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The Tightly Super 2-good-neighbor connectivity and 2-good-neighbor Diagnosability of Crossed Cubes

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***Abstract*— The reliability of an interconnection network is an important issue for multiprocessor systems. We know that connectivity and the diagnosability are two important parameters for measuring the reliability of an interconnection network. In 2012, Peng et al. proposed the g-good-neighbor diagnosability, which has been widely accepted as a new measure of the diagnosability by restricting that every fault-free vertex contains at least g fault-free neighbors. As an important variant of the hypercube, the n-dimensional crossed cube CQ_n has many good properties. In this paper, we show that (1) the 2-good-neighbor connectivity of CQ_n is $4n-8$ for $n \ge 4$, (2) CQ_n is tightly $(4n-8)$ super 2-good-neighbor connected for $n \ge 6$ and (3) the 2-good-neighbor diagnosability of CQ_n is $4n-5$ under the PMC model and MM* model for $n \ge 5$.**

***Index Terms*—Interconnection network, Crossed cube, Connectivity, Diagnosability**

I. INTRODUCTION

Mass data processing and complex problem solving have higher and higher demands for performance of multiprocessor systems. Many multiprocessor systems have interconnection networks (networks for short) as underlying topologies and a network is usually represented by a graph where nodes represent processors and links represent communication links between processors. The network determines the performance of a multiprocessor system. So study of topological properties of its network is important. However, a system of nodes may be faulty when the system is in operation. The fault diagnosis is used to identify faulty processors in a system. All the faulty nodes are replaced by fault-free nodes after a system has been diagnosed. The diagnosability of a system is the maximum number of faulty nodes that can be found during the fault diagnosis. For a diagnosable system, Dahbura and Masson [2] proposed an algorithm with time complex $O(n^{2.5})$, which can effectively identify the set of faulty processors.

To diagnose a system, several different models have been proposed. Two important diagnosis models are the Preparata, Metze, and Chien's (PMC) model [9] and the Malek and Maeng's (MM) model [7]. In the PMC model, only neighboring processors are allowed to test each other. In the MM model, a node tests its two neighbors, and then compares their responses. Sengupta and Dahbura [11] suggested a special case of the MM model, namely the MM* model and each node must test its any pair of adjacent nodes in the MM*. They also presented a polynomial algorithm for identifying faulty nodes in a system under the MM* model if the system is diagnosable.

A new measure of a system called the g-good-neighbor diagnosability was introduced by Peng et al. [8] in 2012, which restricts that every fault-free node contains at least g fault-free neighbors. In [8] they proved that the g-good-neighbor diagnosability of the n-dimensional hypercube under the PMC model is $2^g(n-g)+2^g-1$ for $0 \le g \le n-3$. In 2016, Wang and Han [12] showed the g-good-neighbor diagnosability of the n-dimensional hypercube under the MM* model. In [5], Liu et al. determined that the g-good-neighbor diagnosability of the exchanged hypercube under the PMC model is $2^g(s+2-g)-1$ for $1 \le s \le t$ and $0 \le g \le s$. In 2016, Xu et al. [18] showed the g-good-neighbor diagnosability of complete cubic networks under the PMC model and MM* model. In 2016, Ren and Wang [10] gave some properties of the g-good-neighbor diagnosability of a multiprocessor system. Yuan et al. [19,20] studied that the g-good-neighbor diagnosability of the k-ary n-cube $(k \ge 3)$ under the PMC model and MM* model. In [13,14], Wang et al. proved that the g-good-neighbor diagnosability of the Cayley graph generated by the transposition tree under the PMC model and MM* model for $g \in \{1,2\}$. In 2017, Wang et al. [15] determined that the 2-good-neighbor connectivity and 2-good-neighbor diagnosability of the bubble-sort star graph.

The n-dimensional hypercube is a major type of interconnection networks. As an important variant of the hypercube, the n-dimensional crossed cube [3] (denoted by CQ_n) has better properties such as smaller degree, diameter and average distance. In this paper, we proved that (1) the 2-good-neighbor connectivity of CQ_n is $4n-8$ for $n \ge 4$; (2) CQ_n is tightly $(4n-8)$ super 2-good-neighbor connected for $n \ge 6$; (3) the 2-good-neighbor diagnosability of CQ_n is $4n-5$ under the PMC model for $n \ge 5$; (4) the 2-good-neighbor diagnosability of CQ_n is $4n-5$ under the MM* model for $n \ge 5$.

II. PRELIMINARIES

A. Notations

A multiprocessor system is modeled as an undirected simple graph $G=(V,E)$, whose vertices (nodes) represent processors and edges (links) represent communication links. The degree $d_G(v)$ of a vertex v is the number of edges incident with v. The minimum degree of a vertex in G is denoted by $\delta(G)$. For a vertex v, $N_G(v)$ is the set of vertices adjacent to v in G. Given a nonempty vertex subset V' of V, the induced subgraph by V' in G, denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of G with both endpoints in V'. For $S \subseteq V(G)$, let $N_G(S) = \cup_{v\in S} N_G(v) \setminus S$. A cycle with length n is called an n-cycle. We use $P = v_1v_2\cdots v_n$ to denote a path that begins with v_1 and ends with v_n. A path of the length n is denoted by n-path. A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X,Y) is called a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition (X,Y) in which each vertex of X is joined to each vertex of Y. If $|X|=m$ and $|Y|=n$, such a graph is denoted by $K_{m,n}$. The connectivity $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left when G is complete. Let F_1 and F_2 be two distinct subsets of V, and let the symmetric difference $F_1 \Delta F_2 = (F_1 \setminus F_2) \bigcup (F_2 \setminus F_1)$. For graph-theoretical terminology and notation not defined here we follow [1].

Definition 2.1 [19]. Let $G=(V,E)$ ba an undirected simple graph. A faulty set $F \subseteq V$ is called a g-good-neighbor faulty set if $|N(v) \bigcap (V \setminus F)| \ge g$ for every vertex v in $V \setminus F$.

Definition 2.2 [19]. A g-good-neighbor cut of a connected graph G is a g-good-neighbor faulty set F such that $G-F$ is disconnected. The minimum cardinality of g-good-neighbor cuts is said to be the g-good-neighbor connectivity of G, denoted by $\kappa^{(g)}(G)$.

In [4], Hsieh et al. showed that 2-good-neighbor connectivity of the n-dimensional locally twisted cubes is $4n-8$ for $n \ge 4$, and showed that 3-good-neighbor connectivity is equal to $8n-24$ for $n \ge 5$. In [17], Wei and Hsieh studied that the g-good-neighbor connectivity of locally twisted cubes is $2^g(n-g)$ for $0 \le g \le n-2$.

B. The crossed cube CQ_n

Definition 2.3 [16]. Let $R = \{(00,00),(10,10),(01,11),(11,01)\}$. Two digit binary strings $u = u_1u_0$ and $v = v_1v_0$ are pair related, denoted as $u \sim v$, if and only if $(u,v) \in R.$

Definition 2.4 [16]. The vertex set of a crossed cube CQ_n is $\{v_{n-1}v_{n-2}\cdots v_0 : 0 \le i \le n-1, v_i \in \{0,1\}\}$. Two vertices $u = u_{n-1}u_{n-2}\cdots u_0$ and $v = v_{n-1}v_{n-2}\cdots v_0$ are adjacent if and only if one of the following conditions is satisfied.

1. There exists an integer l $(1 \le l \le n-1)$ such that

(1) $u_{n-1}u_{n-2}\cdots u_l = v_{n-1}v_{n-2}\cdots v_l$;

(2) $u_{l-1} \ne v_{l-1}$;

(3) if l is even, $u_{l-2} = v_{l-2}$;

(4) $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$, for $0 \le i < \left\lfloor \frac{l-1}{2} \right\rfloor$.

2.

(1) $u_{n-1} \ne v_{n-1}$;

(2)if n is even, $u_{n-2} = v_{n-2}$;

(3) $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$ for $0 \le i < \left\lfloor \frac{l-1}{2} \right\rfloor$.

Let $n \ge 2$. We define two graphs CQ_n^0 and CQ_n^1 as follows. If $u = u_{n-2}u_{n-3}\cdots u_0 \in V(CQ_{n-1})$, then $u^0 = 0u_{n-2}u_{n-3}\cdots u_0 \in V(CQ_n^0)$ and $u^1 = 1u_{n-2}u_{n-3}\cdots u_0 \in V(CQ_n^1)$. If $uv \in E(CQ_{n-1})$, then $u^0v^0 \in E(CQ_n^0)$ and $u^1v^1 \in E(CQ_n)^1$. Then $CQ_n^0 \cong CQ_{n-1}$ and $CQ_n^1 \cong CQ_{n-1}$. Define the edges between the vertices of CQ_n^0 and CQ_n^1 according to the following rules. The vertex $u = 0u_{n-2}u_{n-3}\cdots u_0 \in V(CQ_n^0)$ and the vertex $v = 1v_{n-2}v_{n-3}\cdots v_0 \in V(CQ_n^1)$ are adjacent if and only if

1. $u_{n-2} = v_{n-2}$ if n is even;
2. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, for $0 \le i < \left\lfloor \frac{n-1}{2} \right\rfloor$.

The edges between the vertices of CQ_n^0 and CQ_n^1 are said to be cross edges.

Proposition 2.1 [16]. All cross edges of CQ_n is a perfect matching.

By Proposition 2.1, CQ_n can be recursively defined as follows.

Definition 2.5 [16]. Define that $CQ_1 \cong K_2$ and $V(CQ_1) = \{0,1\}$. For $n \ge 2$, CQ_n is obtained by CQ_n^0 and CQ_n^1, and a perfect matching between the vertices of CQ_n^0 and CQ_n^1 according to the following rules (see Fig.1). The vertex $u = 0u_{n-2}u_{n-3}\cdots u_0 \in V(CQ_n^0)$ and the vertex $v = 1v_{n-2}v_{n-3}\cdots v_0 \in V(CQ_n^1)$ are adjacent in CQ_n if and only if

1. $u_{n-2} = v_{n-2}$ if n is even;
2. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, for $0 \le i < \left\lfloor \frac{n-1}{2} \right\rfloor$.

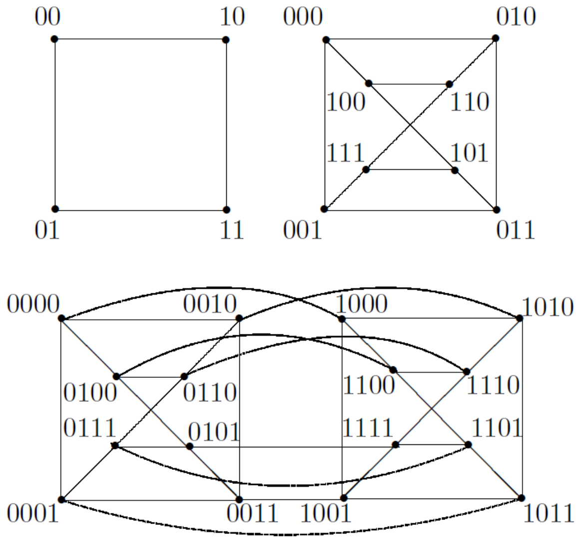

Fig. 1. CQ_2 , CQ_3 , and CQ_4

C. The PMC Model and the MM Model*

Table 1. Comparison results under the PMC model

testor u	tested v	result
faulty	fault-free or faulty	0 or 1
fault-free	faulty	1
fault-free	fault-free	0

Table 2. Comparison results under the MM* model

testor w	tested u,v	result
faulty	fault-free or faulty	0 or 1
fault-free	At least one is faulty	1
fault-free	both are fault-free	0

Let $G=(V(G),E(G))$ be a system. In the PMC model, a processor (vertex) can perform tests on its neighbors. For two adjacent vertices u and v in $V(G)$, the ordered pair (u,v) represents u test v. In this case, u is a tester and v is a tested. Because the faults considered here are permanent, the result of a test is reliable if and only if u is fault-free. A test assignment T for G is a collection of tests and thus can be modeled as a directed graph $T=(V(G),L)$, where $(u,v)\in L$ if and only if $uv\in E(G)$. The collection of all test results from T is called a syndrome. Formally, a syndrome of T is a mapping $\sigma: L\to\{0,1\}$. Table 1 shows all possible test results of the test $\sigma((u,v))$. For a given syndrome σ, a subset of vertices $F\subseteq V(G)$ is said to be consistent with σ if syndrome σ can be produced from the situation that, for any $(u,v)\in L$ such that $u\in V\setminus F$, $\sigma(u,v)=1$ if and only if $v\in F$. Let $\sigma(F)$ denote the set of all syndromes which F is consistent with. Two distinct vertex sets F_1 and F_2 are indistinguishable (respectively, distinguishable) if $\sigma(F_1)\bigcap\sigma(F_2)\neq\varnothing$ (respectively, $\sigma(F_1)\bigcap\sigma(F_2)=\varnothing$), then we say (F_1,F_2) is an indistinguishable pair (respectively, distinguishable pair).

In the MM model, the comparison scheme of a system $G=(V(G),E(G))$ is modeled as a multigraph, denoted by $M=(V(G),L)$, where L is the labeled edge set. If (u,v) is an edge labeled by w, then the labeled edge $(u,v)_w$ belongs to L, which implies that vertices u and v are being compared by vertex w. If the comparator w is faulty, then the result of comparison is unreliable. For $(u,v)_w\in L$, we use $\sigma^*((u,v)_w)$ denote the result of comparing vertices u and v by w. The collection of all comparison result is given by a function $\sigma^*: L\to\{0,1\}$, which is called the syndrome of the diagnosis. Table 2 shows all possible test results of the test $\sigma^*((u,v)_w)$. The MM * model is a special case of the MM model. In the MM * model, all comparisons of G are in the comparison scheme of G, i.e., if $uw,vw\in E(G)$, then $(u,v)_w\in L$. Similarly to the PMC model, we can define a subset of vertices $F\subseteq V(G)$ to be consistent with a given syndrome σ^* and two distinct sets F_1 and F_2 in $V(G)$ to be indistinguishable (resp. distinguishable) under the MM * model.

III. THE CONNECTIVITY OF THE CROSSED CUBE CQ_n

Lemma 3.1 [3]. $\kappa(CQ_n)=n$ for $n\geq 1$.

Lemma 3.2 [6]. $\kappa^1(CQ_n)=2n-2$ for $n\geq 3$.

Lemma 3.3 [6]. There are at most two common neighbors for any pair of vertices in the crossed cube CQ_n for $n\geq 2$.

Lemma 3.4 [16]. Let $F\subseteq V(CQ_n)$ $(n\geq 3)$ with $n\leq |F|\leq 2n-3$. If CQ_n-F is disconnected, then CQ_n-F has exactly two components, one of which is an isolated vertex.

Lemma 3.5 [16]. Let $F\subseteq V(CQ_n)$ $(n\geq 5)$ with $2n-2\leq |F|\leq 3n-6$. If CQ_n-F is disconnected, then CQ_n-F satisfies one of the following conditions:

(1) CQ_n-F has two components, one of which is a K_2;

(2) CQ_n-F has two components, one of which is an isolated vertex;

(3) CQ_n-F has three components, two of which are isolated vertices.

A connected graph G is super g-extra connected if every minimum g-extra cut F of G isolates one connected subgraph of order $g+1$. In addition, if $G-F$ has two components, one of which is the connected subgraph of order $g+1$, then G is tightly $|F|$ super g-extra connected.

Lemma 3.6 [16]. For $n \geq 5$, the crossed cube CQ_n is tightly $(3n-5)$ super 2-extra connected.

Lemma 3.7 [1]. Let G be a graph. If $\delta(G) \geq 2$, then G contains a cycle.

A connected graph G is super 2-good-neighbor connected if every minimum 2-good-neighbor cut F of G isolates one connected subgraph of minimum degree 2. If, in addition, $G-F$ has two components, one of which is a connected subgraph of minimum degree 2, then G is tightly $|F|$ super 2-good-neighbor connected.

Lemma 3.8. Let CQ_n be the crossed cube, and let H be a connected subgraph of CQ_n with $\delta(H)=2$ such that it contains $V(CQ_n)$ as least as possible. Then H is a 4-cycle.

Proof. Since $\delta(H) \geq 2$, by Lemma 3.7, CQ_n contains a cycle. Note that CQ_n does not have triangle. So $|V(CQ_n)| \geq 4$. Since CQ_n contains 4-cycles, we have that H is a 4-cycle. The proof is complete.

Lemma 3.9. Let C be a 4-cycle in the crossed cube $CQ_n\,(n \geq 3)$. Then any pair of vertices in C have no common neighbors outside C.

Proof. Clearly, $CQ_n[V(C)] \cong CQ_2$. Since CQ_n has no triangle, there is no common neighbor for any pair of adjacent vertices in C. By Lemma 3.3, there are at most two common neighbors for any pair of vertices in CQ_n. Combining this with the 4-cycle C, we have that any pair of nonadjacent vertices in C has no common neighbor outside C. Therefore, any pair of vertices in C has no common neighbor outside C. The proof is complete.

Lemma 3.10. Let C be a 5-cycle in the crossed cube $CQ_n\,(n \geq 3)$. The $|N_{CQ_n}(V(C))| \geq 5n-12$.

Proof. Let $C = v_1v_2v_3v_4v_5v_1$. We prove the lemma by induction on n. When $n=3$, it is easy to see that $|N_{CQ_3}(V(C))| = 3 = 5\times 3-12$ (see Fig.1). We assume that the lemma is true for $n-1$, i.e., $|N_{CQ_{n-1}}(V(C))| \geq 5(n-1)-12 = 5n-17$. We will show that the lemma is true for $n\,(n \geq 4)$.We decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}.

Case 1. $V(C) \cap V(CQ_n^0) = \varnothing$ or $V(C) \cap V(CQ_n^1) = \varnothing$.

Without loss of generality, let $V(C) \cap V(CQ_n^1) = \varnothing$. Then $V(C) \subset V(CQ_n^0)$. By the inductive hypothesis, $|N_{CQ_n^0}(V(C))| \geq 5n-17$. By Proposition 2.1, C has five neighbors in CQ_n^1.

Thus, $|N_{CQ_n}(V(C))| \geq 5n-17+5 = 5n-12$.

Case 2. $V(C) \cap V(CQ_n^0) \neq \varnothing$ and $V(C) \cap V(CQ_n^1) \neq \varnothing$.

By Proposition 2.1, $|\ V(C) \cap V(CQ_n^0)\ | = 2$ or $|\ V(C) \cap V(CQ_n^1)\ | = 2$. Without loss of generality, let $|\ V(C) \cap V(CQ_n^0)\ | = 2$. Then $|\ V(C) \cap V(CQ_n^1)\ | = 3$. Let $V(C) \cap V(CQ_n^0) = \{v_1, v_2\}$ and $V(C) \cap V(CQ_n^1) = \{v_3, v_4, v_5\}$. Then $v_1v_2 \cong K_2$ and $P = v_3v_4v_5$ is a 2-path. By Lemma 3.3, v_3 and v_5 have at most two common neighbors in CQ_n^1, one of which is v_4. Thus, v_3 and v_5 may have another common neighbor in CQ_n^1. By Proposition 2.1, v_4 has a neighbor v_4' in CQ_n^0. Since CQ_n has no triangle, v_4' may be adjacent to v_1 or v_2. So C has at most two common neighbors in CQ_n.Thus, $|N_{CQ_n}(V(C))| \geq 5(n-2)-2 = 5n-12$. The proof is complete.

Lemma 3.11. Let CQ_n be the crossed cube and let $A = \{0\cdots000, 0\cdots001, 0\cdots010, 0\cdots011\}$. If $n \geq 4$, $F_1 = N_{CQ_n}(A)$, $F_2 = A \cup N_{CQ_n}(A)$, then $|F_1| = 4n-8$, $|F_2| = 4n-4$, F_1 is a 2-good-neighbor cut of CQ_n, and $CQ_n - F_1$ has two components $CQ_n - F_2$ and $CQ_n[A]$.

Proof. By the definition of crossed cube, $CQ_n[A]$ is a 4-cycle. By Lemma 3.9, we get that any two vertices in A have no common neighbors outside A. Thus, $|F_1| = |N_{CQ_n}(A)| = 4(n-2) = 4n-8$ and $|F_2| = |A|+|F_1| = 4n-4$. We will prove that $CQ_n - F_2$ is connected and $\delta(CQ_n - F_2) \geq 2$ by induction on n. When $n=4$, it is easy to see that $CQ_4 - F_2$ is connected and $\delta(CQ_4 - F_2) \geq 2$ (see Fig. 1). We assume that the result is true for $n-1$, i.e., $CQ_{n-1} - F_2$ is connected an $\delta(CQ_{n-1} - F_2) \geq 2$. Now we show that the result is also true for $n\,(n \geq 5)$. We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_n. Let $F_2^0 = F_2 \cap V(CQ_n^0)$ and $F_2^1 = F_2 \cap V(CQ_n^1)$.

Then $|F_2^0|+|F_2^1| = |F_2|$. Note that $A \subseteq V(CQ_n^0)$. By the inductive hypothesis, $CQ_n^0 - F_2^0$ is connected and $\delta(CQ_n^0 - F_2^0) \geq 2$. Note that $A = \{0\cdots000, 0\cdots001, 0\cdots010, 0\cdots011\}$ and $A \subseteq V(CQ_n^0)$. By Proposition 2.1 and Definition 2.4, we have $F_2^1 = N_{CQ_n}(A) \cap V(CQ_n^1) = \{1\cdots000, 10\cdots011, 10\cdots010, 10\cdots01\}$. By the definition of crossed cube, $CQ_n[F_2^1]$ is a 4-cycle.

Case 1. $CQ_n^1 - F_2^1$ is connected.

Since $|V(CQ_n^0 - F_2^0| = 2^{n-1}-(4n-8) \geq 1\ \ (n \geq 5)$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_2^0) \cup V(CQ_n^1 - F_2^1)] = CQ_n - F_2$ is connected. Note that $CQ_n[F_2^1]$ is a 4-cycle. By Lemma 3.9, every vertex in $CQ_n^1 - F_2^1$ has at most one neighbor in F_2^1. Thus, $\delta\left(CQ_n^1 - F_2^1\right) \geq n-1-1 \geq 2\left(n \geq 5\right)$. Note that $\delta\left(CQ_n^0 - F_2^0\right) \geq 2$ and $CQ_n - F_2$ is connected. We can get $\delta\left(CQ_n - F_2\right) \geq 2$.

Case 2. $CQ_n^1 - F_2^1$ is disconnected.

By Lemma 3.1, we have $\kappa\left(CQ_n^1\right) = n-1 \geq 4\left(n \geq 5\right)$. So we get $CQ_n^1 - F_2^1$ is connected when $n \geq 6$, a contradiction. We consider CQ_5^1. Note that $CQ_5^1 \cong CQ_4$ and $\left|F_2^1\right| = 4$. By Lemma 3.4, $CQ_5^1 - F_2^1$ has two components, one of which is an isolated vertex. Let u be the isolated vertex. Since $N_{CQ_5^1}\left(u\right) \subseteq F_2^1$ and $\left|N_{CQ_5^1}\left(u\right)\right| = \left|F_2^1\right| = 4$, we have $N_{CQ_5^1}\left(u\right) = F_2^1$. Note that $CQ_n\left[F_2^1\right]$ is a 4-cycle. We can get that u and an edge of $CQ_n\left[F_2^1\right]$ form a triangle, a contradiction. Thus, this case does not exist. Note that $CQ_n - F_1$ has two components $CQ_n - F_2$ and $CQ_n\left[A\right]$ with $\delta\left(CQ_n - F_2\right) \geq 2$ and $\delta\left(A\right) = 2$. Therefore, F_1 is a 2-good-neighbor cut of CQ_n. The proof is complete.

Lemma 3.12. The 2-good-neighbor connectivity $\kappa^{(2)}(CQ_n) \leq 4n-8$ for $n \geq 4$.

Proof. Let A be defined in Lemma 3.11, and $F = N_{CQ_n}(A)$. Obviously, $CQ_n - F$ is disconnected, $|F| = 4n-8$, and F is a 2-good-neighbor cut. By the definition of 2-good-neighbor connectivity, $\kappa^2(CQ_n) \leq |F| = 4n-8$. The proof is complete.

Lemma 3.13. Let $F \subseteq V(CQ_4)$ with $|F| = 7$. Suppose that $CQ_4 - F$ is disconnected. Then F is not a 2-good-neighbor cut of CQ_4.

Proof. We can decompose CQ_4 along dimension 3 into CQ_4^0 and CQ_4^1. Then both CQ_4^0 and CQ_4^1 are isomorphic to CQ_3. Let $F_0 = F \cap V(CQ_4^0)$ and $F_1 = F \cap V(CQ_4^1)$ with $|F_0| \leq |F_1|$. Note that $|F| = 7$. Thus, $|F_0| \leq 3$.

Case 1. $CQ_4^0 - F_0$ is connected.

Suppose that $CQ_4^1 - F_1$ is connected. Since $2^3 - 7 = 1$, by Proposition 2.1, $CQ_4[V(CQ_4^0 - F_0) \cup V(CQ_4^1 - F_1)] = CQ_4 - F$ is connected, a contradiction. So we suppose that $CQ_4^1 - F_1$ is disconnected. Let the components of $CQ_4^1 - F_1$ be $C_1, C_2, \ldots, C_k$ $(k \geq 2)$. Note that $|F_0| \leq 3$. If every component C_i $(i \in \{1, \ldots, k\})$ of $CQ_4^1 - F_1$ such that $|V(C_i)| \geq 4$, by Proposition 2.1, then $CQ_4[V(CQ_4^0 - F_0) \cup V(C_1) \cup \cdots \cup V(C_k)] = CQ_4 - F$ is connected, a contradiction. Thus, there is at least a component C_j $(1 \leq j \leq k)$ such that $|V(C_j)| \leq 3$. If every component C_j such that C_j is connected to $CQ_4^0 - F_0$, then $CQ_4 - F$ is connected, a contradiction. Thus, there is a C_j such that $N_{CQ_4}(V(C_j) \cap V(CQ_4^0) \subseteq F_0$. Then C_j is a component of $CQ_4 - F$. Since $|V(C_j)| \leq 3$, by Lemma 3.8, C_j is not a 2-good-neighbor component of $CQ_4 - F$. Thus, F is not a 2-good-neighbor cut of CQ_4.

Case 2. $CQ_4^0 - F_0$ is disconnected.

By Lemma 3.1, we have $\kappa(CQ_4^0) = 3$. Since $CQ_4^0 - F_0$ is disconnected, $|F_0| = 3$. By Lemma 3.4, $CQ_4^0 - F_0$ has two components, one of which is an isolated vertex. Let u be the isolated vertex. If u is connected to one of F_1, then u is an isolated vertex component of $CQ_4 - F$. So F is not a 2-good-neighbor cut of CQ_4. If u is connected to one of $V(CQ_4^1 - F_1)$, then $d_{CQ_4 - F}(u) = 1$. Thus, F is not a 2-good-neighbor cut of CQ_4. The proof is complete.

Lemma 3.14. Let $F \subseteq V(CQ_5)$ with $|F| = 11$. Suppose that $CQ_5 - F$ is disconnected. Then F is not a 2-good-neighbor cut of CQ_5.

Proof. We can decompose CQ_5 along dimension 4 into CQ_5^0 and CQ_5^1. Then both CQ_5^0 and CQ_5^1 are isomorphic to CQ_4. Let $F_0 = F \cap V(CQ_5^0)$ and $F_1 = F \cap V(CQ_5^1)$ with $|F_0| \leq |F_1|$. Since $|F| = 11$, we have $|F_0| \leq 5$. Note that $|F_0| \leq 2(n-1) - 3 = 5$. By Lemma 3.4, $CQ_5^0 - F_0$ is connected or has two components, one of which is an isolated vertex. Suppose that $CQ_5^0 - F_0$ is disconnected. Let u be the isolated vertex. If u is connected to one of F_1, then u is an isolated vertex in $CQ_5 - F$. Thus, F is not a 2-good-neighbor cut. If u is connected to one of $V(CQ_5^1 - F_1)$, then $d_{CQ_5 - F}(u) = 1$. Thus, F is not a 2-good-neighbor cut. Then we suppose that $CQ_5^0 - F_0$ is connected. Suppose that $CQ_5^1 - F_1$ is connected. Since $2^4 - 11 \geq 1$, by Proposition 2.1, $CQ_5[V(CQ_5^0 - F_0) \cup V(CQ_5^1 - F_1)] = CQ_5 - F$ is connected, a contradiction. So we suppose that $CQ_5^1 - F_1$ is disconnected. Let the components of $CQ_5^1 - F_1$ be $C_1, C_2, \ldots, C_k$ $(k \geq 2)$.

Case 1. $|F_0| = 5$.

In this case, $|F_1| = 6$. If every component C_i of $CQ_5^1 - F_1$ such that $|V(C_i)| \geq 6$ for $i \in \{1, \ldots, k\}$, by Proposition 2.1, then $CQ_5[V(CQ_5^0 - F_0) \cup V(C_1) \cup \cdots \cup V(C_k)] = CQ_5 - F$ is connected, a contradiction. Thus, there exists at least one component C_j $(1 \leq j \leq k)$ such that $|V(C_j)| \leq 5$. If every component C_j such that C_j is connected to one of $V(CQ_5^0 - F_0)$, then $CQ_5 - F$ is connected, a contradiction. Thus, there is a C_j such that $N_{CQ_5}(V(C_j)) \cap V(CQ_5^0) \subseteq F_0$. Then C_j is a component of $CQ_5 - F$. If C_j is a 5-cycle, by Lemma 3.10, then $|N_{CQ_5^1}(V(C_j))| \geq 5(n-1) - 12 = 8$. Since C_j is also a component of $CQ_5^1 - F_1$, we have $N_{CQ_5^1}(V(C_j)) \subseteq F_1$. Then $8 \leq |N_{CQ_5^1}(V(C_j))| \leq |F_1| = 6$, a contradiction. Thus, C_j is not a 5-cycle. If C_j is a 4-cycle, by Lemma 3.9, then $|N_{CQ_5^1}(V(C_j))| = 4(n-1-2) = 8$. Similarly, C_j is also not a 4-cycle. By Lemma 3.8, C_j is not a 2-good-neighbor component with $|V(C_j)|\le 5$. Thus, F is not a 2-good-neighbor cut of CQ_5.

Case 2. $|F_0|=4$.

In this case, $|F_1|=7$. If every component C_i $(i\in\{1,\dots,k\})$ of $CQ_5^1-F_1$ such that $|V(C_i)|\ge 5$, then $CQ_5[V(CQ_5^0-F_0)\cup V(C_1)\cup\cdots\cup V(C_k)]=CQ_5-F$ is connected, a contradiction. Thus, there exists at least one component C_j $(1\le j\le k)$ such that $|V(C_j)|\le 4$. If every component C_j such that C_j is connected to one of $V(CQ_5^0-F_0)$, then CQ_5-F is connected, a contradiction. Thus, there is a C_j such that $N_{CQ_5}(V(C_j))\cap V(CQ_5^0)\subseteq F_0$. Then C_j is a component of CQ_5-F. If C_j is a 4-cycle, then it is similar to Case 1. We get $8=|N_{CQ_5^1}(V(C_j))|\le|F_1|=7$, a contradiction. So C_j is not a 4-cycle. By Lemma 3.8, C_j is not a 2-good-neighbor component with $|V(C_j)|\le 4$. Thus, F is not a 2-good-neighbor cut of CQ_5.

Case 3. $|F_0|\le 3$.

By Proposition 2.1, there are at most three vertices in CQ_5^1 such that they are connected to one of F_0, respectively. Since CQ_5-F is disconnected, there is a component C in CQ_5-F such that $|V(C)|\le 3$. By Lemma 3.8, C is not a 2-good-neighbor component in CQ_5-F. Therefore, F is not a 2-good-neighbor cut of CQ_5. The proof is complete.

Lemma 3.15. Let $F\subseteq V(CQ_4)$. If $|F|=6$, then CQ_4-F satisfies one of the following conditions:

(1) CQ_4-F is connected;

(2) CQ_4-F has two components, one of which is a K_2;

(3) CQ_4-F has two components, one of which is an isolated vertex;

(4) CQ_4-F has three components, two of which are isolated vertices;

(5) CQ_4-F has two components H_1, H_2, and $|V(H_i)|=5$ and $\delta(H_i)=1$ for $i=1,2$.

Proof . We can decompose CQ_4 along dimension 3 into CQ_4^0 and CQ_4^1. Then both CQ_4^0 and CQ_4^1 are isomorphic to CQ_3. Let $F_0=F\cap V(CQ_4^0)$ and $F_1=F\cap V(CQ_4^1)$ with $|F_0|\le|F_1|$. Since $|F|=6$, we have $|F_0|\le 3$.

Case 1. $|F_0|=3$.

In this case, $|F_0|=|F_1|=3=2(n-1)-3$. By Lemma 3.4, $CQ_4^i-F_i$ $(i\in\{0,1\})$ is connected or has two components, one of which is an isolated vertex. Let u_i be the isolated vertex and let B_i be the other component for $i\in\{0,1\}$. Then $|V(B_i)|=|V(CQ_4^i)$, $(F_i\cup\{u_i\})|=2^3-(3+1)=4$.

Case 1.1. u_0 is connected to u_1.

Note that $|F_i|=3<4=|V(B_i)|$ for $i\in\{0,1\}$. By Proposition 2.1, CQ_4-F satisfies the condition (2).

Case 1.2. u_0 is connected to one of F_1.

If u_1 is connected to one of F_0, by Proposition 2.1, CQ_4-F satisfies the condition (4).

If u_1 is connected to one of $V(B_0)$, by Proposition 2.1, CQ_4-F satisfies the condition (3).

Case 1.3. u_0 is connected to one of $V(B_1)$.

If u_1 is connected to one of F_0, by Proposition 2.1, CQ_4-F satisfies the condition (3). If u_1 is connected to one of $V(B_0)$, by Proposition 2.1, CQ_4-F satisfies the condition (1) or (5).

Case 2. $|F_0|\le 2$.

In this case, $CQ_4^0-F_0$ is connected. Note that $|F_1|=|F|-|F_0|\ge 4$. By Lemma 3.1, $|F_1|\ge 4>3=\kappa(CQ_4^1)$. Then $CQ_4^1-F_1$ is connected or disconnected. Suppose that $CQ_4^1-F_1$ is connected. Since $2^3-6\ge 1$, by Proposition 2.1, CQ_4-F is connected. Then we suppose that $CQ_4^1-F_1$ is disconnected. Let the components of $CQ_4^1-F_1$ be $C_1,C_2,\dots,C_k$ $(k\ge 2)$. Note that $|F_0|\le 2$. If every component C_i $(i\in\{1,\dots,k\})$ of $CQ_4^1-F_1$ such that $|V(C_i)|\ge 3$, then $CQ_4[V(CQ_4^0-F_0)\cup V(C_1)\cup\cdots\cup V(C_k)]=CQ_4-F$ is connected, a contradiction. Thus, there exists at least one component C_i $(1\le i\le k)$ such that $|V(C_i)|\le 2$. Thus, CQ_4-F satisfies one of the conditions (1)-(4). The proof is complete.

Lemma 3.16. Let $F\subseteq V(CQ_5)$. If $|F|=10$, then CQ_5-F satisfies one of the following conditions:

(1) CQ_5-F is connected;

(2) CQ_5-F has two components, one of which is a K_2;

(3) CQ_5-F has two components, one of which is a 2-path;

(4) CQ_5-F has two components, one of which is an isolated vertex;

(5) CQ_5-F has three components, two of which are isolated vertices;

(6) CQ_5-F has four components, three of which are isolated vertices;

(7) CQ_5-F has three components, one of which is an isolated vertex and the other is a K_2.

Proof. We can decompose CQ_5 along dimension 4 into CQ_5^0 and CQ_5^1. Then both CQ_5^0 and CQ_5^1 are isomorphic to CQ_4. Let $F_0=F\cap V(CQ_5^0)$ and $F_1=F\cap V(CQ_5^1)$ with $|F_0|\le|F_1|$. Since $|F|=10$, we have $|F_0|\le 5$.

Case 1. $|F_0|=5$.

In this case, $|F_0|=|F_1|=5=2(n-1)-3$. By Lemma 3.4, $CQ_5^i-F_i$ $(i\in\{0,1\})$ is connected or has two components, one of which is an isolated vertex. Since $2^4-10-2\ge 1$, by Proposition 2.1, CQ_5-F satisfies one of the conditions (1)-(7).

Case 2. $|F_0|=4$.

Note that $|F_0|=4=n-1$. By Lemma 3.4, $CQ_5^0-F_0$ is connected or has two components, one of which is an isolated vertex. Let u be the isolated vertex and B be the other component.

Then $|V(B)|=|V(CQ_5^0)\setminus(F_0\cup\{u\})|=2^4-(4+1)=11$.

In this case, $|F_1|=6$. By Lemma 3.15, $CQ_5^1-F_1$ satisfies one of the following conditions:

(a) $CQ_5^1-F_1$ is connected;

(b) $CQ_5^1-F_1$ has two components, one of which is a K_2;

(c) $CQ_5^1-F_1$ has two components, one of which is an isolated vertex;

(d) $CQ_5^1-F_1$ has three components, two of which are isolated vertices;

(e) $CQ_5^1-F_1$ has two components, which are two components of order 5.

Case 2.1. Both $CQ_5^0-F_0$ and $CQ_5^1-F_1$ are connected.

Since $2^4-10\geq 1$, by Proposition 2.1, $CQ_5[V(CQ_5^0-F_0)\cup V(CQ_5^1-F_1)]=CQ_5-F$ is connected.

Case 2.2. $CQ_5^0-F_0$ is disconnected and $CQ_5^1-F_1$ is connected. Since $2^4-10-1\geq 1$, by Proposition 2.1, $CQ_5[V(B)\cup V(CQ_5^1-F_1)]$ is connected. Thus, CQ_5-F satisfies the condition (1) or (4).

Case 2.3. $CQ_5^0-F_0$ is connected and $CQ_5^1-F_1$ is disconnected.

Suppose that $CQ_5^1-F_1$ satisfies one of the conditions (b)-(d). Since $2^4-10-2\geq 1$, by Proposition 2.1, CQ_5-F satisfies one of the conditions (1)-(7). Note that $|F_0|=4$. Suppose that $CQ_5^1-F_1$ satisfies the condition (e). By Proposition 2.1, CQ_5-F is connected.

Case 2.4. Both $CQ_5^0-F_0$ and $CQ_5^1-F_1$ are disconnected.

If $CQ_5^1-F_1$ satisfies one of the conditions (b)-(d), by Proposition 2.1, then CQ_5-F satisfies one of the conditions (1)-(7). Suppose that $CQ_5^1-F_1$ satisfies one of the condition (e). Let C_1 and C_2 be two components of order 5 in $CQ_5^1-F_1$.

Case 2.4.1. u is connected to F_1.

Note that $|F_0|=4<5=|V(C_i)|$ $(i\in\{1,2\})$. By Proposition 2.1, $CQ_5[V(B)\cup V(C_1)\cup V(C_2)]$ is connected. Thus, CQ_5-F satisfies the condition (4).

Case 2.4.2. u is connected to C_1 or C_2.

Without loss of generality, we assume that u is connected to C_1. Note that $|F_0|=4<5=|V(C_2)|$. By Proposition 2.1, $CQ_5[V(B)\cup V(C_2)]$ is connected. If $CQ_5[V(B)\cup V(C_1)]$ is connected, then CQ_5-F is connected. We suppose that $CQ_5[V(B)\cup V(C_1)]$ is disconnected. Then $N_{CQ_5}(V(C_1))\cap V(CQ_5^0)=F_0\cup\{u\}$. Thus, CQ_5-F has two components, one of which is $CQ_5[V(C_1)\cup\{u\}]$ and the other is $CQ_5[V(B)\cup V(C_2)]$ with $|V(C_1)\cup\{u\}|=5+1=6$ and $|V(B)\cup V(C_2)|=11+5=16$. By Lemma 3.6, CQ_5 is tightly 10 super 2-extra connected, i.e., CQ_5-F has two components, one of which is order 3. We get that CQ_5-F should have a component of order 3. This is a contradiction to that $|V(C_1)\cup\{u\}|=6$ and $|V(B)\cup V(C_2)|=16$. So the hypothesis is not true.

Case 3. $|F_0|\leq 3$.

By Lemma 3.1, $CQ_5^0-F_0$ is connected. Suppose that $CQ_5^1-F_1$ is connected. Since $2^4-10\geq 1$, by Proposition 2.1, $CQ_5[V(CQ_5^0-F_0)\cup V(CQ_5^1-F_1)]=CQ_5-F$ is connected. Then we suppose that $CQ_5^1-F_1$ is disconnected. Let the components of $CQ_5^1-F_1$ be C_1, C_2, ..., C_k $(k\geq 2)$. If every component of $CQ_5^1-F_1$ such that $|V(C_i)|\geq 4$ for $i\in\{1,\ldots,k\}$, then $CQ_5[V(CQ_5^0-F_0)\cup V(C_1)\cup\cdots\cup V(C_k)]=CQ_5-F$ is connected. Suppose that there is a components C_i such that $|V(C_i)|\leq 3$. If $N_{CQ_5}(V(C_i))\cap V(CQ_5^0)\subseteq F_0$, then C_i is a component of CQ_5-F. Thus, CQ_5-F satisfies one of the conditions (1)-(7).

Therefore, according to the cases 1-3, we can get that CQ_5-F satisfies one of the conditions (1)-(7). The proof is complete.

Lemma 3.17. Let $F\subseteq V(CQ_n)$ $(n\geq 6)$. If $|F|=3n-5$, then CQ_n-F satisfies one of the following conditions:

(1) CQ_n-F is connected;

(2) CQ_n-F has two components, one of which is a K_2;

(3) CQ_n-F has two components, one of which is a 2-path;

(4) CQ_n-F has two components, one of which is an isolated vertex;

(5) CQ_n-F has three components, two of which are isolated vertices;

(6) CQ_n-F has four components, three of which are isolated vertices;

(7) CQ_n-F has three components, one of which is an isolated vertex and the other is a K_2.

Proof. We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0=F\cap V(CQ_n^0)$ and $F_1=F\cap V(CQ_n^1)$ with $|F_0|\leq|F_1|$. Let B_i be the maximum component of $CQ_n^i-F_i$ (If $CQ_n^i-F_i$ is connected, then let $B_i=CQ_n^i-F_i$) for $i\in\{0,1\}$. We have $0\leq|F_0|\leq\left\lfloor\frac{3n\text{-}5}{2}\right\rfloor\leq 2n-5$ and $n\leq\left\lceil\frac{3n-5}{2}\right\rceil\leq|F_1|\leq 3n-5$ $(n\geq 6)$.

Case 1. $n\leq|F_1|\leq 2n-5$.

In this case, $|F_0|\leq|F_1|\leq 2n-5=2(n-1)-3$.

By Lemma 3.4, $CQ_n^i-F_i$ $(i\in\{0,1\})$ is connected or has two components, one of which is an isolated vertex. Since $2^{n-1}-(3n-5)-2\geq 1$ $(n\geq 6)$, by Proposition 2.1, $CQ_n[V(B_0)\cup V(B_1)]$ is connected. Thus, CQ_n-F satisfies one of the conditions (1)-(7).

Case 2. $|F_1|=2n-4$.

In this case, $|F_0|=3n-5-(2n-4)=n-1$.

By Lemma 3.4, $CQ_n^0-F_0$ is connected or has two components, one of which is an isolated vertex. Note that $|F_1|=2n-4=2(n-1)-2$.

By Lemma 3.5, $CQ_n^1-F_1$ satisfies one of the following conditions:

(a) $CQ_n^1-F_1$ is connected;

(b) $CQ_n^1-F_1$ has two components, one of which is a K_2;

(c) $CQ_n^1-F_1$ has two components, one of which is an isolated vertex;

(d) $CQ_n^1-F_1$ has three components, two of which are isolated vertices.

Since $2^{n-1}-(3n-5)-3\geq 1$ $(n\geq 6)$, by Proposition 2.1, $CQ_n[V(B_0)\cup V(B_1)]$ is connected. Thus, CQ_n-F satisfies one of the conditions (1)-(7).

Case 3. $2n-3\leq|F_1|\leq 3n-9$.

In this case, $|F_0|\leq 3n-5-(2n-3)=n-2$. By Lemma 3.1, $CQ_n^0-F_0$ is connected. Note that $2(n-1)-2<2n-3\leq|F_1|\leq 3n-9=3(n-1)-6$.

By Lemma 3.5, $CQ_n^1-F_1$ satisfies one of the following conditions:

(a) $CQ_n^1-F_1$ is connected;

(b) $CQ_n^1-F_1$ has two components, one of which is a K_2;

(c) $CQ_n^1-F_1$ has two components, one of which is an isolated vertex;

(d) $CQ_n^1-F_1$ has three components, two of which are isolated vertices.

Since $2^{n-1}-(3n-5)-2\geq 1$ $(n\geq 6)$, by Proposition 2.1, $CQ_n[V(B_0)\cup V(B_1)]$ is connected. Thus, CQ_n-F satisfies one of the conditions (1)-(7).

Case 4. $3n-8\leq|F_1|\leq 3n-5$.

In this case, $|F_0|\leq 3n-5-(3n-8)=3$. By Lemma 3.1, $CQ_n^0-F_0$ is connected for $n\geq 6$. Suppose that $CQ_n^1-F_1$ is connected. Since $2^{n-1}-(3n-5)\geq 1$ $(n\geq 6)$, $CQ_n[V(CQ_n^0-F_0)\cup V(CQ_n^1-F_1)]=CQ_n-F$ is connected. Then we suppose that $CQ_n^1-F_1$ is disconnected. Let the components of $CQ_n^1-F_1$ be $C_1,C_2,\ldots,C_k$ $(k\geq 2)$. Note that $|F_0|\leq 3$. If every component C_i $(1\leq i\leq k)$ such that $|V(C_i)|\geq 4$, then $CQ_n[V(CQ_n^0-F_0)\cup V(C_1)\cup\cdots\cup V(C_k)]=CQ_n-F$ is connected. Suppose that there is a component C_i such that $|V(C_i)|\leq 3$. If $N_{CQ_n}(V(C_i))\cap V(CQ_n^0)\subseteq F_0$, then C_i is a component of CQ_n-F. Thus, CQ_n-F satisfies one of the conditions (1)-(7). The proof is complete..

Lemma 3.18. Let $F\subseteq V(CQ_6)$. If $14\leq|F|\leq 15$, then CQ_6-F satisfies one of the following conditions:

(1) CQ_6-F is connected;

(2) CQ_6-F has two components, one of which is a K_2;

(3) CQ_6-F has two components, one of which is a $K_{1,3}$;

(4) CQ_6-F has two components, one of which is a 2-path;

(5) CQ_6-F has two components, one of which is a 3-path;

(6) CQ_6-F has two components, one of which is an isolated vertex;

(7) CQ_6-F has three components, two of which are isolated vertices;

(8) CQ_6-F has four components, three of which are isolated vertices;

(9) CQ_6-F has three components, one of which is an isolated vertex and the other is a K_2;

(10) CQ_6-F has three components, one of which is an isolated vertex and the other is a 2-path.

Proof. We can decompose CQ_6 along dimension 5 into CQ_6^0 and CQ_6^1. Then both CQ_6^0 and CQ_6^1 are isomorphic to CQ_5. Let $F_0=F\cap V(CQ_6^0)$ and $F_1=F\cap V(CQ_6^1)$ with $|F_0|\leq|F_1|$. Let B_i be the maximum component of $CQ_6^i-F_i$ (If $CQ_6^i-F_i$ is connected, then let $B_i=CQ_6^i-F_i$) for $i\in\{0,1\}$. Since $14\leq|F|\leq 15$, we have $0\leq|F_0|\leq\left\lfloor\frac{15}{2}\right\rfloor=7$ and $7=\frac{14}{2}\leq|F_1|\leq 15$.

Case 1. $5\leq|F_0|\leq 7$.

Note that $5=n-1\leq|F_0|\leq 2(n-1)-3=7$.

By Lemma 3.4, $CQ_6^0-F_0$ is connected or has two components, one of which is an isolated vertex.

Since $14\leq|F|\leq 15$, we can get $7\leq|F_1|\leq 10$.

Case 1.1. $|F_1|=7$.

Note that $|F_1|=7=2(n-1)-3$. By Lemma 3.4, $CQ_6^1-F_1$ is connected or has two components, one of which is an isolated vertex. Since $2^5-14-2\geq 1$, by Proposition 2.1, $CQ_6[V(B_0)\cup V(B_1)]$ is connected. Thus, CQ_6-F satisfies one of the conditions (1)-(10).

Case 1.2. $8\leq|F_1|\leq 9$.

Note that $8=2(n-1)-2\leq|F_1|\leq 3(n-1)-6=9$. By Lemma 3.5, $CQ_6^1-F_1$ satisfies one of the following conditions:

(a) $CQ_6^1-F_1$ is connected;

(b) $CQ_6^1-F_1$ has two components, one of which is a K_2;

(c) $CQ_6^1-F_1$ has two components, one of which is an isolated vertex;

(d) $CQ_6^1-F_1$ has three components, two of which are isolated vertices.

Since $2^5-14-3\geq 1$, by Proposition 2.1, $CQ_6[V(B_0)\cup V(B_1)]$ is connected. Thus, CQ_6-F satisfies one of the conditions (1)-(10).

Case 1.3. $|F_1|=10$.

Note that $|F_1|=3(n-1)-5=10$. By Lemma 3.16, $CQ_6^1-F_1$ satisfies one of the following conditions:

(a) $CQ_6^1-F_1$ is connected;

(b) $CQ_6^1-F_1$ has two components, one of which is a K_2;

(c) $CQ_6^1-F_1$ has two components, one of which is a 2-path;

(d) $CQ_6^1-F_1$ has two components, one of which is an isolated vertex;

(e) $CQ_6^1 - F_1$ has three components, two of which are isolated vertices;
(f) $CQ_6^1 - F_1$ has four components, three of which are isolated vertices;
(g) $CQ_6^1 - F_1$ has three components, one of which is an isolated vertex and the other is a K_2.

Since $2^5 - 14 - 4 \geq 1$, by Proposition 2.1, $CQ_6[V(B_0) \cup V(B_1)]$ is connected. Thus, $CQ_6 - F$ satisfies one of the conditions (1)-(10).

Case 2. $|F_0| = 4$.

By Lemma 3.1, $CQ_6^0 - F_0$ is connected.

In this case, $10 \leq |F_1| \leq 11$.

Case 2.1. $|F_1| = 10$.

Note that $|F_1| = 3(n-1) - 5 = 10$. By Lemma 3.16, $CQ_6^1 - F_1$ satisfies one of the conditions (a)-(g) in Case 1.3. Since $2^5 - 14 - 3 \geq 1$, by Proposition 2.1, $CQ_6[V(B_0) \cup V(B_1)]$ is connected. Thus, $CQ_6 - F$ satisfies one of the conditions (1)-(10).

Case 2.2. $|F_1| = 11$.

Suppose that $CQ_6^1 - F_1$ is connected. Since $2^5 - 15 \geq 1$, by Proposition 2.1, $CQ_6 - F$ is connected. Then we suppose that $CQ_6^1 - F_1$ is disconnected. Let the components of $CQ_6^1 - F_1$ be C_1, C_2, ... , C_k $(k \geq 2)$. Note that $|F_0| = 4$. If every component C_i of $CQ_6^1 - F_1$ such that $|V(C_i)| \geq 5$ for $i \in \{1, \ldots, k\}$, then $CQ_6[V(CQ_6^0 - F_0) \cup V(C_1) \cup \cdots \cup V(C_k)] = CQ_6 - F$ is connected. Suppose that there is a components C_i such that $|V(C_i)| \leq 4$. If $N_{CQ_6}(V(C_i)) \cap V(CQ_6^0) \subseteq F_0$, then C_i is a component of $CQ_6 - F$.

When $|V(C_i)| = 4$, C_i is a 4-cycle, 3-path or $K_{1,3}$. Since C_i is also a component of $CQ_6^1 - F_1$, we have $N_{CQ_6^1}(V(C_i)) \subseteq F_1$.

If C_i is a 4-cycle, then $|N_{CQ_6^1}(V(C_i))| = 4(n-1-2) = 4 \times (5-2) = 12$. Note that $|N_{CQ_6^1}(V(C_i))| = 12 > 11 = |F_1|$. We get $N_{CQ_6^1}(V(C_i))$ Ú F_1, a contradiction. So C_i is not a 4-cycle. We get that C_i may be a 3-path or $K_{1,3}$. Thus, $CQ_6 - F$ satisfies the condition (3) or (5). When $|V(C_i)| \leq 3$, $CQ_6 - F$ satisfies one of the conditions (1)-(10).

Case 3. $|F_0| \leq 3$.

By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Let the components of $CQ_6^1 - F_1$ be C_1, C_2, ..., C_k $(k \geq 2)$. If every component C_i of $CQ_6^1 - F_1$ such that $|V(C_i)| \geq 4$ for $i \in \{1, \ldots, k\}$, then $CQ_6[V(CQ_6^0 - F_0) \cup V(C_1) \cup \cdots \cup V(C_k)] = CQ_6 - F$ is connected. Suppose that there is a components C_i such that $|V(C_i)| \leq 3$. If $N_{CQ_6}(V(C_i)) \cap V(CQ_6^0) \subseteq F_0$, then C_i is a component of $CQ_6 - F$. Thus, $CQ_6 - F$ satisfies one of the conditions (1)-(10). The proof is complete.

Lemma 3.19. Let $F \subseteq V(CQ_n)$ $(n \geq 6)$. If $3n - 4 \leq |F| \leq 4n - 9$, then $CQ_n - F$ satisfies one of the following conditions:
(1) $CQ_n - F$ is connected;
(2) $CQ_n - F$ has two components, one of which is a K_2;
(3) $CQ_n - F$ has two components, one of which is a $K_{1,3}$;
(4) $CQ_n - F$ has two components, one of which is a 2-path;
(5) $CQ_n - F$ has two components, one of which is a 3-path;
(6) $CQ_n - F$ has two components, one of which is an isolated vertex;
(7) $CQ_n - F$ has three components, two of which are isolated vertices;
(8) $CQ_n - F$ has four components, three of which are isolated vertices;
(9) $CQ_n - F$ has three components, one of which is an isolated vertex and the other is a K_2;
(10) $CQ_n - F$ has three components, one of which is an isolated vertex and the other is a 2-path.

Proof. We prove the lemma by induction on n. By Lemma 3.18, the lemma is true for $n = 6$. We assume that the lemma is true for $n-1$, i.e., if $3n - 7 \leq |F| \leq 4n - 13$, then $CQ_{n-1} - F$ satisfies one of the conditions (1)-(10). Now we show that the lemma is also true for n $(n \geq 7)$. We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0 = F \cap V(CQ_n^0)$ and $F_1 = F \cap V(CQ_n^1)$ with $|F_0| \leq |F_1|$. Let B_i be the maximum component of $CQ_n^i - F_i$ (If $CQ_n^i - F_i$ is connected, then let $B_i = CQ_n^i - F_i$) for $i \in \{0,1\}$. Since $3n - 4 \leq |F| \leq 4n - 9$, we have

$$0 \leq |F_0| \leq \left\lfloor \frac{4n-9}{2} \right\rfloor = 2n - 5 \text{ and}$$

$$n + 1 \leq \left\lceil \frac{3n-4}{2} \right\rceil \leq |F_1| \leq 4n - 9 \quad (n \geq 7).$$

We consider the following cases.

Case 1. $n + 1 \leq |F_1| \leq 2n - 5$.

Note that $|F_0| \leq |F_1| \leq 2n - 5$. By Lemma 3.4, $CQ_n^i - F_i$ $(i \in \{0,1\})$ is connected or has two components, one of which is an isolated vertex. Since $2^{n-1} - (4n - 9) - 2 \geq 1$ $(n \geq 7)$, by Proposition 2.1, $CQ_n[V(B_0) \cup V(B_1)]$ is connected. Thus, $CQ_n - F$ satisfies one of the conditions (1)-(10).

Case 2. $2n - 4 \leq |F_1| \leq 3n - 9$.

Note that $|F_0| \leq 4n - 9 - (2n - 4) = 2n - 5$. By Lemma 3.4, $CQ_n^0 - F_0$ is connected or has two components, one of which is an isolated vertex. By Lemma 3.5, $CQ_n^1 - F_1$ satisfies one of the following conditions:
(a) $CQ_n^1 - F_1$ is connected;
(b) $CQ_n^1 - F_1$ has two components, one of which is a K_2;
(c) $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex;

(d) $CQ_n^1 - F_1$ has three components, two of which are isolated vertices.

Since $2^{n-1}-(4n-9)-3 \ge 1$ $(n \ge 7)$, by Proposition 2.1, $CQ_n[V(B_0) \cup V(B_1)]$ is connected. Thus, $CQ_n - F$ satisfies one of the conditions (1)-(10).

Case 3. $|F_1| = 3n-8$.

In this case, $|F_0| \le 4n-9-(3n-8) = n-1$. By Lemma 3.4, $CQ_n^0 - F_0$ is connected or has two components, one of which is an isolated vertex. Note that $|F_1| = 3n-8 = 3(n-1)-5$. By Lemma 3.17, $CQ_n^1 - F_1$ satisfies one of the following conditions:

(a) $CQ_n^1 - F_1$ is connected;

(b) $CQ_n^1 - F_1$ has two components, one of which is a K_2;

(c) $CQ_n^1 - F_1$ has two components, one of which is a 2-path;

(d) $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex;

(e) $CQ_n^1 - F_1$ has three components, two of which are isolated vertices;

(f) $CQ_n^1 - F_1$ has four components, three of which are isolated vertices;

(g) $CQ_n^1 - F_1$ has three components, one of which is an isolated vertex and the other is a K_2.

Since $2^{n-1}-(4n-9)-4 \ge 1$ $(n \ge 7)$, by Proposition 2.1, $CQ_n[V(B_0) \cup V(B_1)]$ is connected. Thus, $CQ_n - F$ satisfies one of the conditions (1)-(10).

Case 4. $3n-7 \le |F_1| \le 4n-13$.

In this case, $|F_0| \le 4n-9-(3n-7) = n-2$. By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Note that $3(n-1)-4 = 3n-7 \le |F_1| \le 4n-13 = 4(n-1)-9$. By the inductive hypothesis, $CQ_n^1 - F_1$ satisfies one of the conditions (1)-(10). Since $2^{n-1}-(4n-9)-4 \ge 1$ $(n \ge 7)$, by Proposition 2.1, $CQ_n[V(B_0) \cup V(B_1)]$ is connected. Thus, $CQ_n - F$ satisfies one of the conditions (1)-(10).

Case 5. $4n-12 \le |F_1| \le 4n-9$.

In this case, $|F_0| = |F| - |F_1| \le (4n-9)-(4n-12) = 3$. By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Suppose that $CQ_n^1 - F_1$ is connected. Since $2^{n-1}-(4n-9) \ge 1$ $(n \ge 7)$, $CQ_n[V(CQ_n^0 - F_0) \cup V(CQ_n^1 - F_1)] = CQ_n - F$ is connected. So we suppose that $CQ_n^1 - F_1$ is disconnected. Let the components of $CQ_n^1 - F_1$ be $C_1, C_2, \ldots, C_k$ $(k \ge 2)$. Note that $|F_0| \le 3$. If every component C_i such that $|V(C_i)| \ge 4$ $(1 \le i \le k)$, then $CQ_n[V(CQ_n^0 - F_0) \cup V(C_1) \cup \cdots \cup V(C_k)] = CQ_n - F$ is connected. Suppose that there exists one component C_i such that $|V(C_i)| \le 3$. If $N_{CQ_n}(V(C_i)) \cap V(CQ_n^0) \subseteq F_0$, then C_i is a component of $CQ_n - F$. Thus, $CQ_n - F$ satisfies one of the conditions (1)-(10). The proof is complete.

Lemma 3.20. The 2-good-neighbor connectivity $\kappa^{(2)}(CQ_n) \ge 4n-8$ for $n \ge 4$.

Proof. Let F be the minimum 2-good-neighbor cut of $CQ_n - F$. If $n = 4$ and $|F| \le 7$, then F is not a 2-good-neighbor cut of $CQ_4 - F$ by Lemmas 3.4, 3.13 and 3.15. If $n = 5$ and $|F| \le 11$, then F is not a 2-good-neighbor cut of $CQ_5 - F$ by Lemmas 3.4, 3.5, 3.14 and 3.16. If $n \ge 6$ and $|F| \le 4n-9$, then F is not a 2-good-neighbor cut of $CQ_n - F$ by Lemma 3.19. Thus, $|F| \ge 4n-8$. By the definition of 2-good-neighbor connectivity, $\kappa^{(2)}(CQ_n) = |F| \ge 4n-8$. The proof is complete.

Combining Lemmas 3.12 and 3.20, we have the following theorem.

Theorem 3.1. Let CQ_n be the crossed cube. Then $\kappa^{(2)}(CQ_n) = 4n-8$ for $n \ge 4$.

Theorem 3.2. For $n \ge 6$, the crossed cube CQ_n is tightly $(4n-8)$ super 2-good-neighbor connected.

Proof. Now we consider CQ_n for any minimum 2-good-neighbor cut $F \subseteq V(CQ_n)$. By Theorem 3.1, $|F| = 4n-8$. We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0 = F \cap V(CQ_n^0)$ and $F_1 = F \cap V(CQ_n^1)$ with $|F_0| \le |F_1|$.

Then $|F_1| \ge \frac{4n-8}{2} = 2n-4$. We consider the following cases.

Case 1. $|F_1| = 2n-4$.

In this case, $|F_0| = |F_1| = 2n-4 = 2(n-1)-2$. By Lemma 3.5, $CQ_n^i - F_i$ $(i \in \{0,1\})$ satisfies one of the following conditions:

(1) $CQ_n^i - F_i$ is connected;

(2) $CQ_n^i - F_i$ has two components, one of which is a K_2;

(3) $CQ_n^i - F_i$ has two components, one of which is an isolated vertex;

(4) $CQ_n^i - F_i$ has three components, two of which are isolated vertices.

When $CQ_n^i - F_i$ $(i \in \{0,1\})$ satisfies the condition (2), let $u_i v_i$ be the component K_2 and let B_i be the other component of $CQ_n^i - F_i$. Since $2^{n-1}-(4n-8)-4 \ge 1$ $(n \ge 6)$, by Proposition 2.1, $CQ_n[V(B_0) \cup V(B_1)]$ is connected. If $CQ_n[\{u_0, v_0, u_1, v_1\}]$ is a 4-cycle, then $CQ_n - F$ has two components, one of which is a 4-cycle and the other is $CQ_n[V(B_0) \cup V(B_1)]$. Thus, CQ_n is tightly $(4n-8)$ super 2-good-neighbor connected. Otherwise, F is not a 2-good-neighbor cut of CQ_n.

Case 2. $2n-3 \le |F_1| \le 3n-9$.

In this case, $|F_0| \le 4n-8-(2n-3) = 2n-5$. By Lemma 3.4, $CQ_n^0 - F_0$ is connected or has two components, one of which is an isolated vertex. By Lemma 3.5, $CQ_n^1 - F_1$ satisfies one of the following conditions:

(1) $CQ_n^1 - F_1$ is connected;

(2) $CQ_n^1 - F_1$ has two components, one of which is a K_2;

(3) $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex;

(4) $CQ_n^1-F_1$ has three components, two of which are isolated vertices.

By Proposition 2.1, F is not a 2-good-neighbor cut of CQ_n.

Case 3. $|F_1|=3n-8$

In this case, $|F_0|=4n-8-(3n-8)=n$. By Lemma 3.4, $CQ_n^0-F_0$ is connected or has two components, one of which is an isolated vertex. Note that $|F_1|=3n-8=3(n-1)-5$. By Lemma 3.17, $CQ_n^1-F_1$ satisfies one of the following conditions:

(1) $CQ_n^1-F_1$ is connected;

(2) $CQ_n^1-F_1$ has two components, one of which is a K_2;

(3) $CQ_n^1-F_1$ has two components, one of which is a 2-path;

(4) $CQ_n^1-F_1$ has two components, one of which is an isolated vertex;

(5) $CQ_n^1-F_1$ has three components, two of which are isolated vertices;

(6) $CQ_n^1-F_1$ has four components, three of which are isolated vertices;

(7) $CQ_n^1-F_1$ has three components, one of which is an isolated vertex and the other is a K_2.

By Proposition 2.1, F is not a 2-good-neighbor cut of CQ_n.

Case 4. $3n-7\le|F_1|\le 4n-13$.

In this case, $|F_0|=4n-8-(3n-7)=n-1$. By Lemma 3.4, $CQ_n^0-F_0$ is connected or has two components, one of which is an isolated vertex. By Lemma 3.19, $CQ_n^1-F_1$ satisfies one of the following conditions:

(1) $CQ_n^1-F_1$ is connected;

(2) $CQ_n^1-F_1$ has two components, one of which is a K_2;

(3) $CQ_n^1-F_1$ has two components, one of which is a $K_{1,3}$;

(4) $CQ_n^1-F_1$ has two components, one of which is a 2-path;

(5) $CQ_n^1-F_1$ has two components, one of which is a 3-path;

(6) $CQ_n^1-F_1$ has two components, one of which is an isolated vertex;

(7) $CQ_n^1-F_1$ has three components, two of which are isolated vertices;

(8) $CQ_n^1-F_1$ has four components, three of which are isolated vertices;

(9) $CQ_n^1-F_1$ has three components, one of which is an isolated vertex and the other is a K_2;

(10) $CQ_n^1-F_1$ has three components, one of which is an isolated vertex and the other is a 2-path.

By Proposition 2.1, F is not a 2-good-neighbor cut of CQ_n.

Case 5. $|F_1|=4n-12$

In this case, $|F_0|=4n-8-(4n-12)=4$. By Lemma 3.1, $CQ_n^0-F_0$ is connected. By Proposition 2.1, there are four vertices v_1, v_2, v_3, v_4 in CQ_n^1 such that $N_{CQ_n}(\{v_1,v_2,v_3,v_4\})\cap V(CQ_n^0)=F_0$. Suppose that $CQ_n[\{v_1,v_2,v_3,v_4\}]$ is a 4-cycle in $CQ_n^1-F_1$. Let $C=CQ_n[\{v_1,v_2,v_3,v_4\}]$. Then C is a 2-good-neighbor component in CQ_n-F. By Proposition 2.1, $CQ_n[V(CQ_n^0-F_0)\cup V(CQ_n^1-F_1-V(C))]$ is connected. Thus, CQ_n-F has two components, one of which is a 4-cycle and the other is $CQ_n[V(CQ_n)-F-V(C)]$. Thus, CQ_n is tightly $(4n-8)$ super 2-good-neighbor connected. Otherwise, CQ_n is not tightly $(4n-8)$ super 2-good-neighbor connected.

Case 6. $4n-11\le|F_1|\le 4n-9$.

In this case, $|F_0|\le 4n-8-(4n-11)=3$. By Proposition 2.1, there are at most three vertices in $CQ_n^1-F_1$ such that they are connected to F_0. By Lemma 3.8, there is not a 2-good-neighbor component in CQ_n-F. This is a contradiction to that F is a 2-good-neighbor cut of CQ_n.

Case 7. $|F_1|=4n-8$

In this case, $|F_0|=0$. By Proposition 2.1, CQ_n-F is connected. This is a contradiction to that F is a 2-good-neighbor cut of CQ_n. The proof is complete.

IV. THE 2-GOOD-NEIGHBOR DIAGNOSAILITY OF THE CROSSED CUBE CQ_n UNDER THE PMC MODEL

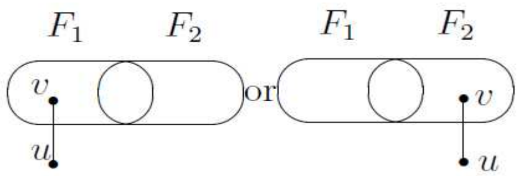

Fig. 2 Illustration of a distinguishable pair (F_1,F_2) under the PMC model

Theorem 4.1 [19]. A system $G=(V,E)$ is g-good-neighbor t-diagnosable under the PMC model if and only if there is an edge $uv\in E$ with $u\in V\setminus(F_1\cup F_2)$ and $v\in F_1\Delta F_2$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of $V(CQ_n)$ with $|F_1|\le t$ and $|F_2|\le t$ (see Fig.2).

Lemma 4.1. Let $n\ge 4$. Then the 2-good-neighbor diagnosability of the crossed cube CQ_n under PMC model is less than or equal to $4n-5$, i.e., $t_2(CQ_n)\le 4n-5$.

Proof. Let A be defined in Lemma 3.11, $F_1=N_{CQ_n}(A)$ and $F_2=A\cup N_{CQ_n}(A)$. By Lemma 3.11, $|F_1|=4n-8$, $|F_2|=4n-4$, F_1 is a 2-good-neighbor cut of CQ_n, and CQ_n-F_1 has two components CQ_n-F_2 and $CQ_n[A]$. Thus, F_1 and F_2 are both 2-good-neighbor faulty sets of CQ_n with $|F_1|=4n-8$ and $|F_2|=4n-4$. Since $A=F_1\Delta F_2$ and $N_{CQ_n}(A)=F_1\subset F_2$, there is no edge of CQ_n between $V(CQ_n)\setminus(F_1\cup F_2)$ and $F_1\Delta F_2$. By Theorem 4.1, CQ_n is not 2-good-neighbor $(4n-4)$-diagnosable under PMC model. By the definition of 2-good-neighbor diagnosability, we can deduce that the 2-good-neighbor diagnosability of CQ_n is less than or equal to $4n-5$, i.e., $t_2(CQ_n)\le 4n-5$. The proof is complete.

Lemma 4.2. Let $n \ge 5$. Then the 2-good-neighbor diagnosability of the crossed cube CQ_n under PMC model is more than or equal to $4n-5$, i.e., $t_2(CQ_n) \ge 4n-5$.

Proof. By the definition of 2-good-neighbor diagnosability, it is sufficient to show that CQ_n is 2-good-neighbor $(4n-5)$-diagnosable. By Theorem 4.1, we need to prove that there is an edge $uv \in E$ with $u \in V(CQ_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of $V(CQ_n)$ with $|F_1| \le 4n-5$ and $|F_2| \le 4n-5$.

Suppose, on the contrary, that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of $V(CQ_n)$ with $|F_1| \le 4n-5$ and $|F_2| \le 4n-5$, but there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \ne \varnothing$. Suppose that $V(CQ_n) = F_1 \cup F_2$. $2^n = |V(CQ_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le |F_1| + |F_2| \le 2(4n-5) = 8n-10$, a contradiction to $n \ge 5$. Therefore, $V(CQ_n) \ne F_1 \cup F_2$. Since there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$, $CQ_n - F_1$ has two parts $CQ_n \setminus (F_1 \cup F_2)$ and $CQ_n[F_2 \setminus F_1]$. Note that F_1 is a 2-good-neighbor faulty set. Thus, every component B_i of $CQ_n \setminus (F_1 \cup F_2)$ such that $\delta(B_i) \ge 2$ and every component C_i of $CQ_n[F_2 \setminus F_1]$ such that $\delta(C_i) \ge 2$. If $F_1 \setminus F_2 = \varnothing$, then $F_1 \cap F_2 = F_1$. Thus, $F_1 \cap F_2$ is a 2-good-neighbor faulty set. If $F_1 \setminus F_2 \ne \varnothing$, similarly, every component D_i of $CQ_n[F_1 \setminus F_2]$ such that $\delta(D_i) \ge 2$. Therefore, $F_1 \cap F_2$ is also a 2-good-neighbor faulty set. Since there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut of CQ_n. By Theorem 3.1, $|F_1 \cap F_2| \ge 4n-8$. Since $\delta(CQ_n[F_2 \setminus F_1]) \ge 2$, by Lemma 3.8, $|F_2 \setminus F_1| \ge 4$. Thus, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \ge 4 + 4n - 8 = 4n-4$. This is a contradiction to that $|F_2| \le 4n-5$. Therefore, CQ_n is 2-good-neighbor $(4n-5)$-diagnosable, i.e., $t_2(CQ_n) \ge 4n-5$. The proof is complete.

Combining Lemmas 4.1 and 4.2, we have the following theorem.

Theorem 4.2. Let $n \ge 5$. Then the 2-good-neighbor diagnosability of the crossed cube CQ_n under PMC model is $4n-5$, i.e., $t_2(CQ_n) = 4n-5$.

V. THE 2-GOOD-NEIGHBOR DIAGNOSAILITY OF THE CROSSED CUBE CQ_n UNDER THE MM* MODEL

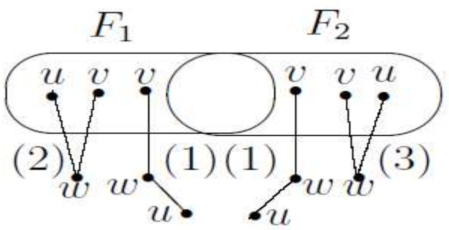

Fig. 3 Illustration of a distinguishable pair (F_1, F_2) under the MM* model

Theorem 5.1 [19]. A system $G=(V,E)$ is g-good-neighbor t-diagnosable under the MM* model if and only if each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \le t$ and $|F_2| \le t$ satisfies one of the following conditions (see Fig.3):

(1) There exist two vertices $u, w \in V(G) \setminus (F_1 \cup F_2)$ and there exists a vertex $v \in F_1 \Delta F_2$ such that $uw, vw \in E(G)$.

(2) There exist two vertices $u, v \in F_1 \setminus F_2$ and there exists a vertex $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw, vw \in E(G)$.

(3) There exist two vertices $u, v \in F_2 \setminus F_1$ and there exists a vertex $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw, vw \in E(G)$.

Lemma 5.1. Let $n \ge 4$. Then the 2-good-neighbor diagnosability of the crossed cube CQ_n under MM* model is less than or equal to $4n-5$, i.e., $t_2(CQ_n) \le 4n-5$.

Proof. Let A be defined in Lemma 3.11, $F_1 = N_{CQ_n}(A)$, and $F_2 = A \cup N_{CQ_n}(A)$. By Lemma 3.11, $|F_1| = 4n-8$, $|F_2| = 4n-4$, F_1 is a 2-good-neighbor cut of CQ_n, and $CQ_n - F_1$ has two components $CQ_n - F_2$ and $CQ_n[A]$. Thus, F_1 and F_2 are both 2-good-neighbor faulty sets of CQ_n with $|F_1| = 4n-8$ and $|F_2| = 4n-4$. Since $A = F_1 \Delta F_2$ and $N_{CQ_n}(A) = F_1 \subset F_2$, there is no edge of CQ_n between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Theorem 5.1, CQ_n is not 2-good-neighbor $(4n-4)$-diagnosable under MM* model. By the definition of 2-good-neighbor diagnosability, we can deduce that the 2-good-neighbor diagnosability of CQ_n is less than $4n-4$, i.e., $t_2(CQ_n) \le 4n-5$. The proof is complete.

Lemma 5.2. Let $n \ge 5$. Then the 2-good-neighbor diagnosability of the crossed cube CQ_n under MM* model is more than or equal to $4n-5$, i.e., $t_2(CQ_n) \ge 4n-5$.

Proof. By the definition of 2-good-neighbor diagnosability, it is sufficient to show that CQ_n is 2-good-neighbor $(4n-5)$-diagnosable. On the contrary, there are two distinct 2-good-neighbors faulty subsets F_1 and F_2 of CQ_n with $|F_1| \le 4n-5$ and $|F_2| \le 4n-5$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 5.1. Without loss of generality, assume that $F_2 \setminus F_1 \ne \varnothing$. Similarly to the discussion on $V(CQ_n) = F_1 \cup F_2$ in Lemma 4.2, we can deduce $V(CQ_n) \ne F_1 \cup F_2$.

Claim 1. $CQ_n - (F_1 \cup F_2)$ has no isolated vertex.

We suppose, on the contrary, that $CQ_n - (F_1 \cup F_2)$ has at least one isolated vertex w. Since F_1 is a 2-good-neighbor faulty set, there are two vertices u, v in $F_2 \setminus F_1$ such that w is adjacent to u and v. Thus, (F_1, F_2) is satisfied with condition (3). This contradicts with our hypothesis. Similarly to the discussion on F_2 is a 2-good-neighbor faulty set. Therefore, $CQ_n - (F_1 \cup F_2)$ has no isolated vertex. The proof of Claim 1 is complete.

Let $u \in V(CQ_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $CQ_n - (F_1 \cup F_2)$. Since (F_1, F_2) is not satisfied with any one condition in Theorem 5.1, u has no neighbor in $F_1 \Delta F_2$. By the arbitrariness of u, there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. Since F_1 and F_2 are two 2-good-neighbor faulty set, every component H_i of $CQ_n - (F_1 \cup F_2)$ has $\delta(H_i) \geq 2$, every component B_i of $CQ_n([F_2 \setminus F_1])$ has $\delta(B_i) \geq 2$, and every component C_i of $CQ_n([F_1 \setminus F_2])$ has $\delta(C_i) \geq 2$ when $F_1 \setminus F_2 \neq \emptyset$. Thus, $F_1 \cap F_2$ is also a 2-good-neighbor faulty set. Since $\delta(CQ_n[F_2 \setminus F_1]) \geq 2$, by Lemma 3.8, $|F_2 \setminus F_1| \geq 4$. Since there is no edge between $CQ_n - (F_1 \cup F_2)$ and $F_1 \Delta F_2$, we have $F_1 \cap F_2$ is a 2-good-neighbor cut of CQ_n. By Theorem 3.1, we have $|F_1 \cap F_2| \geq 4n-8$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 4 + (4n-8) = 4n-4$, which contradicts $|F_2| \leq 4n-5$. Therefore, CQ_n is 2-good-neighbor $(4n-5)$-diagnosable, i.e., $t_2(CQ_n) \geq 4n-5$. The proof is complete.

Combining Lemmas 5.1 and 5.2, we can get the following theorem.

Theorem 5.2. Let $n \geq 5$. Then the 2-good-neighbors diagnosability of the crossed cube CQ_n under MM* model is $4n-5$, i.e., $t_2(CQ_n) = 4n-5$.

IV. Conclusion

We prove that the 2-good-neighbor connectivity of CQ_n is $4n-8$ for $n \geq 4$. Moreover, CQ_n is tightly $(4n-8)$ super 2-good-neighbor connected for $n \geq 6$. Then we determine that the 2-good-neighbor diagnosability of CQ_n is $4n-5$ under the PMC model and MM* model for $n \geq 5$. On the basis of this study, the researchers can continue to study the g-good-neighbors connectivity and diagnosability.

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