rangle_q \langle b \rangle_q \langle c \rangle_q \subseteq Q$ . Since  $Q$  is a prime quasi-PO-ternary ideal of  $T$ .

Therefore  $a \in Q$  or  $b \in Q$  or  $c \in Q$ . This is a contradiction.

Therefore  $xyz \leq d$  for some  $x \in \langle a \rangle_q, y \in \langle b \rangle_q$  and  $z \in \langle c \rangle_q$ .

Hence  $T \setminus Q$  is an  $m_q$ -system of  $T$ .

Conversely suppose that,  $A, B, C$  are quasi-PO-ternary ideals of  $T$  such that  $ABC \subseteq Q$ . Assume that  $A \not\subseteq Q, B \not\subseteq Q$  and  $C \not\subseteq Q$ . Let  $a \in A \setminus Q, b \in B \setminus Q$  and  $c \in C \setminus Q$ . Then  $a, b, c \in T \setminus Q$ . Since  $T \setminus Q$  is an  $m_q$ -system. Therefore there exist an element  $d \in T \setminus Q$  such that  $xyz \leq d$  for some  $x \in \langle a \rangle_q, y \in \langle b \rangle_q$  and  $z \in \langle c \rangle_q$ . But  $xyz \in \langle a \rangle_q \langle b \rangle_q \langle c \rangle_q \subseteq ABC \subseteq Q$ . This is a contradiction. Hence  $A \subseteq Q$  or  $B \subseteq Q$  or  $C \subseteq Q$ . Therefore  $Q$  is a prime quasi-PO-ternary ideal of  $T$ .

**Theorem IV.23:** Let  $T$  be a PO-ternary semiring and  $Q$  is a quasi-PO-ternary ideal of  $T$ . Then  $Q$  is semiprime quasi-PO-ternary ideal of  $T$  if and only if  $T \setminus Q$  is an  $n_q$ -system of  $T$ .

**Proof:** Similar to the proof of the theorem IV.22.

**Definition IV.24:** A quasi-PO-ternary ideal of a PO-ternary semiring  $T$  is said to be ***T-prime quasi-PO-ternary ideal*** of  $T$  provided  $xTyTz \subseteq Q$  implies  $x \in Q$  or  $y \in Q$  or  $z \in Q$ .

**Definition IV.25:** A quasi-PO-ternary ideal of a PO-ternary semiring  $T$  is said to be ***T-semiprime quasi-PO-ternary ideal*** of  $T$  provided  $xTxTx \subseteq Q$  implies  $x \in Q$ .

**Theorem IV.26:** A quasi-PO-ternary ideal  $Q$  of a PO-ternary semiring  $T$  is **T-prime** if and only if  $RML \subseteq Q$  implies  $R \subseteq Q$  or  $M \subseteq Q$  or  $L \subseteq Q$  for any right PO-ternary ideal  $R$ , lateral PO-ternary ideal  $M$  and left PO-ternary ideal  $L$  of  $T$ .

**Proof:** Let  $Q$  be a **T-prime** quasi-PO-ternary ideal of  $T$  and  $RML \subseteq Q$ . Suppose  $R \not\subseteq Q$  and  $M \not\subseteq Q$ . Then there exist  $x \in R \setminus Q$  and  $y \in M \setminus Q$ . Let  $z \in L$ . Then  $xTyTz \subseteq RTML \subseteq RML \subseteq Q$ . Since  $Q$  is **T-prime**. Therefore,  $x \in Q$  or  $y \in Q$  or  $z \in Q$ . But  $x \notin Q$  and  $y \notin Q$ . Hence  $z \in Q$  and hence  $L \subseteq Q$ .

Conversely, suppose that  $xTyTz \subseteq Q$ . Then  $(xTT)(TyT)(TTz) \subseteq xTyTz \subseteq Q$ . Since  $xTT$  is a right PO-ternary ideal of  $T$ ,  $TyT$  is a lateral PO-ternary ideal of  $T$  and  $TTz$  is a left PO-ternary ideal of  $T$ . Therefore, by hypothesis  $xTT \subseteq Q$  or  $TyT \subseteq Q$  or  $TTz \subseteq Q$ . If  $xTT \subseteq Q$ , then  $x^3 \in xTT \subseteq Q$ .

Now  $\langle x \rangle_r \langle x \rangle_m \langle x \rangle_l = (x \cup xTT)(x \cup xTT \cup xTTxTT)(x \cup xTTx) \subseteq (x)^3 \cup (xTT) \subseteq Q$ . By hypothesis  $\langle x \rangle_r \subseteq Q$  or  $\langle x \rangle_m \subseteq Q$  or

$\langle x \rangle_l \subseteq Q$ . Therefore  $x \in Q$ . Similarly, if  $TyT \subseteq Q \Rightarrow y \in Q$  and if  $TTz \subseteq Q \Rightarrow z \in Q$ . Hence  $Q$  is **T-prime** PO-ternary ideal of  $T$ .

**Notation IV.27:** we use the following set defined as

$$\begin{aligned} L(Q) &= \{x \in Q : (TTx) \subseteq Q\}, \\ M(Q) &= \{x \in Q : (TxTUTTxTT) \subseteq Q\} \\ R(Q) &= \{x \in Q : (xTT) \subseteq Q\} \\ I_L &= \{x \in L(Q) : (TTx) \subseteq L(Q)\} \\ I_M &= \{x \in M(Q) : (TxTUTTxTT) \subseteq M(Q)\} \\ I_R &= \{x \in R(Q) : (xTT) \subseteq R(Q)\}. \end{aligned}$$

**Theorem IV.28:** Let  $Q$  be a quasi-PO-ternary ideal of  $T$ . Then  $L(Q)$  is a left PO-ternary ideal of  $T$  contained in  $Q$  if  $Q$  is non-empty.

**Proof:** Let  $x, y \in L(Q)$  and  $s, t \in T$ . Then  $x, y \in L(Q) \Rightarrow stx \in (TTx) \subseteq Q, sty \in (TTy) \subseteq Q$ .

$\Rightarrow stx, sty \in Q \Rightarrow stx + sty = st(x + y) \in (TT(x + y)) \subseteq Q \Rightarrow x + y \in L(Q)$

Now  $TTstx \subseteq TTx \Rightarrow (TTstx) \subseteq (TTx) \subseteq Q$ . Therefore  $stx \in L(Q)$ .

Consequently,  $TTL(Q) \subseteq Q$ . Hence  $L(Q)$  is a left PO-ternary ideal of  $T$ .

**Theorem IV.29:** Let  $Q$  be a quasi-PO-ternary ideal of  $T$ . Then  $M(Q)$  is a lateral PO-ternary ideal of  $T$  contained in  $Q$  if  $Q$  is non-empty.

**Proof:** Let  $x, y \in M(Q)$  and  $s, t \in T$ . Then  $x, y \in M(Q) \Rightarrow stx \in (TxT) \subseteq Q, sty \in (TyT) \subseteq Q$ .

$\Rightarrow stx, sty \in Q \Rightarrow stx + sty = s(x + y)t \in (T(x + y)T) \subseteq Q$

$\Rightarrow x + y \in M(Q)$

Now  $TsxtT \subseteq TxTUTTxTT \Rightarrow (TsxtT) \subseteq (TxTUTTxTT) \subseteq Q$ . Therefore  $sxt \in M(Q)$ .

Consequently,  $TTM(Q) \subseteq Q$ . Hence  $M(Q)$  is a lateral PO-ternary ideal of  $T$ .

**Theorem IV.30:** Let  $Q$  be a quasi-PO-ternary ideal of  $T$ . Then  $R(Q)$  is a right PO-ternary ideal of  $T$  contained in  $Q$  if  $Q$  is non-empty.

**Proof:** Let  $x, y \in R(Q)$  and  $s, t \in T$ . Then  $x, y \in R(Q) \Rightarrow xst \in (xTx) \subseteq Q, yst \in (yTT) \subseteq Q$ .

$\Rightarrow xst, yst \in Q \Rightarrow xst + yst = (x + y)st \in ((x + y)TT) \subseteq Q$

$\Rightarrow x + y \in R(Q)$

Now  $xstTT \subseteq xTT \Rightarrow (xstTT) \subseteq (xTT) \subseteq Q$ .

Therefore  $xst \in R(Q)$ .

Consequently,  $R(Q)TT \subseteq Q$ .

Hence  $R(Q)$  is a right PO-ternary ideal of  $T$ .

**Theorem IV.31:** Let  $Q$  is a **T-prime** quasi-PO-ternary ideal of a PO-ternary semiring  $T$ . Then  $I_Q$  is a prime quasi-PO-ternary ideal of  $T$ .

**Proof:** Let  $Q$  be a **T-prime** quasi-PO-ternary ideal of a PO-ternary semiring  $T$ .

Suppose  $RML \subseteq I_Q$  for any PO-ternary ideals  $R, M$  and  $L$  of  $T$ . Now  $I_Q \subseteq L(Q) \subseteq Q$  implies  $RML \subseteq Q$ . Since  $Q$  is **T-prime**, therefore, by theorem 4.26, we have  $R \subseteq Q$  or  $M \subseteq Q$  or  $L \subseteq Q$ . Also  $I_Q$  is the largest PO-ternary ideal contained in  $Q$ , therefore,  $R \subseteq I_Q$  or  $M \subseteq I_Q$  or  $L \subseteq I_Q$ . Hence  $I_Q$  is a prime PO-ternary ideal of  $T$ .

**Corollary IV.32:** Let  $Q$  is a Semiprime quasi-PO-ternary ideal of a PO-ternary semiring  $T$ . Then  $I_Q$  is a Semiprime PO-ternary ideal of  $T$ .

**Theorem IV.33:** If a PO-ternary semiring  $T$  is a regular, then every quasi-PO-ternary ideal of  $T$  is **T-semiprime**.

**Proof:** Suppose that T is regular and Q be a quasi-PO-ternary ideal of T. Let  $aTa \subseteq Q$  for  $a \in T$ . Since T is regular, therefore, for  $a \in T$ , there exist  $x, y \in T$  such that  $a \leq axaya$ . Thus  $a \leq axaya \Rightarrow a \in (aTa) \subseteq aTa \subseteq Q$ . Therefore  $a \in Q$ . Hence Q is a T-semiprime.

## V. QUASI- $k$ -PO-TERNARY IDEALS

**Definition V.1:** An additive subsemigroup Q of a PO-ternary semiring T is said to be **quasi- $k$ -PO-ternary ideal** of T provided  $\overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ} \subseteq Q$  and  $(Q] \subseteq Q$ .

**Theorem V.2:** Let T be a PO-ternary semiring and  $A, B, C \subseteq T$ . Then  $\overline{\overline{ABC}} = \overline{\overline{\overline{ABC}}}$ .

**Proof:** Since  $A \subseteq \overline{A}$ ,  $B \subseteq \overline{B}$  and  $C \subseteq \overline{C}$ , therefore,  $ABC \subseteq \overline{\overline{ABC}}$ . Hence  $\overline{\overline{ABC}} \subseteq \overline{\overline{\overline{ABC}}}$ . Again, let  $x \in \overline{A}$ ,  $y \in \overline{B}$  and  $z \in \overline{C}$ . Then there exist  $a_1, a_2 \in A$ ,  $b_1, b_2 \in B$  and  $c_1, c_2 \in C$  such that  $x + a_1 = a_2$ ,  $y + b_1 = b_2$  and  $z + c_1 = c_2$ . Now

$$\begin{aligned} xyz + a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ = xyz + (x + a_1)(y + b_1)c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ = xyz + xyc_1 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ = xyc_2 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ = xyc_2 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + (x + a_1)b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ = x(y + b_1)c_2 + xb_1c_1 + a_1(y + b_1)c_1 + a_1b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ = xb_2c_2 + (x + a_1)b_1c_1 + a_1b_2c_1 + a_1b_1c_2 + a_1b_2c_2 + a_1b_1c_1 \\ = (x + a_1)b_2c_2 + a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2 = a_2b_2c_2 + a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2. \end{aligned}$$

As  $a_i, b_i, c_i \in ABC$ , where  $i = 1, 2$ . Therefore we can prove that  $xyz \in \overline{\overline{ABC}}$  for  $x \in \overline{A}$ ,  $y \in \overline{B}$  and  $z \in \overline{C}$ . Suppose that

$t \in \overline{\overline{ABC}}$ . Then  $t = \sum_{i=1}^n a_i b_i c_i$  for some  $a_i \in \overline{A}$ ,  $b_i \in \overline{B}$ ,

$c_i \in \overline{C}$ . Thus  $t = \overline{\overline{ABC}}$ . Therefore  $\overline{\overline{ABC}} \subseteq \overline{\overline{ABC}}$ . Hence  $\overline{\overline{ABC}} \subseteq \overline{\overline{\overline{ABC}}} = \overline{\overline{ABC}}$ . Therefore  $\overline{\overline{ABC}} = \overline{\overline{ABC}}$ .

**Definition IV.3:** A PO-ternary semiring T is said to be  **$k$ -regular** provided for each  $a \in T$  there exist  $x, y \in T$  such that  $a + axa = aya$ .

**Theorem IV.4:** If a PO-ternary semiring T is  $k$ -regular. Then every quasi- $k$ -PO-ternary ideal Q of T is of the form  $Q = \overline{QTTQ} = \overline{TTQ} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{QTT}$ .

**Proof:** Let Q be a quasi- $k$ -PO-ternary ideal of T. Then  $\overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ} \subseteq Q$  and  $(Q] \subseteq Q$ . Let  $a \in Q$  and T is  $k$ -regular, then there exist  $x, y \in T$  such that  $a + axa = aya \Rightarrow axa + axaya = aya$ . Since  $axaya, aya \in QTQ$ . Therefore  $axa \in \overline{QTQ}$ . Similarly,  $aya \in \overline{QTQ}$ . Since  $\overline{QTQ}$  is  $k$ -closed and hence  $a \in \overline{QTQ} = \overline{QTQ}$ . Therefore  $Q \subseteq \overline{QTQ}$ . Again  $QTQ \subseteq Q$ ,  $T \subseteq QTT$  and  $QTQ \subseteq TTQ$  and  $QTQ \subseteq TTQTT$  and hence  $QTQ \subseteq \overline{TTQ}$ ,  $QTQ \subseteq \overline{QTT}$  and  $QTQ \subseteq \overline{TQT} \cup \overline{TTQTT}$  as  $0 \subseteq \overline{TQT}$ .

Thus we have Q

$\subseteq \overline{QTQ} \subseteq \overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ} \subseteq Q$  as Q is quasi- $k$ -PO-ternary ideal of T. Hence  $Q = \overline{QTQ} = \overline{QTT} \cap (\overline{TQT} \cup \overline{TTQTT}) \cap \overline{TTQ}$ .

## VI. CONCLUSION

In this paper mainly we studied about quasi po- $k$ -ternary ideals and full quasi po- $k$ -ternary ideals in PO-ternary semiring.

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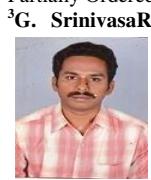


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