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A Prey-Predator Model with a Linear Cover the Prey

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***Abstract*— A two species syn-ecological model with a Prey and a predator is considered for analytical study. Here, the prey is protected with a linear cover to some extent from the predator and the predator is supplied with an additional food.**

First order coupled non-linear ordinary differential equations are used to form the model. Possible latent roots for the system are obtained and their stability established. The linearized equations are solved completely and results are noted.

***Index Terms*— Equilibrium points, Linear cover, Normal Study State, Prey, Predator, Stability, Trajectories.**

I. Introduction

Prey - Predator models were very popular in population dynamics since early 20th century. The interaction between prey and predators benefits predator and a loss for prey. The increase in prey population results increase in predator. When prey population decrease below a certain level, the predator would migrate to some other place in search of food. Some of the prey-predator models were discussed by Kapur [1], Michale Olinnck [2], May [3], Varma [4] Colinvaux [5], Freedman [6], Lakshmi Narayan K at.el. [7,8]. The model under consideration is characterized by a coupled first order non-linear ordinary differential equations. All the five equilibrium points of the model are identified and stability criteria are discussed..

II. Basic Equations:

Nomenclature:

N_1 , N_2 : Strength of species,

a_1, a_2 : natural growth rate of the species,

α_{11}, α_{22} : rates of mortality due to internal competition,

α_{12} : Prey's death rate due to attacks of predator,

α_{21} : growth rate of predator due to interaction with the prey,

Governing equations are

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12}[N_1 - (a + bN_1)]N_2 \qquad (2.1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21}[N_1 - (a + bN_1)]N_2 \qquad (2.2)$$

III. Latent Roots:

We have five latent roots.

I. Extinct point $\overline{N}_1 = 0; \overline{N}_2 = 0$ (3.1)

II. $\overline{N}_1 = \frac{a_1}{\alpha_{11}}; \overline{N}_2 = 0$ (3.2)

prey exists, predator extinct.

III. Interior state:

$$\overline{N}_1 = \frac{p}{2q}; \overline{N}_2 = \frac{2qr + \alpha_{21}(1-b)p}{2q\alpha_{22}} \qquad (3.3)$$

which possible when

$$p^2 = 4qa\alpha_{12}(a\alpha_{21} - a_2) \qquad (3.4)$$

IV. Interior state:

$$\overline{N}_1 = \frac{p^2 + a\alpha_{12}qr}{pq};$$

$$\overline{N}_2 = \frac{pqr + \alpha_{21}(1-b)[p^2 + a\alpha_{12}qr]}{\alpha_{22}pq} \qquad (3.5)$$

V. Interior state:

$$\overline{N}_1 = \frac{a\alpha_{12}(a\alpha_{21} - a_2)}{p};$$

$$\overline{N}_2 = \frac{pr - ar\alpha_{12}\alpha_{21}(1-b)}{\alpha_{22}p} \qquad (3.6)$$

The latent roots of equations (3.5) & (3.6) exists

If $p^2 > 4qa\alpha_{12}(a\alpha_{21} - a_2)$ (3.7)

Here

$$p = a_1\alpha_{22} - \alpha_{12}(1-b)(a_2 - 2a\alpha_{21});$$

$$q = \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-b)^2; \; r = a_2 - a\alpha_{21} \qquad (3.8)$$

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IV. THE STABILITY OF THE LATENT ROOTS:

Let $N = (N_1, N_2)^T = \overline{N} + U$

(4.1)

With U = $(u_1, u_2)^T$ is the perturbation over the latent root $\overline{N} = (\overline{N}_1, \overline{N}_2)^T$.

$$\frac{dU}{dt} = AU \qquad (4.2)$$

Where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2 & a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1 \\ \alpha_{21}(1-b)\overline{N}_2 & -\alpha_{22}\overline{N}_2 \end{bmatrix}$$

(4.3) the secular equation for the system is

$$det\left[A - \lambda I\right] = 0 \qquad (4.4)$$

It is stable only when the critical values negative in case they are real or complex with negative real parts.

4. 1. Stability of the latent root I:

The trajectories extinct state is

$$u_1 = a\alpha_{12}\frac{u_{20}\, e^{(a_2 - a\alpha_{21})t}}{a_2 - a\alpha_{21} - a_1} + \{u_{10} - \frac{a\alpha_{12}u_{20}}{a_2 - a\alpha_{21} - a_1}\}e^{a_1 t}$$

(4.5)

$$u_2 = u_{20}\, e^{(a_2 - a\alpha_{21})t}$$

(4.6)

Here u_{10}, u_{20} are the starting values of u_1, u_2.

The solution curves are given in figures 1 to 4

Case 1: predator's dominance throughout as shown in Fig.1

Case 2: Initially prey dominates, after some time situation reverses (i.e. $a_2 < a\alpha_{21}$ and $u_{10} > u_{20}$) as shown in Fig. 2

Case 3: Initially prey dominates, after some time situation reverses (i.e. $u_{10} < u_{20}$ and $a_2 > a\alpha_{21}$; $a_1 > a_2$) At

$$t = t^* = \frac{1}{(a_2 - a\alpha_{21} - a_1)}\left(\frac{u_{10}(a_2 - a\alpha_{21} - a_{11}) - a\alpha_{21}u_{20}}{u_{20}(a_2 - a\alpha_{21} - a_{11}) - a\alpha_{21}u_{20}}\right)$$

(4.7)

both are with equal strength as shown in fig 3

Case 4: Prey's dominance continues throughout as shown in fig 4.

4. 2. Trajectories of perturbed species for latent root I:

The trajectories in the $u_1 - u_2$ plane are given by

$$u_1 = \frac{cu_2^{\,p_3} - q_3 u_2}{p_3 - 1} \qquad (4.8)$$

Here $p_3 = \dfrac{a_1}{a_2 - a\alpha_{21}}$; $q_3 = \dfrac{a\alpha_{21}}{a_2 - a\alpha_{21}}$

(4.9)

and c is a constant. These are given in Fig.5.

4.3. Stability of the latent root II:

The trajectories extinct state is

Case A: If $d_2 = 0 \Rightarrow b = 1$

$$u_1 = \left[\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}}\right]\frac{u_{20}}{a_1} + \left[u_{10} - \left\{\left(\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}}\right)\frac{u_{20}}{a_1}\right\}\right]e^{-a_1 t}$$

(4.10)

and

$$u_2 = u_{20} \qquad (4.11)$$

The results are given in figures 6 & 7.

CaseA$_1$: Initially prey dominates, after some time situation reverses (i.e. $u_{10} > u_{20}$). In the course of time u_1 is asymptotic to $u_1^* = \dfrac{a\alpha_{11}\alpha_{12}u_{20}}{a_1\alpha_{11}}$ which is given in Fig.6

$$t = t^* = \frac{1}{a_1} ln\left\{\frac{a_1 u_{10} - \left(\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}}\right)u_{20}}{u_{20} - \left(\frac{a\alpha_{11}\alpha_{12}}{\alpha_{11}}\right)u_{20}}\right\}$$

(4.12)

CaseA$_2$: Initially predator dominates, after some time situation reverses (*i.e.* $u_{10} < u_{20}$). In the course of time u_1 is asymptotic to

$u_1^* = \dfrac{a\alpha_{11}\alpha_{12}u_{20}}{a_1\alpha_{11}}$ as is given in Fig.7.

Case B: If $d_2 > 0 \Rightarrow b < 1$

$$u_1 = \left[\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}}\right]\frac{u_{20}e^{d_2 t}}{(d_2 + a_1)} + \left[u_{10} - \left\{\left(\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}}\right)\frac{u_{20}}{(d_2 + a_1)}\right\}\right]e^{-a_1 t}$$

(4.13)

$$u_2 = u_{20}\, e^{d_2 t} \qquad (4.14)$$

Here $d_2 = \dfrac{a_1\alpha_{21}(1-b)}{\alpha_{11}}$ (4.15)

The solution curves are given in the figures 8 to 11

Case B$_1$: Initially prey dominates, after some time situation reverses (i.e. $u_{10} > u_{20}$), as shown in Fig. 8

$$t = t^* = \frac{1}{a_1} ln\left\{\frac{(a_1 + d_2)u_{10} - \left(\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}}\right)u_{20}}{a_1 + d_2 - \left(\frac{a\alpha_{11}\alpha_{12} - a_1\alpha_{12}(1-b)}{\alpha_{11}}\right)}\right\}$$

(4.16)

Case B$_2$: The predator dominates the prey in natural growth and in its initial population strength. i.e. $u_{20} > u_{10}$; $d_2 > a_1$ as shown in Fig. 9

Case B$_3$: The predator dominates the prey in natural growth and in its initial population strength and $d_2 < a_1$. i.e. $u_{20} > u_{10}$ and $d_2 < a_1$ which is given in Fig. 10

Case B$_4$: Initially the prey dominates and $d_2 < a_1$ i.e. $u_{20} < u_{10}$. In this case, the prey out number the predator till the time-instant t* given by the equation (4.16), after which the predator out number the prey and grows unboundedly while the prey asymptotically approaches to the latent root $\overline{N}_1$ given in (3.2), as shown in Fig. 11.

Case C: $d_2 < 0 \Rightarrow b > 1$ the trajectories are as same in Case B, but the state is stable

The solution curves are given in the figures 12 & 13.

Case C$_1$: The predator dominates the prey in natural growth as well as in its initial population strength. i.e. $u_{10}<u_{20}$ However both converge asymptotically to the latent root $(\overline{N}_1, \overline{N}_2)$ given by (3.2). Hence the latent root is **stable** as shown in Fig. 12

Case C$_2$: The prey dominates the predator in its initial strength. i.e. $u_{10}>u_{20}$. In this case $u_1(t) = u_2(t)$ is possible at time t* given by (4.16) as shown in Fig. 13. Hence the latent root is **stable.**

4.4 Trajectories of perturbed species for latent root II:

The trajectories in the u_1-u_2 plane are given by

$$u_1 = \frac{c\,u_2^{\,p_2} - q_2 u_2}{p_2 - 1} \qquad (4.17)$$

here

$$p_2 = \frac{-a_1\alpha_{11}}{a_1\alpha_{21}(1-b)}; \quad q_2 = \frac{a\alpha_{11}\alpha_{22} - a_1\alpha_{12}(1-b)}{a_1\alpha_{21}(1-b)} \qquad (4.18)$$

and c = constant. The solution curves are given in Fig.14.

4.5. Stability of the latent root III:

The trajectories for the co-existence state are

$$u_1 = \left[\frac{\left(\lambda_1 + \alpha_{22}\overline{N}_2\right)u_{10} + \left(a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1\right)u_{20}}{\lambda_1 - \lambda_2}\right]e^{\lambda_1 t} + \left[\frac{\left(\lambda_2 + \alpha_{22}\overline{N}_2\right)u_{10} + \left(a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1\right)u_{20}}{\lambda_2 - \lambda_1}\right]e^{\lambda_2 t} \qquad (4.19)$$

$$u_2 = \left[\frac{\left\{\lambda_1 - \left(a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2\right)\right\}u_{20} + \alpha_{21}(1-b)\overline{N}_2 u_{10}}{\lambda_1 - \lambda_2}\right]e^{\lambda_1 t} + \left[\frac{\left\{\lambda_2 - \left(a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2\right)\right\}u_{20} + \alpha_{21}(1-b)\overline{N}_2 u_{10}}{\lambda_2 - \lambda_1}\right]e^{\lambda_2 t} \qquad (4.20)$$

The solution curves are given in figures 15 & 16

Case 1: The prey dominates the predator in natural growth as well as in its initial population strength i.e. $u_{10}>u_{20}$, which is given in Fig.15.

Case 2: The prey dominates the predator in natural growth rate but its initial strength is less than that of the predator i.e. $u_{10}<u_{20}$. In this case, the predator out number the prey till the time-instant t*, after which the prey out number the predator as shown in Fig.17.

$$t = t^* = \frac{1}{\lambda_1 - \lambda_2} ln\left\{\frac{(\lambda_2 - D_1) + B_1u_{20} - (\lambda_2 - A_1)u_{20} - C_1u_{10}}{(\lambda_1 - D_1) + B_1u_{20} - (\lambda_1 - A_1)u_{20} - B_1u_{10}}\right\} \qquad (4.21)$$

here

$$A_1 = a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2;\; B_1 = a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1;$$
$$C_1 = \alpha_{21}(1-b)\overline{N}_2; \quad D_1 = -\alpha_{22}\overline{N}_2 \qquad (4.22)$$

Case 3: If $(A_1 + D_1)^2 < 4(A_1 D_1 - B_1 C_1)$, (4.23)

the roots are complex with negative real part. Hence the latent root is **stable.** The solution curves are given in Fig.17

4.6. Trajectories of perturbed species for latent root III:

The trajectories in the u_1-u_2 plane are given by

$$c u_2 = \left(\frac{u_2}{u_1} - v_1\right)^{\left(\frac{\alpha_{12}(1-b)\overline{N}_2 v_1 - \alpha_{22}\overline{N}_2}{v_2 - v_1}\right)} \left(\frac{u_2}{u_1} - v_2\right)^{\left(\frac{\alpha_{12}(1-b)\overline{N}_2 v_2 - \alpha_{22}\overline{N}_2}{v_1 - v_2}\right)} \qquad (4.24)$$

here v_1 & v_2 are roots of the equation

$$\alpha_{12}(1-b)\overline{N}_2 v^2 - \left(a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-b)\overline{N}_2 + \alpha_{22}\overline{N}_2\right) v - a\alpha_{12} - \alpha_{12}(1-b)\overline{N}_1 = 0 \qquad (4.25)$$

If $(A_1 + D_1)^2 < 4(A_1 D_1 - B_1 C_1)$, the roots are complex with negative real part the curve is a concentric spiral as shown in Fig 18 Hence the latent root is **stable.**
One can easily find the similarities in the results for latent roots IV & V as observed in latent root III.

V. Trajectories

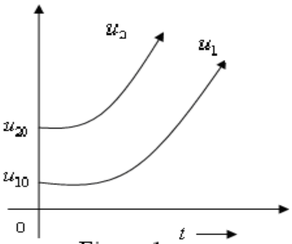


Figure.1

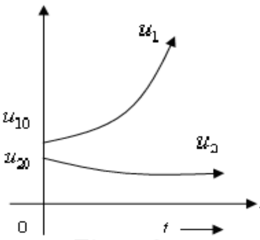


Figure.2

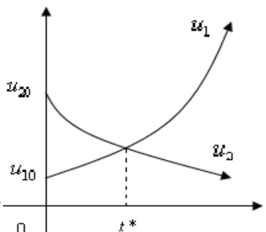


Figure.3

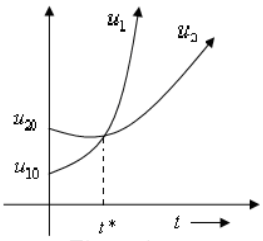


Figure.4

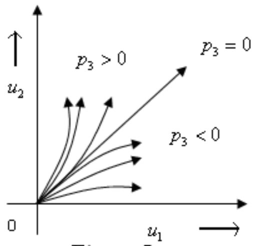


Figure.5

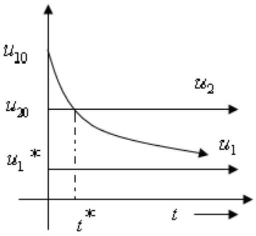
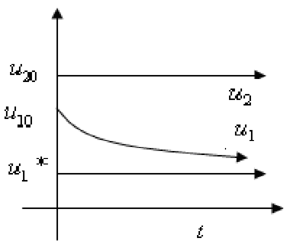


Figure.6



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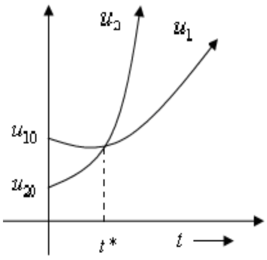

Figure.8

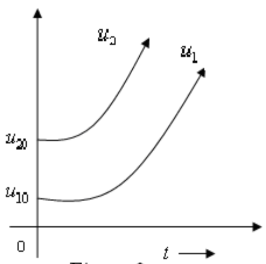

Figure.9

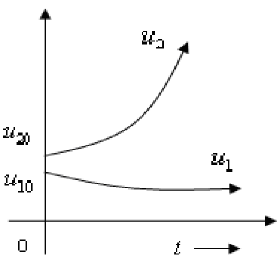

Figure.10

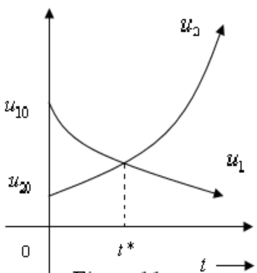

Figure.11

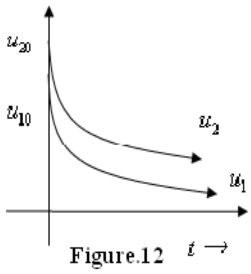

Figure.12

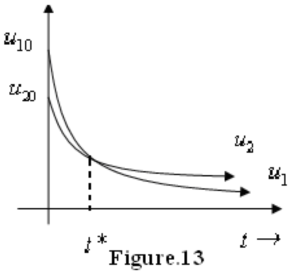

Figure.13

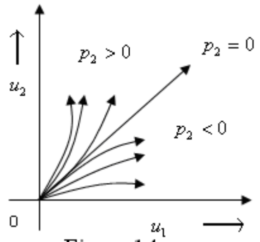

Figure.14

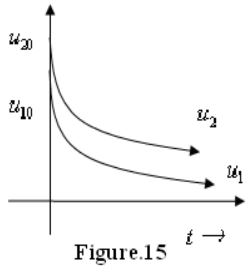

Figure.15

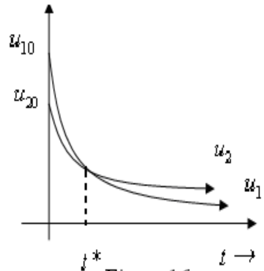

Figure.16

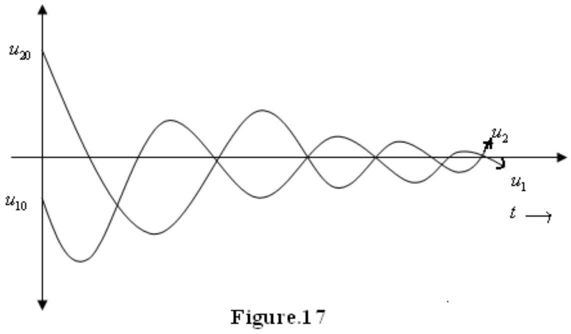

Figure.17

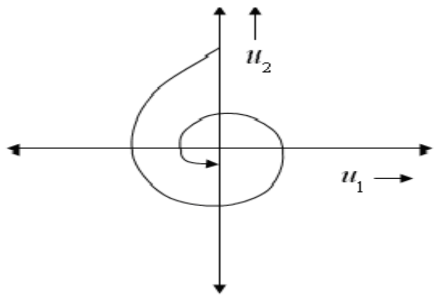

Figure.18

VI. Future Works

In the present paper it is investigated that a Prey-Predator model with a cover linearly varying with the size of prey is provided to protect it from the predator and the predator provided with an alternative food in addition to the prey. There is a scope to study the model by taking $a = 0$ or $b = 0$ or both can be taken as zeros. Also one can introduce harvesting in this problem. One can construct Lypunov's function to study the global stability of the model and also threshold results can be illustrated.

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