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Surface Particle Motion of Rayleigh Waves in Prestressed Heterogeneous Orthotropic Elastic Half Space

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Abstract— **The motion of the surface particles of Rayleigh waves in the prestressed heterogeneous elastic half space is discussed in detail. The analytical expressions for displacement components are derived and showed that these components are effected by initial stress and inhomogeneity factor. It is also proved that motion of the surface particles becomes retrograde elliptical in unstressed heterogeneous half space. For λ=0, the ratio of the major and minor axes are derived and showed that the ratio becomes unity at the cut-off frequency.**

Index Terms— **Rayleigh waves, prestressed heterogeneous medium, orthotropic medium, retrograde elliptic, retrograde circular..**

I. Introduction

The surface wave method is most widely used to find the mechanical properties of the medium, because it is non-destructive, has less inspection time, low cost and has wide range of applications in various fields like seismology, geophysics and materials science.

The study of propagation of Rayleigh waves in heterogeneous half space have been studied among others Stonely(1934), Wilson(1942), Newlands(1950), Hook(1961), Dutta(1963), Karlson(1963), Singh(1965) and Sidhu(1970). A good amount of literature can be found in the standard books like pilant (1979) and A. Ben-Menahem and S.J. Singh (2000) etc.

In most of above studies, the effect of inhomogenity factor in different forms are discussed for the study of propagation of Rayleigh waves. Though, earth is prestressed heterogeneous medium due to many factors like gravity field and variation in temperature etc.

Biot (1965) was the first that has shown anisotropy was developed due to initial stress present in the medium. Using basic equation of Biot, some researchers like Sidhu and Singh (1983) hasshown the effect of prestress on the propagation of P, SV,SH seismic waves.

Due to complexity of the problem, the study of transmission of Rayleigh waves in the prestressed heterogeneous half space was not enough. Das et al. (1992), abd-alla et al. (2009) and Kakar and Kakar (2013) have tried to drive the dispersive equationsof Rayleigh waves in prestressed heterogeneous media. Most of them used potential method to solve these problems. Norries (1983) also has pointed out that potential method is not suitable for prestressed media

In the present paper authors want to discuss the motion of surface particles in the prestressed heterogeneous elastic half space using matrix method. The expressions for the components of displacement are derived. For that purpose the basic equations are taken from their previous paper (Gupta and Kumar, proceeding ISTAM, 2014). Here it is assumed that material properties and initial stress components are varying as λ= $\lambda^0 e^{az}$,μ= $\mu^0 e^{az}$,ρ= $\rho^0 e^{az}$, $S_{11} = S_{11}^0 e^{az}$, $S_{22} = S_{22}^0 e^{az}$, $S_{33} = S_{33}^0 e^{az}$, $p = p^0 e^{az}$ (a>0).

II. BASIC EQUATION

Consider a semi-infinite, perfectly elastic prestressed, heterogeneous medium .The materials is either isotropic in finite strain or anisotropic with orthotropic symmetry .The principle directions of prestressed are chosen to coincide with the direction of elastic symmetry and the co-ordinate axes . The general equations of motion for prestressed solid in the absence of external forces are given by Gupta and Kumar (proceedingISTAM, 2014).

$$B_{11}^0 \frac{\partial^2 u}{\partial x^2} + \left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 w}{\partial x \partial z} + \left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 v}{\partial x \partial y} + \left(Q_2^0 + \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial y^2} + \left(Q_2^0 + \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial z^2} + a\left(Q_2^0 + \frac{P^0}{2}\right)\frac{\partial u}{\partial z} + a\left(Q_2^0 - \frac{P^0}{2}\right)\frac{\partial w}{\partial x} = \rho^0 \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$\left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial x \partial y} + \left(Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 v}{\partial x^2} + B_{22}^0 \frac{\partial^2 v}{\partial y^2} + (B_{23}^0 + Q_1^0)\frac{\partial^2 w}{\partial y \partial z} + Q_1^0 \frac{\partial^2 v}{\partial z^2} + aQ_1^0 \frac{\partial w}{\partial y} + aQ_1^0 \frac{\partial v}{\partial z} = \rho^0 \frac{\partial^2 v}{\partial t^2}, \tag{2}$$

$$\left(Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 w}{\partial x^2} + \left(B_{12}^0 + Q_2^0 - \frac{P^0}{2}\right)\frac{\partial^2 u}{\partial x \partial z} + Q_1^0 \frac{\partial^2 w}{\partial y^2} + (B_{23}^0 + Q_1^0)\frac{\partial^2 v}{\partial y \partial z} + B_{22}^0 \frac{\partial^2 w}{\partial z^2} + a(B_{12}^0 - P^0)\frac{\partial u}{\partial x} + aB_{23}^0 \frac{\partial v}{\partial y} + aB_{22}^0 \frac{\partial w}{\partial z} = \rho^0 \frac{\partial^2 w}{\partial t^2}, \tag{3}$$

where $B_{11}^0, B_{22}^0, B_{12}^0, B_{23}^0, Q_1^0, Q_2^0$ and S_{11}^0, S_{33}^0 are elastic coefficients and initial stresses in homogeneous orthotropic prestressed medium .(λ^0, μ^0) ,ρ^0 are Lami's constant and density of material at the free surface.

Suppose the solutions of Eqs. (1-3) in the form of displacement components are as follow:

$$u = u_1(z) e^{-i(x\xi + y\eta - pt)},$$

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$$v = v_1(z)e^{-i(x\xi + y\eta - pt)}, \qquad (4)$$
$$w = w_1(z)e^{-i(x\xi + y\eta - pt)},$$

where $i = \sqrt{-1}$. Then eq. (4) define a plane harmonic wave propagating in the direction of the normal to the plane $x\xi + y\eta = \text{constant}$, (5)

as shown in Fig. 1

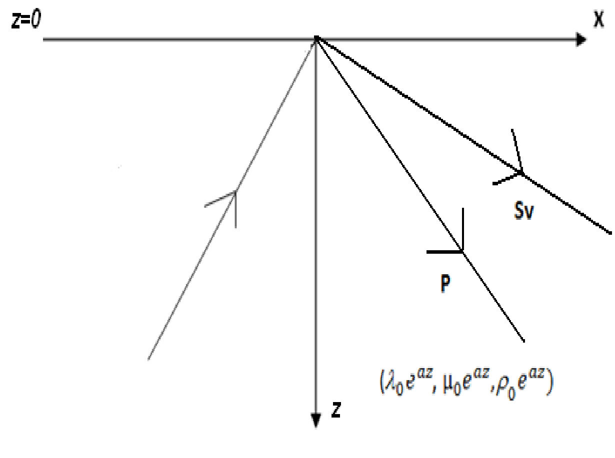


Figure 1

withperiod $\frac{2\pi}{p}$, wave length $\frac{2\pi}{\sigma}$ and phase velocity $c = \frac{p}{\sigma}$,

where

$$\sigma^2 = \xi^2 + \eta^2. \qquad (6)$$

On simplification of Eqs. (1-6) , we get

$$Cu_1 + A_3\xi\eta v_1 + iaA_2\xi\eta w_1 + i\xi A_3\frac{\partial w_1}{\partial z} - aA_2\frac{\partial u_1}{\partial z} - A_1\frac{\partial^2 u_1}{\partial z^2} = 0,$$

(7)

$$A_3\xi\eta u_1 + Dv_1 + iaQ_1^0\eta w_1 + i\eta A_4\frac{\partial w_1}{\partial z} - aQ_1^0\frac{\partial v_1}{\partial z} - Q^0\frac{\partial^2 v_1}{\partial z^2} = 0,$$

$$ia(B_{12}^0 - P^0)\xi u_1 + iaB_{23}^0\eta v + Ew_1 + i\xi A_3\frac{\partial u_1}{\partial z} + i\eta A_4\frac{\partial v_1}{\partial z} - al\frac{\partial w_1}{\partial z} - l\frac{\partial^2 w_1}{\partial z^2} = 0,$$

where

$$C = B_{11}^0\xi^2 + A_1\eta^2 - \rho^0 p^2,$$
$$D = B_{22}^0\eta^2 + A_2\xi^2 - \rho^0 p^2,$$
$$E = Q_1^0\eta^2 + A_2\xi^2 - \rho^0 p^2, \qquad (8)$$
$$A_1 = Q_2^0 + \frac{P^0}{2},$$
$$A_2 = Q_2^0 - \frac{P^0}{2},$$
$$A_3 = B_{12}^0 + Q_2^0 - \frac{P^0}{2}, \qquad A_4 = B_{23}^0 + Q_1^0,$$
$$l = B_{22}^0.$$

The set of Eq.(7) can be written in the following matrix form

$$\frac{dT}{dz} = AT, \qquad (9)$$

where T is column matrix with elements $T_1, T_2, T_3, T_4, T_5, T_6$ and A is 6x6 matrix(given by Gupta and Kumar) . Let the solution of Eq. (9)is of the form $T = e^{sz}C$

which is satisfied for all z if $[A - sI]\vec{C} = 0$, (10)

where I denoting six order unit matrix.

The set Eq.(10) has a non –vanishing solution vector $\vec{C}$ if and only if

$$|A - sI| = 0 \qquad (11)$$

The determinant of Eq.(11)is a sixth order equation in s and gives the following six distinct values,

$$s_1 = \frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) - 2\sqrt{(p_1 - p_2)^2 - 4p_3}\right)^{1/2} - a\right],$$
$$s_2 = \frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) + 2\sqrt{(p_1 - p_2)^2 - 4p_3}\right)^{1/2} - a\right],$$
$$s_3 = -\frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) - 2\sqrt{(p_1 - p_2)^2 - 4p_3}\right)^{1/2} + a\right],$$

(12)

$$s_4 = -\frac{1}{2}\left[\left(a^2 + 2(p_1 + p_2) - 2\sqrt{(p_1 + p_2)^2 - 4p_3}\right)^{1/2} + a\right],$$
$$s_5 = \frac{1}{2}\left[\sqrt{a^2 + 4p_2} - a\right]$$
$$s_6 = \frac{1}{2}\left[\sqrt{a^2 + 4p_2} + a\right]$$

.

Where

$$p_1 = \sigma^2 - \frac{p^2}{c_p^2}, p_2 = \sigma^2 - \frac{p^2}{c_s^2}, p_3 = a^2\sigma^2\left(1 - 2\frac{c_s^2}{c_p^2}\right), \; C_p^2 = \frac{B_{22}^0}{\rho_0}, C_s^2 = \frac{Q_2^0 + \frac{P^0}{2}}{\rho_0} \text{ and } \gamma^2 = \frac{Q_2^0 + \frac{P^0}{2}}{B_{22}^0}$$

. (13)

The general solution of Eq.(9) is ,therefore

$$\vec{T} = \sum_{i=1}^{6} e^{s_i z}\vec{C_i} \qquad (14)$$

Writing the column vector $\vec{T}$ and $\vec{C_i}$ in the symbolic form as $\vec{T} = [T_j]$, $\vec{C_i} = [C_{ij}], i, j = (1,2,3, \ldots ,6)$ in Eq.(14) and equating the corresponding elements ,we get

$$u_1 = T_1 = \sum_{i=1}^{6} e^{s_i z}C_{i1}$$
$$, v_1 = T_2 = \sum_{i=1}^{6} e^{s_i z}C_{i2}, w_1 = T_3 = \sum_{i=1}^{6} e^{s_i z}C_{i3}.$$

(15)

We replace s by s_i and $\vec{C}$ by $\vec{C_i}$ in Eq. (10) ,we get

$$[A - s_i I][C_j] = 0. \qquad (16)$$

Here we assume, that each of the sets of six C_{ij} for any given i must be consistent. It means all six solution are L.D. ,we can express one in terms of L.C. of other five. For that, fixed i , and express them in such a way that none of the values of i=1,2,3…..6, the coefficient determinate becomes zero. It will be possible, if we write C_{ij}(j=2,3,….,6) as C_{i1}=K_i(say). For each i, Eq. (16) gives a set of six homogeneous equations in six unknowns C_{ij}(j=1,2,3,….,6)as

$$C_{i2} = \frac{\eta}{\xi}\left[\frac{(C - aA_2 s_i - s_i^2 A_1)(aQ_1^0 + A_4 s_i) - A_3\xi^2(aA_2 + s_i A_3)}{(D - aQ_1^0 s_i - s_i^2 Q_1^0)(aA_2 + A_3 s_i) - A_3\eta^2(aQ_1^0 + A_4 s_i)}\right]K_i, \qquad (17)$$

$$or \;\; C_{i2} = \frac{\eta}{\xi}TK_i, \qquad (18)$$

where

$$T = \left[\frac{(C - aA_2 s_i - s_i^2 A_1)(aQ_1^0 + A_4 s_i) - A_3\xi^2(aA_2 + s_i A_3)}{(D - aQ_1^0 s_i - s_i^2 Q_1^0)(aA_2 + A_3 s_i) - A_3\eta^2(aQ_1^0 + A_4 s_i)}\right] \qquad (19)$$

and $$C_{i3} = \frac{i}{\xi}\left[\frac{\xi^2\{a(B_{12}^0 - P^0) - A_3 s_i\} + T\eta^2\{aB_{23}^0 - A_4 s_i\}}{(E - als_i - ls_i^2)}\right]K \qquad (20)$$

When the elastic half space is unstressed heterogeneous infinite isotropic solidthen put P=R=0 in Eq. (8), we get

$$l = B_{11}^0 = B_{22}^0 = \lambda_0 + 2\mu_0, B_{12}^0 = B_{23}^0 = \lambda_0, Q_1^0 = Q_2^0 = A_1 = A_2 = \mu_0$$
$$\alpha^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}, \beta^2 = \frac{\mu_0}{\rho_0}, \; \gamma^2 = \left(\frac{\mu_0}{\lambda_0 + 2\mu_0}\right), \qquad (21)$$

$\mathrm{C} = (\lambda_0 + 2\mu_0)\xi^2 + \mu_0\eta^2 - \rho^0 p^2$,
$\mathrm{D} = (\lambda_0 + 2\mu_0)\eta^2 + \mu_0\xi^2 - \rho^0 p^2$,
and
$\mathrm{E} = \mu_0\sigma^2 - \rho p^2$.

After some manipulation, we obtain

$$C_{i2} = \begin{Bmatrix} -K_i\frac{\xi}{\eta}\ ,\ i = 5,6 \\ K_i\frac{\eta}{\xi}\ ,\ i = 1,2,3,4 \end{Bmatrix}, \qquad (22)$$

$$C_{i3} = \begin{Bmatrix} 0\ ,\ i = 5,6 \\ \frac{i m_i K_i}{\xi_i}\ ,\ i = 1,2,3,4 \end{Bmatrix}, \qquad (23)$$

where

$$m_i = -\frac{(a\lambda_0 + A_3 s_i)\sigma^2}{(E - l s_i^2 - a l s_i)}\ , i = 1,2,3,4 \qquad (24)$$

and

$$A_3 = (\lambda_0 + \mu_0). \qquad (25)$$

Substituting for these in Eq.(15), we obtain

$$u_1 = \sum_{i=1}^{6} e^{s_i z} K_i,$$
$$v_1 = \frac{\eta}{\xi}\sum_{i=1}^{6} e^{s_i z} K_i - \frac{\xi}{\eta}\sum_{i=5}^{6} e^{s_i z} K_i, \qquad (26)$$
$$w_1 = \frac{i}{\xi}\sum_{i=1}^{4} e^{s_i z} m_i K_i.$$

Appropriate solution to be used from Eq.(26) which satisfy the radiation as $z \to \infty$ are for $K_1 = K_2 = K_5 = 0$, where we must ensure that $Re(s_1, s_2, s_3) > 0$. Therefore putting $K_1 = K_2 = K_5 = 0$ in Eq.(26), we get

$$u_1 = \sum_{i=3}^{4} K_i e^{s_i z} + K_6\, e^{s_6 z},$$
$$v_1 = \frac{\eta}{\xi}\sum_{i=3}^{4} e^{s_i z} K_i - \frac{\xi}{\eta}\, e^{s_6 z} K_6\,, \qquad (27)$$
$$w_1 = \frac{i}{\xi}\sum_{i=3}^{4} e^{s_i z}.\, m_i K_i.$$

The boundary conditions are(Gupta and Kumar, proceeding ISTAM,2014)

$$\Delta f_x = (Q_2^0 + \frac{P^0}{2})u_z + (Q_2^0 - \frac{R^0}{2})\omega_x = 0,$$
$$\Delta f_y = (Q_1^0 - s_{33}^0)\omega_y + Q_1^0 v_z = 0, \text{at } z = 0 \qquad (28)$$
$$\Delta f_z = (B_{12}^0 + s_{11}^0)u_x + (B_{23}^0 + s_{33}^0)v_y + B_{22}^0\omega_z = 0,$$

where $R = R^0$ ($S_{11}^0 + S_{33}^0$) and $\omega_x = -\omega_{23}\ , \omega_y = -\omega_{31}\, , \omega_z = -\omega_{12}$ defined by $\omega_{ij} = (\frac{u_{i,j} - u_{j,i}}{2}), u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$.

Using Eq. (27) in Eq. (28), we get

$[(Q_2^0 + P^0/2)s_3 + (Q_2^0 - R^0/2)m_3]K_3 + [(Q_2^0 + P^0/2)s_4 + (Q_2^0 - R^0/2) m_4]K_4 + (Q_2^0 + P^0/2)s_6 K_6 = 0,$

$\frac{\eta}{\xi}[(Q_1^0 - S_{33}^0)m_3 + Q_1^0 s_3]K_3 + [(Q_1^0 - S_{33}^0)m_4 + Q_1^0 s_4]K_4 + \frac{\xi}{\eta}(Q_2^0 - P^0/2)s_6 K_6 = 0,$

(29)

$[(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_3 s_3]K_3 + [(B_{12}^0 + S_{11}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_4 s_4]K_4 = 0$

If we put $Q_2^0 = Q_1^0\ , S_{11}^0 = S_{33}^0$ and $K_6 = 0$ then first two Eqs.Of Eqs. (29) becomes identical and from Eq. (29) , we get

$[Q_2^0 s_3 + (Q_2^0 - S_{33}^0)m_3]K_3 + [Q_2^0 s_4 + (Q_2^0 - S_{33}^0)m_4]K_4 = 0,$
$[(B_{12}^0 + S_{33}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_3 s_3]K_3 + [(B_{12}^0 + S_{33}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_4 s_4]K_4 = 0 \qquad (30)$

Elimination of K_3 and K_4 from Eq. (30) gives

$$\begin{vmatrix} Q_2^0 s_3 + (Q_2^0 - S_{33}^0)m_3 & Q_2^0 s_4 + (Q_2^0 - S_{33}^0)m_4 \\ (B_{12}^0 + S_{33}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_3 s_3 & (B_{12}^0 + S_{33}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2 - B_{22}^0 m_4 s_4 \end{vmatrix} = 0, \qquad (31)$$

which is the frequency equation of Rayleigh waves propagating over the free surface of heterogeneous prestressed medium.

On Simplification, the determinate value of Eq. (31) becomes

$$\frac{(s_3 - s_4)\sigma^2}{D_3 D_4}\left[\frac{L D_3 D_4}{\sigma^2} - \mathrm{N}\{\lambda_0\, l(s_3 + s_4) + a^2\lambda_0\, l + m\mathrm{E} + m l s_3 s_4\}\right] = 0, \qquad (32)$$

Where

$\mathrm{L} = \mathrm{A}Q_2^0 + B_{12}^0(Q_2^0 - S_{33}^0)m_3 m_4,$
$\mathrm{N} = \mathrm{A}(Q_2^0 - S_{33}^0) + B_{22}^0 Q_2^0 s_3 s_4. \qquad (33)$
$\mathrm{A} = (B_{12}^0 + S_{33}^0)\xi^2 + (B_{23}^0 + S_{33}^0)\eta^2,$
$D_i = \mathrm{E} - l s_i^2 - a l s_i$, (i =3,4)

D_3 and D_4 can be expressed as follow

$$D_3 = -\frac{1}{2}\left[m\left(2\sigma^2 - \frac{p^2}{\beta^2}\right) - l\left\{(p_1 - p_2)^2 - 4p_3\right\}^{\frac{1}{2}}\right] \qquad (34)$$

$$D_4 = -\frac{1}{2}\left[m\left(2\sigma^2 - \frac{p^2}{\beta^2}\right) - l\left\{(p_1 + p_2)^2 - 4p_3\right\}^{\frac{1}{2}}\right]$$

Since σ and $s_6 \neq 0$, we must have $K_6 = 0$, the Eq. (27) reduces to

$$u_1 = K_3 e^{s_3 z} + K_4 e^{s_4 z}$$
$$v_1 = \frac{\eta}{\xi}[K_3 e^{s_3 z} + K_4 e^{s_4 z}] \qquad (35)$$
$$w_1 = \frac{i}{\xi}[m_3 K_3 e^{s_3 z} + m_4 K_4 e^{s_4 z}]$$

Putting the value of u_1, v_1 and w_1 in Eq.(4) and taking real part , we get

$$u = \left[K_3 e^{s_3 z} + K_4 e^{s_4 z}\right]\cos(x\xi + y\eta - pt),$$
$$v = \frac{\eta}{\xi}\left[K_3 e^{s_3 z} + K_4 e^{s_4 z}\right]\cos(x\xi + y\eta - pt),$$
$$w = \frac{1}{\xi}\left[m_3 K_3 e^{s_3 z} + m_4 K_4 e^{s_4 z}\right]\sin(x\xi + y\eta - pt), \qquad (36)$$

which give the displacements everywhere. For surface displacements we put z=0. FromEq.(30)

$$K_3 = \frac{Q_2^0 s_4 + \left(Q_2^0 - S_{33}{}^0\right)m_4}{Q_2^0 s_3 + \left(Q_2^0 - S_{33}{}^0\right)m_3} K_4. \qquad (37)$$

Using Eq. (37) in Eq. (36), we obtain

$$u = A^*\cos(x\xi + y\eta - pt),$$
$$v = \frac{\eta A^*}{\xi}\cos(x\xi + y\eta - pt)\ , \qquad (38)$$
$$w = \frac{B^*}{\xi}\sin(x\xi + y\eta - pt), \qquad (39)$$

where

$$A^* = K_3 + K_4 = D^*\left[\sigma^2(Q_2^0 - S_{33}{}^0)\{a\lambda_0 l(s_3 + s_4 + a) + m(E + l s_3 s_4)\} - D_3 D_4 Q_2^0\right], \qquad (40)$$

and

$$\mathrm{A}^* = D^*\left[\sigma^2\mu_0\left(1 - \frac{S_{33}{}^0}{\mu_0}\right)\{a\lambda_0 l(s_3 + s_4 + a) + m(E + l s_3 s_4\} - D_3 D_4 \mu_0\right], \text{(for } S_{11} = S_{33} \text{ case)}$$

a
nd

$$B^* = m_3 K_3 + m_4 K_4 = D^*\sigma^2\left(Q_2^0 + \frac{P^0}{2}\right)\left[a\lambda_0\left\{E - l\left(s_4^2 + s_3^2 + s_3 s_4\right) - al(s_3 + s_4)\right\} - m l s_3 s_4\left(s_3 + s_4 + a\right)\right], \qquad (41)$$

and

$$\mathrm{B}^* = \mu_0 \mathrm{D}^*\sigma^2\left[a\lambda_0\left\{E - l(s_4{}^2 + s_3{}^2 + s_3 s_4 - al(s_3 + s_4) - m l s_3 s_4(s_3 + s_4 + a)\right\}\right], \text{(for } S_{11} = S_{33} \text{ case)} \qquad (42)$$

where

$$D^* = \frac{K_4\left(s_4 - s_3\right)}{D_3 D_4\left[Q_2^0 s_3 + \left(Q_2^0 - S_{33}{}^0\right)m_3\right]}\ . \qquad \text{(for } S_{11} = S_{33} \text{ case)} \qquad (43)$$

When the medium is heterogeneous unstressed ($i.e. P^0=0$, $R^0=0$), then Eqs. (40),(42) and (43) reduce to

$$A^* = D^*\left[\sigma^2\left\{a\lambda_0 l(s_3+s_4+a)+m(E+ls_3s_4)\right\}-D_3D_4\right], \quad (44)$$

$$B^* = D^*\sigma^2\left[a\lambda_0\left\{E-l\left(s_4^2+s_3^2+s_3s_4\right)-al(s_3+s_4)\right\}-mls_3s_4\left(s_3+s_4+a\right)\right], \quad (45)$$

and

$$D^* = \frac{K_4\left(s_4-s_3\right)}{D_3D_4\left(s_3+m_3\right)} \quad . \qquad (46)$$

From Eqs.(21), (44) to (46), we have

$$A^* = D^*\left[\sigma^2\left\{a\lambda_0(\lambda_0+2\mu_0)(s_3+s_4+a)+(\lambda_0+\mu_0)(\mu_0\sigma^2-\rho p^2+(\lambda_0+\mu_0)s_3s_4)\right\}-D_3D_4\right] \quad (47)$$

$$B^* = D^*\sigma^2\left[a\lambda_0\left\{(\mu_0\sigma^2-\rho p^2)-(\lambda_0+\mu_0)(s_4^2+s_3^2+s_3s_4)-a(\lambda_0+2\mu_0)(s_3+s_4)\right\}-(\lambda_0+\mu_0)(\lambda_0+2\mu_0)s_3s_4(s_3+s_4+a)\right], \quad (48)$$

and

$$D^* = \frac{K_4(s_4-s_3)}{D_3D_4\left[s_3-\dfrac{a\lambda_0+(\lambda_0+\mu_0)s_3}{(\mu_0\sigma^2-\rho p^2)-(\lambda_0+2\mu_0)s_3^2-a(\lambda_0+\mu_0)s_3}\right]}. \quad (49)$$

For the case $\lambda_0=0$,we will make use of following notations

$$\frac{p^2}{\sigma^2}=C_R^2=\frac{C_s^2}{\psi}, \quad \frac{p^2}{C_s^2}=\zeta \text{ so that } \sigma^2=\zeta\psi \text{ and } \psi>0,\zeta>0. \quad (50)$$

From Eqs.(12) and (13), we obtain

$$C_p^2=2C_s^2, p_3=0,$$

$$s_3=-\frac{1}{2}\left[a^2+4\zeta(\psi-1)+a\right] \text{ and}$$

$$s_4=-\frac{1}{2}\left[a^2+4\zeta(\psi-\frac{1}{2})+a\right]. \qquad (51)$$

From Eq. (32), we get

$$2s_3s_4=\zeta \qquad (52)$$

After some simplification and removal of irrelevant factors s_1+a=0, we get

$$s_3=-\frac{a(2\psi-1)}{\left[4(\psi-1)(\psi-\frac{1}{2})-1\right]}, \qquad (53)$$

$$s_4=-\frac{a\psi}{\left[4(\psi-1)(\psi-\frac{1}{2})-1\right]}, \qquad (54)$$

for s_3 and s_4 are real and positive iff

$$1<\psi<\frac{3+\sqrt{5}}{4}. \qquad (55)$$

From Eq. (34) in this case

$$D_3=-\mu_0\zeta(\psi-1)$$

$$D_4=-\mu_0\zeta\psi \qquad (56)$$

Using Eqs. (51) to (56) in Eqs. (44) and (45), we get

$$A^*=D_0^*\mu_0^2\zeta^2\psi, \qquad (57)$$

$$B^*=-\sigma^2D_0^*\mu_0^2\zeta^2\frac{\left(4\psi^2-6\psi+1\right)}{2a}, \qquad (58)$$

where D_0^* is the value of D^* for λ=0.

Elimination of t from first two of Eq. (38) gives

$$\eta u-\xi v=0 \qquad (59)$$

and from first and third gives

$$\frac{u^2}{A^{*2}}+\frac{w^2}{B^{*2}/\xi^2}=1 \qquad (60)$$

It is clear that the particles lie on the curve obtain from two Eqs. (59) and (60).

If we rotate the (u,v) axes to (u',v') through an angle of θ where $\cos\theta=\frac{\eta}{\sigma}$, $\sin\theta=\frac{\xi}{\sigma}$ then the relation between two co-ordinate axes (u,v) and (u',v') as follows

$$u=u'\frac{\eta}{\sigma}+v'\frac{\xi}{\sigma},$$

$$v=-u'\frac{\xi}{\sigma}+v'\frac{\eta}{\sigma}. \qquad (61)$$

Where v' is the direction of propagation of the Rayleigh waves and u' is perpendicular to it. Then the curve represented by Eqs.(59) and (60) is equivalent to

$$u'=0, \quad \frac{v'^2}{\sigma^2A^{*2}/\xi^2}+\frac{w^2}{B^{*2}/\xi^2}=1 \qquad (62)$$

Using Eq.(61) and first two equations of Eq.(38) , we get

$$v'=\frac{\sigma A^*}{\xi}\cos(x\xi+y\eta-pt)=A\cos(x\xi+y\eta-pt), \quad (63)$$

where $A=\frac{\sigma A^*}{\xi}$

Third equation of Eq.(38) can be written as

$$w=\frac{B^*}{\xi}\sin(x\xi+y\eta-pt)=B\sin(x\xi+y\eta-pt), \quad (64)$$

where $B=\frac{B^*}{\xi}$

Eq. (62) shows that, the particle paths are ellipses in vertical planes to the direction of Rayleigh wave propagation. (51)

Eqs. (63) and (64) gives

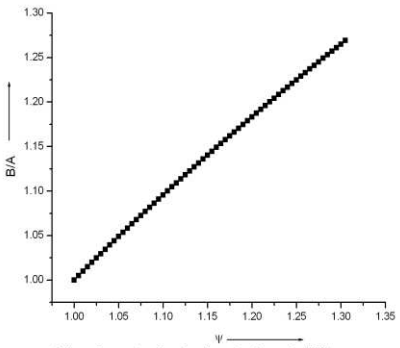
$$\frac{v'^2}{A^2}+\frac{w^2}{B^2}=1 \qquad (65) \qquad (52)$$

Hence Eq. (65), shows that the motion is elliptical. The ratio of the major and minor axes is given by Eqs.(57) and (58) is

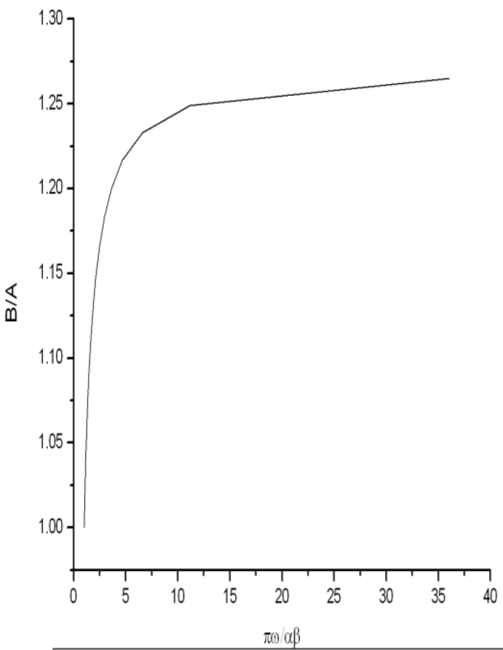
$$\frac{B}{A}=\frac{B^*}{\sigma A^*}=\sqrt{2\psi-1}, \text{For } \lambda=0 \qquad (66)$$

(53)

Where the range of ψ is taken from the Eq. (55). Graphically we can see that the ratio of major and minor is increase with ψ as shown in figure (2). Again we have drawn a graph between $\frac{B}{A}$ and $\frac{\pi\omega}{\alpha\beta}$ taking $\frac{B}{A}$ along y-axis and $\frac{\pi\omega}{\alpha\beta}$ along x-axis for λ=0. (54)



variation of ψ w.r.t. ratio of major and minor axis(B/A)



variation of frequency w.r.t. ratio of major and minor axis (B/A).

III. Stationary Rayleigh waves:

For the surface displacements due to stationary Rayleigh waves we can take Eqs.(63) and (64) and change the sign of p and add, we get

$$v' = 2A\cos(x\xi + y\eta)\cos pt\,, \qquad (67)$$

$$w = 2B\sin(x\xi + y\eta)\cos pt\,. \qquad (68)$$

The particle motion is seen to be rectilinear simple harmonic and of the same frequency as that of the wave. if we put

$$x\xi + y\eta = \sigma\left(x\frac{\xi}{\sigma} + y\frac{\eta}{\sigma}\right) = \sigma p_1, \qquad (69)$$

where p_1 is the distance from the origin in the direction of the wave propagation. Eqs.(67) and (68) can be written as

$$v' = 2A\cos(\sigma p_1)\cos pt\,, \qquad (70)$$

$$w = 2B\sin(\sigma p_1)\cos pt\,. \qquad (71)$$

From Eq. (6), the wavelength is $K=\frac{2\Pi}{\sigma}$, then Eqs (67) and (68) give

$$v' = 0 \quad , \text{ when} \quad p_1 = \frac{K}{4}, \frac{3K}{4}, \frac{5K}{4}, \ldots\ldots\ldots$$

$$w = 0 \quad , \text{ when} \quad p_1 = 0, \frac{K}{2}, \frac{2K}{2}, \frac{3K}{2}, \ldots\ldots\ldots$$

Hence, at interval of one fourth of wavelength , the disturbance is alternately vertical and horizontal.

IV. Conclusion

it is clear that the particles lie on the curve obtained by intersection of two surfaces represented by two Eqs. (59) and (60). From Eq. (65) we conclude that the particle paths are ellipses in vertical planes parallel to the direction of Rayleigh wave motion. The motion is retrograde elliptical if the signs of A^* and B^* are same for λ =0. For different value of ψ lying in the range $1 < \psi < \frac{3+\sqrt{5}}{4}$,we can draw the graph of $\frac{B}{A}$ in unstressed state of medium. When λ =0, then ψ =1 and $\frac{B}{A}$ becomes unity at the cut-off frequency. Hence the particle motion is retrograde circular. Other than this value of the frequency , the particle paths are retrograde elliptical. It is clear from the figure (3) that the increasing value of frequency the ratio of $\frac{B}{A}$ elongated in the vertical direction. At $\left(\psi = \frac{3+\sqrt{5}}{4}\right)$ upper most value, of the frequency, phase velocities approach Rayleigh wave velocities of the homogeneous case equal to $\sqrt{\frac{\sqrt{5}+1}{2}}$.

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