# A New Approach To Homothetic Motions and Surfaces with Tessarines

**Faik BABADAG**

***Abstract*- This paper is a detailed study on homothetic motions, surfaces and Lie groups by considering the product and addition rules and conjugates of the tessarines which is given according to the arbitrary unit $i_1$ . To do these, we define a matrix that is similar to Hamilton operators and give some algebraic properties of this matrix. And then, we introduce different the surfaces and different hyperquadrics in $R_2^4$ with the help of the matrix. Also by using the homothetic motions we give some special subgroups of Lie groups and reparametrize the surfaces**

**The study gives some formulas, facts and properties about homothetic motion and Lie groups that are obtained by using tessarines product and addition in $R_2^4$, which are not generally known.**

***Index Terms*— Curves and Surfaces, Homothetic motion, Lie group, Tessarines;**

## I. INTRODUCTION

First time, James Cockle defined the tessarines in 1848, using more modern notation for complex numbers as a successor to complex numbers and algebra similar to the quaternions. The tessarines are coincided with 4 -dimensional vector space $R^4$ over real numbers. Cockle used tessarines to isolate the hyperbolic cosine series and the hyperbolic sine series in the exponential series. He also showed how zero divisors arise in tessarines, inspiring him to use the term "impossibles." The tessarines are now best known for their subalgebra of real tessarines t=w+yj, also called split-complex numbers, which express the parametrization of the unit hyperbola , Cockle (1848) .

Homothetic motion is general form of Euclidean motion. It is crucial that homothetic motions are regular motions. These motions have been studied in kinematic and differential geometry in recent years.

In this study, we describe the tessarines and give some algebraic properties of them. Then, we show a matrix that is similar to Hamilton operators and a hyperquadric Γ in $R_2^4$ acquired by this matrix. We determine two different types of homothetic motions by using two different orthonormal matrices in $R_2^4$ and obtain the surfaces $M_1$ and $M_2$ by means of these homothetic motions.

## II. PRELIMINARY

This section is devoted to some basic and fundamental concepts of tessaries. The tessarines are given by

$$W = w_0 + w_1 i + w_2 j + w_3 k$$

where the imaginary units $i_1, i_2$ and $i_3$ are governed by the rules:

$$i^2 = -1, j^2 = +1, k^2 = -1$$

and

$$ij = ji = k;\ ik = ki = -j;\ jk = kj = i.$$

Let $W$ and $U$ be tessarines. The addition, subtraction of these numbers are given by

$$W \mp U = (w_0 \mp u_0) + (w_1 \mp u_1)\, i + (w_2 \mp u_2)\, j + (w_3 \mp u_3)\, k$$

and multiplication of these numbers as follows

$$W.U = (w_0 + w_1 i + w_2 j + w_3 k).(u_0 + u_1 i + u_2 j + u_3 k) = \begin{cases} w_0u_0 - w_1u_1 - w_2u_2 + w_3u_3 \\ +i(w_0u_1 + w_1u_0 - w_2u_3 - w_3u_2) \\ +j(w_0u_2 + w_2u_0 - w_3u_1 - w_1u_3) \\ +k(w_0u_3 + w_3u_0 + w_1u_2 + w_2u_1). \end{cases}$$

It is easy to see that the multiplication of two tessarines is commutative. It is also convenient to write the set of tessarines as

$$T = \{W \mid W = w_0 + w_1 i + w_2 j + w_3 k \mid (w_{0-3}) \in R\}. \quad (2)$$

**Definition**: (The conjugate of the tessarine): The conjugate of the tessarine $W$ is shown by $W^*$ and also there are different conjugations of tessarines according to the imaginary units $i$, $j$ and $k$={$i$ and $k$} as follows:

1. $W^* = (w_0 - w_1 i) + j(w_2 - w_3 i), = w_0 - w_1 i + w_2 j - w_3 k.$
2. $W^* = (w_0 + w_1 i) - j(w_2 + w_3 i), = w_0 + w_1 i - w_2 j - w_3 k).$
3. $W^* = (w_0 - w_1 i) - j(w_2 - w_3 i), = w_0 - w_1 i - w_2 j + w_3 k.$

The conjugation of $W$ plays an important role both for algebraic and geometric properties for tessarines.

Multiplication of the tessarine with conjugate is given

1. $W W^* = w_0^2 + w_1^2 + w_2^2 + w_3^2 + 2j\,(w_0 w_2 + w_1 w_3),$
2. $W W^* = w_0^2 - w_1^2 - w_2^2 + w_3^2 + 2i\,(w_0 w_1 - w_2 w_3),$
3. $W W^* = w_0^2 + w_1^2 - w_2^2 - w_3^2 + 2k\,(w_0 w_3 - w_1 w_2).$

The system T is a commutative algebra. From equation (2), we can give the representation to show a mapping into 4x4 matrix as follows

$$\varphi: W \to \varphi(W),$$

$$\varphi(W) = \begin{bmatrix} w_0 & -w_1 & w_2 & -w_3 \\ w_1 & w_0 & w_3 & w_2 \\ w_2 & -w_3 & w_0 & -w_1 \\ w_3 & w_2 & w_1 & w_0 \end{bmatrix},$$

Yayli and Bükçü (1995), $T$ is algebraically isomorphic to the matrix algebra

$$A = \begin{bmatrix} w_0 & -w_1 & w_2 & -w_3 \\ w_1 & w_0 & w_3 & w_2 \\ w_2 & -w_3 & w_0 & -w_1 \\ w_3 & w_2 & w_1 & w_0 \end{bmatrix}$$

and $\varphi(W)$ is a faithful real matrix representation of $T$. Moreover, $\forall W, U \in T$ and $\forall \lambda \in R$, we obtain

$i.\ W = U \Leftrightarrow \varphi(W) = \varphi(U)$
$ii.\ \varphi(WU) \Leftrightarrow \varphi(W).\varphi(U)$
$iii.\ \varphi(\lambda W) = \lambda.\varphi(W)\ ;\ \lambda \in IR$
$iv.\ \varphi(1) = I_4$

## III. HOMOTHETIC MOTIONS AND LIE GROUPS WITH TESSARINES

Let us consider two different types of surfaces $M_1$ and $M_2$ as follows,

$$M_1 = \{W = (w_0, w_1, w_2, w_3) \mid w_0w_2 + w_1w_3 = 0\},$$
$$M_2 = \{W = (w_0, w_1, w_2, w_3) \mid w_0w_1 - w_2w_3 = 0\},$$

here by considering tessarines product and addition, if we denote

$$w_0 = \cos\rho, w_1 = \sin\rho, w_2 = 0, w_3 = 0$$

and

$$w_0 = \cosh\rho, w_1 = 0,\ w_2 = \sinh\rho, w_3 = 0,$$

respectively. The matrix forms of he surfaces $M_1$ and $M_2$ are defined by

$$H_1 = \begin{bmatrix} \cos\rho & -\sin\rho & 0 & 0 \\ \sin\rho & \cos\rho & 0 & 0 \\ 0 & 0 & \cos\rho & -\sin\rho \\ 0 & 0 & \sin\rho & \cos\rho \end{bmatrix}$$

and

$$H_1 = \begin{bmatrix} \cosh\rho & 0 & \sinh\rho & 0 \\ 0 & \cosh\rho & 0 & \sinh\rho \\ \sinh\rho & 0 & \cosh\rho & 0 \\ 0 & \sinh\rho & 0 & \cosh\rho \end{bmatrix},$$

respectively. We consider surfaces $M_1$ and $M_2$ by using the homothetic motions. Now let us defined as follow:

$$\varphi_1(\rho, s) = \mu(\rho) H_1\, \upsilon(s) + \gamma(\rho), \qquad (3)$$

$$\left\{ = \mu(\rho) \begin{bmatrix} \cos\rho & -\sin\rho & 0 & 0 \\ \sin\rho & \cos\rho & 0 & 0 \\ 0 & 0 & \cos\rho & -\sin\rho \\ 0 & 0 & \sin\rho & \cos\rho \end{bmatrix} \begin{bmatrix} \upsilon_1(s) \\ \upsilon_2(s) \\ \upsilon_3(s) \\ \upsilon_4(s) \end{bmatrix} + \begin{bmatrix} \gamma_1(\rho) \\ \gamma_2(\rho) \\ \gamma_3(\rho) \\ \gamma_4(\rho) \end{bmatrix} \right\}$$

and

$$\varphi_2(\rho, s) = \mu(\rho) H_2\, \upsilon(s) + \gamma(\rho), \qquad (4)$$

$$\varphi_1(t, s) =$$

$$\left\{ \mu(\rho) \begin{bmatrix} \cosh\rho & 0 & \sinh\rho & 0 \\ 0 & \cosh\rho & 0 & \sinh\rho \\ \sinh\rho & 0 & \cosh\rho & 0 \\ 0 & \sinh\rho & 0 & \cosh\rho \end{bmatrix} \begin{bmatrix} \upsilon_1(s) \\ \upsilon_2(s) \\ \upsilon_3(s) \\ \upsilon_4(s) \end{bmatrix} + \begin{bmatrix} \gamma_1(\rho) \\ \gamma_2(\rho) \\ \gamma_3(\rho) \\ \gamma_4(\rho) \end{bmatrix} \right\}.$$

Now let $\varphi_1 : M_1 \to E_2^4$ and $\varphi_2 : M_2 \to E_2^4$ be immersion of the surfaces, respectively. Here the homothetic scale of the motion is given as $h(\rho)$.

$\gamma(\rho) = (\gamma_1(\rho), \gamma_2(\rho), \gamma_3(\rho), \gamma_4(\rho))$ is the translation vector and a profile curve is defined by $\upsilon(s) = (\upsilon_1(s), \upsilon_2(s), \upsilon_3(s), \upsilon_4(s))$.

**Proposition**: Let $\varphi_1 : M_1 \to E_2^4$ be an immersion of a surface $M_1$ in the semi-Euclidean 4-space and then from equation(3), $M_1$ can be rewritten by using tessarins product and addition rules as

$$\varphi(\rho, s) = \delta_1(\rho) \times \upsilon(s) + \gamma(\rho),$$

where " $\times$ " tessarine product, " $+$ " tessarine addition,

$$\begin{cases} \delta_1(\rho) = (\mu(\rho)\cos\rho, \mu(\rho)\sin\rho, 0, 0), \\ \upsilon(s) = (\upsilon_1(s), \upsilon_2(s), \upsilon_3(s), \upsilon_4(s)) \end{cases}$$

are the curves and $\gamma(\rho) = (\gamma_1(\rho), \gamma_2(\rho), \gamma_3(\rho), \gamma_4(\rho))$ is the translation vector.

**Proof**: We can define the curves $\delta_1, \upsilon$ and the translation vector $\gamma(t)$ by using tessarines. Then we can redefine the curves $\delta, \upsilon$ as follows:

$\delta_1(t) = \mu(\rho)\cos\rho + (h(\rho)\sin\rho)i,$
$\upsilon(s) = \upsilon_1(s) + \upsilon_2(s)i + \upsilon_3(s)j + \upsilon_4(s)k,$
$\gamma(\rho) = \gamma_1(\rho) + \gamma_2(\rho)i + \gamma_3(\rho)j + \gamma_4(\rho)k.$

Using the tessarine product and addition, the surface $M_1$ is written as

$\varphi(\rho, s) = \delta_1(\rho) \times \upsilon(s) + \gamma(\rho).$

**Proposition**: Let $\varphi_2 : M_2 \to E_2^4$ be an immersion of a surface $M_2$ in the semi-Euclidean 4-space and assume that $M_2$ is a surface in equation (4), then $M_2$ can be rewritten as

$$\varphi(\rho, s) = \delta_2(\rho) \times \upsilon(s) + \gamma(\rho),$$
$$\begin{cases} \delta_2(\rho) = (\mu(\rho)\cosh\rho, 0, \mu(\rho)\sinh\rho, 0), \\ \upsilon(s) = (\upsilon_1(s), \upsilon_2(s), \upsilon_3(s), \upsilon_4(s)) \end{cases}$$

are the curves.

**Proof** We denote the curves $\upsilon, \delta_2$ and the translation vector $\gamma(\rho)$ by using tessarine. Then we can rewrite the curves $\upsilon, \delta_2$ as follows:

$$\delta_2(\rho) = \mu(\rho)\cosh\rho + (\mu(\rho)\sinh\rho)j,$$
$\upsilon(s) = \upsilon_1(s) + \upsilon_2(s)i + \upsilon_3(s)j + \upsilon_4(s)k,$

Using the tessarine product and addition, the surface $M_2$ is obtained as

$\varphi(\rho, s) = \delta_2(\rho) \times \upsilon(s) + \gamma(\rho).$

**Corollary**: Consider $H_1$ the matrix representation of tessarine

$$\delta_1(\rho) = \mu(\rho)\cos t + (\mu(\rho)\sin\rho)i.$$

Then we get the surface $M_1$ defined by in equations (3) and (5) as

$$\varphi(\rho, s) = H_1\upsilon(s) + \gamma(\rho).$$

**Corollary:** Consider $H_2$ the matrix representation of tessarine

$\delta_2(\rho) = \mu(\rho)\cosh\rho + (\mu(\rho)\sinh\rho)j$. Then we get the surface $M_2$ defined by in equations (4) and (6) as

$\varphi(\rho, s) = H_2\upsilon(s) + \gamma(\rho)$

The surfaces $M_1$, $M_2$ obtained by the parametrization from equatios (3) and (4) are rewritten as tessarines product of two curves in four dimensional semi-Euclidean space. Now we can rewrie the surfaces $M_1$, $M_2$ as tessarine product of a curve and a surface.

**Corollary:** Let $\varphi_1 : M_1 \to E_2^4$ be an immersion of a surface $M_1$ in the semi-Euclidean 4-space and $M_1$ defined by equation (3) is a surface. Then the surface $M_1$ can be rewritten by

$$\varphi(t, s) = \xi_1(t) \times r(t, s) + \gamma(\rho)$$

or

$$\varphi(t, s) = H_1(t)\, r(t, s) + \gamma(\rho),$$

where $\xi_1(\rho) = (\cot\rho, \sin\rho, 0,0)$ is a circle, $H_1(t)$ is the matrix represantation of curve $\xi_1$,

$r(\rho, s) = \mu(\rho)\upsilon(s)$ is a.

**Corollary:** Let $\varphi_2 : M_2 \to E_2^4$ be an immersion of a surface $M_2$ in the semi-Euclidean 4-space and given a surface, $M_2$, defined by equation (4). Then the surface $M_2$ can be rewritten by

$$\varphi(\rho, s) = \xi_2(\rho) \times r(\rho, s) + \gamma(\rho)$$

or

$$\varphi(\rho, s) = H_2(\rho) r(\rho, s) + \gamma(\rho),$$

where $\xi_2(\rho) = (\cosh\rho, 0, \sinh\rho, 0)$ is a circle, $H_2(t)$ is the matrix represantation of curve $\xi_2, r(\rho, s) = h(\rho)\upsilon(s)$ is a surface.

## IV. LIE GROUPS AND SOME SPECIAL SUBGROUPS WITH TESSARINES

In this section, Consider a hyperquadric $\Gamma$ and a unit hyperquadric sphere $S^3$, respectively as follows:

$$\begin{cases} \Gamma = \{X = (x_0, x_1, x_2, x_3) \neq 0 \mid \\ x_0x_1 - x_2x_3 = 0,; \ f(X,X) \neq 0\}, \end{cases} \quad (5)$$

and

$$\begin{cases} S^3 = \{X = (x_0, x_1, x_2, x_3) \mid \\ x_0^2 - x_1^2 - x_2^2 + x_3^2 = 1\}. \end{cases}$$

The set of tessarines can be given as follows:

$$\Gamma = \begin{Bmatrix} X = x^0 + x_1i_1 + x_2i_2 + x_3i_3 \mid \\ x_0x_1 - x_2x_3 = 0 \end{Bmatrix}, \quad (6)$$

Corresponding to the operator Γ is represented by the following matrix for tessarines:

$$\Gamma_1 = \{A = \begin{bmatrix} x_0 & -x_1 & x_2 & -x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & -x_3 & x_0 & -x_1 \\ x_3 & x_2 & x_1 & x_0 \end{bmatrix};$$

$$x_0x_1 - x_2x_3 = 0,; \ f(X,X) \neq 0\}.$$

Here $f(X,X)$ is pseudo Euclidean metric and it is given by

$$f(X,X) = x_0^2 - x_1^2 - x_2^2 + x_3^2$$

**Remark:** The norm of any element w on the hyperquadric $\Gamma_1$ is define by

$N_X = XX^* = f(X,X)$.

**Theorem:** In equations (3)and (4) given the set of $\Gamma$ together with tessarine product is a Lie group.

**Proof:** $\Gamma_1$ is a differentiable manifold and at the same time a group with group operation obtained by matrix multiplication with tessaries. Let the group function be $.: \Gamma_1 x\, \Gamma_1 \to \Gamma_1$. Then $(X, Y) \to XY$ is differentiable. Therefore, $(\Gamma, .)$ can be defined a Lie group in order to made a isomorphism f. We consider the set of all unit tessarines on Γ by $\Gamma_2$. In that case $\Gamma_2$ is given by

$$\Gamma_2 = \{X \in \Gamma;\ \|X\| = 1\}$$
$$= \{X \in \Gamma;\ x_0^2 - x_1^2 - x_2^2 + x_3^2 = 1.$$

We show the matrix form of the group $\Gamma_1$ by $\Gamma_3$ .

$$\Gamma_3 = \{X \in \Gamma_1;\ \|X\| = 1\} \quad (7)$$

with the group operation of tessarine product, $\Gamma_1$ compose of a subgroup of $\Gamma$.

**Lemma:** In equation (7), $\Gamma_1$ is 2-dimensional Lie subgroup of $\Gamma$.

**Theorem:** In equation (6), consider a curve $\xi$ which is obtained by using the homothetic motion with the homothetic function $h(t) = e^{\gamma t}$ and the profile curve

$\upsilon(t) = e^{\mu t}(\coth t; 0, \sinh t, 0)$,

where $\lambda, \mu$ are real constants. Then, in a Lie group $\Gamma$, a one-parameter subgroup are obtained by the curve $\xi$.

**Proof:** We can give the curve $\xi$ as follows:

$$\xi(t) = \left\{\begin{bmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & \cosh t & 0 & \sinh t \\ \sinh t & 0 & \cosh t & 0 \\ 0 & \sinh t & 0 & \cosh t \end{bmatrix}\begin{bmatrix} e^{\mu t}\cos t \\ 0 \\ e^{\mu t}\sin t \\ 0 \end{bmatrix}\right\}$$

$$= e^{(\lambda+\mu)t}(\cosh 2t, 0, \sinh 2t, 0).$$

It can be obtained that

$$\xi(t_1) \times \xi(t_2) = \xi(t_1 + t_2)$$

The unit element of curve $\xi$ is $\xi(0) = (1,0,0,0)$ and for all $t_1, t_2 \in R$ the invers element of $\xi^{-1}(t) = \xi(-t)$. So, $\xi$ is a one parameter Lie subgroup of Γ.

**Theorem:** In equation (5), consider a curve $\upsilon$ which is obtained by using the homothetic motion in equation with the homothetic function $h(t) = e^{\gamma t}$ and the profile curve $\upsilon(t) = e^{\mu t}(\cos t, \sin t, 0, 0)$, where $\gamma, \beta$ are real constants. Then the curve $\xi$ is a one-parameter Lie subgroup in Lie group $\Gamma$.

**Proof:** We can write the curve $\xi$ as follows:

$$\xi(t) = e^{\gamma t}\begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix}\begin{bmatrix} e^{\mu t}\cos t \\ 0 \\ e^{\mu t}\sin t \\ 0 \end{bmatrix}$$
$$= e^{(\lambda+\mu)t}(\cos 2t, \sin 2t, 0, 0).$$

It can be showed that

$$\xi(t_1) \times \xi(t_2) = \xi(t_1 + t_2).$$

The unit element of curve $\xi$ compose $\xi(0) = (1,0,0,0)$ and for all $t_1, t_2 \in R$ the invers element of $\xi^{-1}(t) = \xi(-t)$. Thus $\xi$ compose one parameter Lie subgroup of $\Gamma$.

**Theorem:** In equation (3), consider a curve υ which is obtained by using the homothetic motion in equation with the homothetic function $h(t) = e^{\gamma t}$ and the profile curve $\upsilon(t) = e^{\mu t}(\cosh t, 0, \sinh t, 0)$,

where $\lambda, \mu$ are real constants. Then a one-parameter subgroup in a Lie group $\Gamma$ are made up of a curve $\xi$.

**Proof:** We can write the curve $\xi$ as follows:

$$\xi(t) = \left\{e^{\gamma t}\begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix}\begin{bmatrix} e^{\mu t}\cosh t \\ 0 \\ e^{\mu t}\sinh t \\ 0 \end{bmatrix}\right\}$$

$$= e^{(\lambda+\mu)t}(\cos t\cosh t, \sin t\cosh t, \cos t\sinh t, \sin t\sinh t).$$

It can be showed that

$$\xi(t_1) \times \xi(t_2) = \xi(t_1 + t_2)$$

for all $t_1, t_2 \in R$. Thus $\xi$ is one parameter Lie subgroup of $\Gamma$.

**Corollary:** In equation (4), consider a curve $\xi$ which is obtained by using the homothetic motion in equation with the homothetic function $h(t) = 1$ and the profile curve $\upsilon(t) = (\cosh t, 0, \sinh t, 0)$. Then the curve ξ compose a one-parameter Lie subgroup in Lie group $\Gamma_2$.

**Proof:** By using homothetic motion is given by (4) for $h(t) = 1$ and the profile curve

$\upsilon(t) = (\cosh t, 0, \sinh t, 0)$,

we get

$\xi(t) = (\cosh 2t, 0, \sinh 2t, 0)$.

Since $\|\xi(t)\| = 1$, it implies that $\xi(t) \subset \Gamma_2$. So it is a one-parameter Lie subgroup of $\Gamma_2$.

**Corollary:** In equation (3), consider a curve ξ which is obtained by using the homothetic motion in equation with the homothetic function $h(t) = 1$ and the profile curve $\upsilon(t) = (\cos t, \sin t, 0, 0)$.

Then curve ξ compose a one-parameter Lie subgroup in Lie group $\Gamma_2$.

**Proof:** By using homothetic motion is given by (3) for $h(t) = 1$ and the profile curve $\upsilon(t) = (\cos t, \sin t, 0, 0)$, we get $\xi(t) = (\cos 2t, \sin 2t, 0, 0)$. $\|\xi(t)\| = 1$, it implies that $\xi(t) \subset \Gamma_2$. So it is a one-parameter Lie subgroup of $\Gamma_2$.

**Corollary:** A curve which is obtained by using the homothetic motion given by equation (3) with the homothetic function $h(t) = 1$ and the profile curve $\upsilon(t) = (cosht, 0, sinht, 0)$ is $\xi$ . Then curve $\xi$ is a one-parameter Lie subgroup in Lie group $\Gamma_2$.

**Proof:** For $h(t) = 1$ and the profile curve $\upsilon(t) = (cosht, 0, sinht, 0)$, we get

$$\xi(t) = (costcosht, sintcosht, costsinht, sintsinht).$$

Since $\|\xi(t)\| = 1$, it implies that $\xi(t) \subset \Gamma_2$. So it is a one-parameter Lie subgroup of $\Gamma_2$.

**Theorem:** In equation (3) , let $M_1$ be a surface which is obtained by using the homothetic motion with the homothetic function $h(t) = e^{\alpha t}$ and the profile curve $\upsilon(t) = e^{\beta s}(coshs, 0, sinhs, 0)$. Then the surface $M_1$ compose 2-dimensional Lie subgroup of $\Gamma$.

**Proof:** We obtain

$$\xi(t) = e^{at}\begin{bmatrix} cost & -sint & 0 & 0 \\ sint & cost & 0 & 0 \\ 0 & 0 & cost & -sint \\ 0 & 0 & sint & cost \end{bmatrix}\begin{bmatrix} e^{bs}cosht \\ 0 \\ e^{bs}sinht \\ 0 \end{bmatrix}$$

$$= e^{(a+b)t}(cos\,tcoshs, sintcoshs, costsinhs, sintsinhs).$$

Since

$$\xi(t_1, s_1) \times \xi(t_2, s_2) = \xi(t_1 + t_2, s_1 + s_2) \text{ for all } t_1, t_2, s_1, s_2 \in R,$$

the closure property is satisfied. The identity element of $M_1$ is $\xi(0,0)=(1,0,0,0)$ and for all $t, s \in R$ and the inverse element of $M_1$ is $\xi^{-1}(t, s) = \xi(-t, -s)$. Then $M_1$ is a subgroup of $\Gamma$,On the other hand, since $M_1$ is a submanifold of $\Gamma$, it is a 2-dimensional Lie subgroup of $\Gamma$.

**Corollary:** In equation (3), let $M_1$ be a surface which is obtained by using the homothetic motion with the homothetic function $h(t) = 1$ and the profile curve $\upsilon(t) = (cosht, 0, sinht, 0)$. Then the surface $M_1$ is 2-dimensional Lie subgroup of $\Gamma_2$.

**Proof:** By using homothetic motion in equation (4) for $h(t) = 1$ and the profile curve $\upsilon(t)= (\cosh t,0,\sinh t,0)$ , we obtain

$$\xi(t, s) = (costcoshs, sintcoshs, costsinhs, sintsinhs).$$

Since $\|\xi(t, s)\| = 1$, it follows that $\xi(t, s) \subset \Gamma_2$. Hence the surface $M_1$ is a 2- dimensional Lie subgroup of $\Gamma_2$.

## CONCLUSION

Study give us further Contributions to Homothetic Motions and surfaces with Tessarines to be Lie groups and one parameter Lie subgroups.


## ACKNOWLEDGEMENTS

The author is very grateful to the editor and anonymous referees for their careful reading of the manuscript and valuable suggestions improving version of this article by using tessarines . This work was'n financially supported.


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