International Journal of New Technology and Research (IJNTR)
ISSN:2454-4116, Volume-3, Issue-11, November 2017 Pages 143-152

Regionally Gradient Efficient actuators and Sensors

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Abstract— **In this paper, we introduce and characterize the notions of regional gradient remediability and regionally gradient efficient actuators. We study their relationship with regional gradient controllability and sensors. As an application, we consider the case where the domain is one and two dimension.**

Index Terms—**Efficient actuators, gradient controllability, gradient remediability, sensors.**

I. INTRODUCTION

Systems analysis concerns a set of concepts that leads to a better knowledge and understanding of the system and their evolution. One of the most important concepts is the remediability, which consists in studying the existence of a convenient operator (efficient actuators) ensuring compensation of any disturbance acting in the considered systems. This problem, particularly motivated by pollution problems, has been introduced and studied first for a class of parabolic systems in the case of finite time horizon [1] and there for hyperbolic systems [2]. After, it was largely developed for other situations (boundary actions of disturbance, regional, asymptotic, regional asymptotic, enlarged and enlarged asymptotic cases) [3-7]. Subsequently, motivated by practical applications, the concept of remediability was extended to the gradient remediability for parabolic systems [8].

Since the importance of the so-called regional analysis, we have extended the gradient remediability to the regional case.

This paper is organized as follows: In the second paragraph, we start by presenting the problem statement. Then, we introduce the definitions of the new notions of exact regional gradient remediability and weak regional gradient remediability and their characterizations.

By analogy with the relation between the gradient remediability and the gradient controllability examined in previous work [8], it is then natural to study, in the paragraph 3, the relationship between the regional gradient remediability and the regional gradient controllability. We show that the regional gradient remediability is weaker and more supple than the regional gradient controllability of the parabolic systems, that is to say, if a parabolic system is regionally gradient controllable, then it is regionally gradient remediable.

In paragraph 4, we give a characterization of regional gradient remediability, which shows that the regional gradient remediability of any parabolic system may depend on the structure of the actuators and sensors. Then we introduce and we characterize the notion of regionally gradient efficient actuators. The paragraph 5 is consecrated to the applications in one and two space dimension.

II. REGIONAL GRADIENT REMEDIABILITY

A. *Problem Statement and definitions:*

Let Ω be an open and bounded subset of IR^n ($(n=1,2,3)$ with a regular boundary $\partial\Omega$. Fix $T>0$ and let denoted by $Q=\Omega\times\left]0,T\right[$, $\Sigma=\partial\Omega\times\left]0,T\right[$.

Consider the system described by the parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t)=A\,y(x,t)+Bu(t)+f(x,t) & Q \\ y(x,0)=y^0(x) & \Omega \\ y(\xi,t)=0 & \Sigma \end{cases} \tag{1}$$

where A is a second order linear differential operator, which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$. $(S^*(t))_{t\geq 0}$ is considered for the adjoint semi-group of $(S(t))_{t\geq 0}$.

$B\in\mathscr{L}(U,X), u\in L^2(0,T;U)$ where U is a Hilbert space representing the control space and $X=H_0^1(\Omega)$ the state space. The disturbance term $f\in L^2(0,T;X)$ is generally unknown.

The system (1) admits a unique solution $y\in C\left(0,T;H_0^1(\Omega)\right)\cap C^1\left(0,T;L^2(\Omega)\right)$ [9] given by

$$y_{u,f}(t)=S(t)y^0+\int_0^t S(t-s)Bu(s)ds+\int_0^t S(t-s)f(s)ds$$

Remark .1

The disturbance function f has a support that can be, in practical applications, a part ω of the domain Ω $(\omega\subset\Omega)$.

For $\omega\subset\Omega$ an open subregion of Ω with positive Lesbegue measure, we consider the operators

$$\chi_\omega:\left(L^2(\Omega)\right)^n\to\left(L^2(\omega)\right)^n$$
$$y\to y|_\omega$$

and

$$\tilde{\chi}_\omega:L^2(\Omega)\to L^2(\omega)$$
$$y\to y|_\omega$$

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while their adjoints, denoted by χ_ω^* and $\tilde{\chi}_\omega^*$ respectively, are defined by

$$\chi_\omega^* : \left(L^2(\omega)\right)^n \to \left(L^2(\Omega)\right)^n$$
$$y \to \chi_\omega^* y = \begin{cases} y & \text{on } \omega \\ 0 & \text{on } \Omega \setminus \omega \end{cases}$$

and

$$\tilde{\chi}_\omega^* : L^2(\omega) \to L^2(\Omega)$$
$$y \to \tilde{\chi}_\omega^* y = \begin{cases} y & \text{on } \omega \\ 0 & \text{on } \Omega \setminus \omega \end{cases}$$

Consider also the operator ∇ defined by

$$\nabla : H_0^1(\Omega) \to \left(L^2(\Omega)\right)^n$$
$$y \to \nabla y = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n}\right)$$

while ∇^* its adjoint operator.

The system (1) is augmented by the regional output equation

$$z_{u,f}^\omega(t) = C\chi_\omega \nabla y_{u,f}(t) \qquad (2)$$

where $C \in \mathscr{L}\left(\left(L^2(\omega)\right)^n, O\right)$, O is a Hilbert space (observation space). In the case of an observation on $[0,T]$ with q sensors, we take generally $O = IR^q$.

In the autonomous case, without disturbance ($f = 0$) and without control ($u = 0$) the regional observation on $[0,T]$ is given by

$$z_{0,0}^\omega(t) = C\chi_\omega \nabla S(t) y^0$$

It is then normal. However, if $f \neq 0$ and $u \neq 0$ the observation is particularly disturbed on ω .

The problem consists to study the existence of an input operator B (actuators), with respect to the output operator C (sensors) only on a given subregion ω , $\omega \subset \Omega$, ensuring the gradient compensation at finite time T , of any disturbance acting on the system, that is

For any $f \in L^2(0,T;X)$, there exists $u \in L^2(0,T;U)$ such that

$$z_{u,f}^\omega(t) = C\chi_\omega \nabla S(T) y^0$$

this is equivalent to

$$C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0$$

where H and F are two operators defined by

$$H : L^2(0,T;U) \to X$$
$$u \to Hu = \int_0^T S(T-s) Bu(s) ds$$

$$F : L^2(0,T;X) \to X$$
$$f \to Ff = \int_0^T S(T-s) f(s) ds$$

This leads to the following definitions

Definition .1

1- We say that the system (1) augmented by the output equation (2) is exactly regionally ω -gradient f - remediable on $[0,T]$, if there exists a control $u \in L^2(0,T;U)$ such that

$$C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0$$

2- We say that the system (1) augmented by the output equation (2) is weakly ω -gradient f - remediable on $[0,T]$, if for every $\varepsilon > 0$, there exists a control $u \in L^2(0,T;U)$ such that

$$\left\| C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff \right\|_{IR^q} < 0$$

3- We say that the system (1) augmented by the output equation (2) is exactly (resp. weakly) ω -gradient remediable on $[0,T]$, if for every $f \in L^2(0,T;X)$ the system (1)–(2) is exactly (resp. weakly) ω -gradient f - remediable on $[0,T]$.

B. *Characterizations*

We can characterize the notions of exact regional gradient remediability and of weak regional gradient remediability by the following propositions

Proposition .1

Let $f \in L^2(0,T;X)$

1- The system (1) – (2) is exactly ω -gradient f - remediable on $[0,T]$ if and only if

$$C\chi_\omega \nabla Ff \in \operatorname{Im}(C\chi_\omega \nabla H)$$

2- The system (1) – (2) is weakly ω -gradient f - remediable on $[0,T]$ if and only if

$$C\chi_\omega \nabla Ff \in \overline{\operatorname{Im}(C\chi_\omega \nabla H)}$$

Proof

1- We assume that the system (1) – (2) is exactly ω -gradient f - remediable on $[0,T]$, then there exists $u \in L^2(0,T;U)$ such that

$$C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0$$
$$C\chi_\omega \nabla Ff = -C\chi_\omega \nabla Hu = C\chi_\omega \nabla H(-u) = C\chi_\omega \nabla Hu_1$$

with $(u_1 = -u)$ $u_1 \in L^2(0,T;U)$ then $C\chi_\omega \nabla Ff \in \operatorname{Im} C\chi_\omega \nabla H$.

Conversely, we assume that $C\chi_\omega \nabla Ff \in \operatorname{Im} C\chi_\omega \nabla H$, then there exists $u \in L^2(0,T;U)$ such that

$C\chi_\omega \nabla Ff = C\chi_\omega \nabla Hu$ that is $C\chi_\omega \nabla Ff - C\chi_\omega \nabla Hu = 0$ this gives $C\chi_\omega \nabla Ff + C\chi_\omega \nabla H(-u) = 0$.

We put $u_1 = -u \in L^2(0,T;U)$ where the system (1) – (2) is exactly ω-gradient f - remediable on $[0,T]$.

2- We assume that the system (1) – (2) is weakly ω-gradient f - remediable on $[0,T]$,then
$\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$ such that
$\|C\chi_\omega \nabla Ff + C\chi_\omega \nabla Hu\|_{IR^q} < \varepsilon$, that is to say
$\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$ such that
$\|C\chi_\omega \nabla Ff - C\chi_\omega \nabla H(-u)\|_{IR^q} < \varepsilon$.
We put $u_1 = -u \in L^2(0,T;U)$, then
$\forall \varepsilon > 0, \exists u_1 \in L^2(0,T;U)$ such that
$\|C\chi_\omega \nabla Ff - C\chi_\omega \nabla Hu_1\|_{IR^q} < \varepsilon$, this gives
$C\chi_\omega \nabla Ff \in \overline{\mathrm{Im}(C\chi_\omega \nabla H)}$.

Conversely, we assume that $C\chi_\omega \nabla Ff \in \overline{\mathrm{Im}(C\chi_\omega \nabla H)}$, then

$\forall \varepsilon > 0, \exists u_1 \in L^2(0,T;U)$ such that

$\|C\chi_\omega \nabla Ff - C\chi_\omega \nabla Hu_1\|_{IR^q} < \varepsilon$.

We put $u_1 = -u \in L^2(0,T;U)$, then

$\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$ such that

$\|C\chi_\omega \nabla Ff + C\chi_\omega \nabla Hu\|_{IR^q} < \varepsilon$ where the system (1) – (2) is weakly ω-gradient f - remediable on $[0,T]$.

Proposition .2

1- The system (1) – (2) is exactly ω-gradient remediable on $[0,T]$ if and only if

$$\mathrm{Im}\, C\chi_\omega \nabla F \subset \mathrm{Im}\, C\chi_\omega \nabla H$$

2- The system (1) – (2) is weakly ω-gradient remediable on $[0,T]$ if and only if

$$\mathrm{Im}\, C\chi_\omega \nabla F \subset \overline{\mathrm{Im}\, C\chi_\omega \nabla H}$$

Proof

1- We assume that the system (1) – (2) is exactly

ω -gradient remediable on $[0,T]$ then $\forall f \in L^2(0,T;X)$ the system (1) – (2) is exactly ω-gradient f - remediable on $[0,T]$ and from the Proposition .1, we have, $\forall f \in L^2(0,T;X)$: $C\chi_\omega \nabla Ff \in \mathrm{Im}\, C\chi_\omega \nabla H$, this gives

$$\mathrm{Im}\, C\chi_\omega \nabla F \subset \mathrm{Im}\, C\chi_\omega \nabla H$$

Conversely, we assume that

$\mathrm{Im}\, C\chi_\omega \nabla F \subset \mathrm{Im}\, C\chi_\omega \nabla H$ and we show that the system (1)–(2) is exactly ω-gradient remediable on $[0,T]$. Let $f \in L^2(0,T;X)$ then $C\chi_\omega \nabla Ff \in \mathrm{Im}\, C\chi_\omega \nabla F$

since $\mathrm{Im}\, C\chi_\omega \nabla F \subset \mathrm{Im}\, C\chi_\omega \nabla H$ we have

$C\chi_\omega \nabla Ff \in \mathrm{Im}\, C\chi_\omega \nabla H$ then there exists

$u \in L^2(0,T;U)$ such that $C\chi_\omega \nabla Ff = C\chi_\omega \nabla Hu$ that is $C\chi_\omega \nabla Ff - C\chi_\omega \nabla Hu = 0$ this gives

$$C\chi_\omega \nabla Ff + C\chi_\omega \nabla H(-u) = 0$$

We put $u_1 = -u \in L^2(0,T;U)$ where the system (1) – (2) is exactly ω-gradient remediable on $[0,T]$.

2- We assume that the system (1) – (2) is weakly ω -gradient remediable on $[0,T]$, then $\forall f \in L^2(0,T;X)$ the system (1) – (2) is weakly ω-gradient f - remediable on $[0,T]$ and from the Proposition .1, we have, $\forall f \in L^2(0,T;X)$: $C\chi_\omega \nabla Ff \in \overline{\mathrm{Im}\, C\chi_\omega \nabla H}$, this gives

$$\mathrm{Im}\, C\chi_\omega \nabla F \subset \overline{\mathrm{Im}\, C\chi_\omega \nabla H}$$

Conversely, we assume that $\mathrm{Im}\, C\chi_\omega \nabla F \subset \overline{\mathrm{Im}\, C\chi_\omega \nabla H}$ and we show that the system (1) – (2) is weakly ω-gradient remediable on $[0,T]$.

Let $f \in L^2(0,T;X)$ then $C\chi_\omega \nabla Ff \in \mathrm{Im}\, C\chi_\omega \nabla F$ since $\mathrm{Im}\, C\chi_\omega \nabla F \subset \overline{\mathrm{Im}\, C\chi_\omega \nabla H}$ then $C\chi_\omega \nabla Ff \in \overline{\mathrm{Im}(C\chi_\omega \nabla H)}$ this leads to $\forall \varepsilon > 0, \exists u \in L^2(0,T;U)$ such that $\|C\chi_\omega \nabla Ff - C\chi_\omega \nabla Hu\|_{IR^q} < \varepsilon$
by putting $u_1 = -u \in L^2(0,T;U)$ this gives
$\forall \varepsilon > 0, \exists u_1 \in L^2(0,T;U)$ such that
$\|C\chi_\omega \nabla Ff + C\chi_\omega \nabla Hu_1\|_{IR^q} < \varepsilon$ where the system (1) – (2) is weakly ω-gradient remediable.

III. REGIONAL GRADIENT REMEDIABILITY AND REGIONAL GRADIENT CONTROLABILITY

Firstly, we recall the definition of exact and weak regional gradient controllability given in [10].

Definition .2

1- The system (1) is said to be exactly ω-gradient controllable on $[0,T]$, if $\forall y^d \in (L^2(\omega))^n$ there exist a control $u \in L^2(0,T;U)$ such that

$$\chi_\omega \nabla y(T) = y^d$$

2- The system (1) is said weakly ω-gradient controllable on $[0,T]$ if $\forall y^d \in (L^2(\omega))^n$, $\forall \varepsilon > 0$,

there exist a control $u \in L^2(0,T;U)$ such that

$$\|\chi_\omega \nabla y(T) - y^d\|_{(L^2(\omega))^n} < \varepsilon$$

Then, we have the following proposition

Proposition .3

If the system (1) – (2) is exactly ω-gradient controllable on $[0,T]$, then it is exactly ω-gradient remediable on $[0,T]$.

2- If the system (1) – (2) is weakly ω -gradient controllable on $[0,T]$, then it is weakly ω -gradient remediable on $[0,T]$.

Proof

1- We assume that the system (1) – (2) is exactly ω -gradient controllable on $[0,T]$ and let $y^d = \chi_\omega \nabla S(T)y^0$ then there exists $u \in L^2(0,T;U)$ such that $\chi_\omega \nabla y(T) = \chi_\omega \nabla S(T)y^0$ and since the linearity of the operator restriction χ_ω we obtain $\chi_\omega \nabla S(T)y^0 + \chi_\omega \nabla Hu + \chi_\omega \nabla Ff = \chi_\omega \nabla S(T)y^0$ this leads to $\chi_\omega \nabla Hu + \chi_\omega \nabla Ff = 0$ then let $C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff = 0$ and then the system (1) - (2) is exactly ω -gradient remediable on $[0,T]$.

We assume that the system (1) – (2) is weakly ω -gradient controllable on $[0,T]$ and let $y^d = \chi_\omega \nabla S(T)y^0$ then $\forall \varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that $\left\|\chi_\omega \nabla y(T) - \chi_\omega \nabla S(T)y^0\right\|_{(L^2(\omega))^n} < \varepsilon$, that is to say $\left\|\chi_\omega \nabla Hu + \chi_\omega \nabla Ff\right\|_{(L^2(\omega))^n} < \varepsilon$.

Since the operator C is continue, consequently we have $\left\|C\chi_\omega \nabla Hu + C\chi_\omega \nabla Ff\right\|_{IR^q} \leq k\left\|\chi_\omega \nabla Hu + \chi_\omega \nabla Ff\right\|_{(L^2(\omega))^n}$ with $k > 0$ where the system (1) – (2) is weakly ω -gradient remediable on $[0,T]$.

IV. REGIONALLY GRADIENT EFFICIENT ACTUATORS AND SENSORS

We suppose that the system (1) is excited by p zone actuators $(\Omega_i, g_i)_{1\leq i\leq p}, g_i \in H_0^1(\Omega_i)$, Ω_i in this case the control space is $U = IR^p$ and the operator B is given by

$$B : IR^p \to X$$

$$u(t) = (u_1(t), u_2(t), \ldots, u_p(t)) \mapsto Bu = \sum_{i=1}^{p} \chi_{\Omega_i}(x) g_i(x) u_i(t)$$

Its adjoint is given by

$$B^* z = \left(\langle g_1, z\rangle_{\Omega_1}, \langle g_2, z\rangle_{\Omega_2}, \ldots, \langle g_p, z\rangle_{\Omega_p}\right)^T \in IR^p \qquad (3)$$

Also suppose that the regional output of the system (2) is given by q sensors $(D_i, h_i)_{1\leq i\leq q}, h_i \in L^2(D_i)$, being the spatial distribution, $D_i = \operatorname{supp} h_i \subset \omega$ for $i = 1, \ldots, q$ and $D_i \cap D_j = \phi$ for $i \neq j$, that is the sensors are located regionally, then the operator C is defined by

$$C : (L^2(\omega))^n \to IR^q$$

$$Cy(t) = \left(\sum_{i=1}^{n} \langle h_1, y_i(t)\rangle_{L^2(D_1)}, \ldots, \sum_{i=1}^{n} \langle h_q, y_i(t)\rangle_{L^2(D_q)}\right)^T$$

its adjoint is given by C^* with for $\theta = (\theta_1, \theta_2, \ldots, \theta_q)^T \in IR^q$

$$C^*\theta = \left(\sum_{i=1}^{q} \chi_{D_i}\theta_i h_i(x), \ldots, \sum_{i=1}^{q} \chi_{D_i}\theta_i h_i(x)\right) \in (L^2(\omega))^n \qquad (4)$$

Lemma .1 [11]

Let V, W and Z be reflexive Banach spaces, $P \in \mathscr{L}(V,Z)$ and $Q \in \mathscr{L}(W,Z)$. Then the following properties are equivalent:

i. $\operatorname{Im} P \subset \operatorname{Im} Q$

ii. $\exists \gamma > 0$ such that $\left\|P^* z^*\right\|_{V^*} \leq \gamma \left\|Q^* z^*\right\|_{W^*}$, $\forall z^* \in Z^*$

We have the following characterizations

Proposition .4

1- The system (1) - (2) is exactly ω -gradient remediable on $[0,T]$ if and only if there exists $\gamma > 0$ such that for every $\theta \in IR^q$, we have

$$\left\|S^*(T-.)\nabla^* \chi_\omega^* C^*\theta\right\|_{L^2(0,T;X')} \leq \gamma \left\|B^* S^*(T-.)\nabla^* \chi_\omega^* C^*\theta\right\|_{L^2(0,T;IR^p)}$$

2- The system (1) - (2) is weakly ω -gradient remediable on $[0,T]$ if and only if

$$\ker(B^* F^* \nabla^* \chi_\omega^* C^*) = \ker(F^* \nabla^* \chi_\omega^* C^*)$$

Proof

1- It follows from the fact that $F^* \nabla^* \chi_\omega^* C^* = S^*(T-.)\nabla^* \chi_\omega^* C^*$ and that $H^* \nabla^* \chi_\omega^* C^* = B^* S^*(T-.)\nabla^* \chi_\omega^* C^*$ and since the Proposition .2, we put $P = C\chi_\omega \nabla F$ and $Q = C\chi_\omega \nabla H$ and using the lemma .1.

2- We assume that the system (1) - (2) is weakly ω -gradient remediable on $[0,T]$ and we show that

$$\ker(B^* F^* \nabla^* \chi_\omega^* C^*) = \ker(F^* \nabla^* \chi_\omega^* C^*)$$

Let $\theta \in IR^q$ such that $B^* F^* \nabla^* \chi_\omega^* C^*\theta = 0$, and we have

$$\begin{cases} F^* = S^*(T-.) \\ H^* = B^* S^*(T-.) \end{cases}$$

then,

$$B^* F^* \nabla^* \chi_\omega^* C^*\theta = 0 \Rightarrow H^* \nabla^* \chi_\omega^* C^*\theta = 0$$

this gives $\theta \in \ker(H^* \nabla^* \chi_\omega^* C^*)$ and we have $\overline{\operatorname{Im}(C\chi_\omega \nabla H)} = \left[\ker(H^* \nabla^* \chi_\omega^* C^*)\right]^\perp$.

Since the hypothesis and the Proposition .2, we have $\operatorname{Im} C\chi_\omega \nabla F \subset \overline{\operatorname{Im} C\chi_\omega \nabla H}$ then

$$\operatorname{Im} C\chi_\omega \nabla F \subset \left[\ker(H^* \nabla^* \chi_\omega^* C^*)\right]^\perp$$

$$\Rightarrow \forall f \in L^2(0,T;X) : C\chi_\omega \nabla Ff \in \left[\ker(H^* \nabla^* \chi_\omega^* C^*)\right]^\perp$$

$$\Rightarrow \langle C\chi_\omega \nabla Ff, \theta\rangle = 0 \text{ because } \theta \in \ker\left(H^*\nabla^*\chi_\omega^* C^*\right)$$

$$\Rightarrow \theta \in \left[\operatorname{Im} C\chi_\omega \nabla F\right]^\perp = \ker\left(F^*\nabla^*\chi_\omega^* C^*\right)$$

Where the result.

Conversely, assume that $\ker\left(B^*F^*\nabla^*\chi_\omega^* C^*\right) = \ker\left(F^*\nabla^*\chi_\omega^* C^*\right)$ and we show that

$$\operatorname{Im} C\chi_\omega \nabla F \subset \overline{\operatorname{Im} C\chi_\omega \nabla H}$$

Let $f \in L^2(0,T;X)$ such that $f \in \operatorname{Im} C\chi_\omega \nabla F$, we have $\overline{\operatorname{Im} C\chi_\omega \nabla H} = \left[\ker\left(H^*\nabla^*\chi_\omega^* C^*\right)\right]^\perp$.

For every $\theta \in IR^q$ such that $H^*\nabla^*\chi_\omega^* C^*\theta = 0$, that is $B^*F^*\nabla^*\chi_\omega^* C^*\theta = 0$ we have $F^*\nabla^*\chi_\omega^* C^*\theta = 0$ because $\ker\left(B^*F^*\nabla^*\chi_\omega^* C^*\right) = \ker\left(F^*\nabla^*\chi_\omega^* C^*\right)$,

then $\langle C\chi_\omega \nabla Ff, \theta\rangle = 0$

Where the result.

Corollary .1

The system (1) - (2) is exactly ω - gradient remediable on $[0,T]$ if and only if $\exists \gamma > 0$ such that $\forall \theta \in IR^q$, we have

$$\int_0^T \left\| S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\|^2_{L^2(\Omega)} ds$$

$$\leq \gamma \sum_{i=1}^p \int_0^T \left\langle g_i, S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\rangle^2_{L^2(\Omega_i)} ds$$

Proof

Since the Proposition .1, the system (1) - (2) is exactly ω-gradient remediable on $[0,T]$ if and only if there exists $\gamma > 0$ such that for every $\theta \in IR^q$, we have

$$\left\| S^*(T-.)\nabla^*\chi_\omega^* C^*\theta \right\|^2_{L^2(0,T;X^*)}$$

$$\leq \gamma \left\| B^*S^*(T-.)\nabla^*\chi_\omega^* C^*\theta \right\|^2_{L^2(0,T;IR^p)}$$

By using (3) the formula of the operator B^*, we have

$$\int_0^T \left\| S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\|^2_{X'} ds$$

$$\leq \gamma \sum_{i=1}^p \int_0^T \left\langle g_i, S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\rangle^2 ds$$

Where the result.

In the suite, we consider without loss of generality, the system (1) with a dynamics A of the form

$$Ay = \sum_{m\geq 1} \lambda_m \sum_{j=1}^{r_m} \left\langle y, w_{mj} \right\rangle_{L^2(\Omega)} w_{mj}, \forall y \in D(A)$$

where $(w_{mj})_{\substack{1\leq j\leq r_m \\ m\geq 1}}$ is an orthogonal basis in $H_0^1(\Omega)$ of eigenvectors of A orthonormal in $L^2(\Omega)$, associated to eigenvalues $\lambda_m < 0$ with a multiplicity r_m. It is well known that A generates on the Hilbert space $L^2(\Omega)$ a strongly continuous semi-group $(S(t))_{t\geq 0}$ given by [12, 13]

$$S(t)\,y = \sum_{m\geq 1} e^{\lambda_m t} \sum_{j=1}^{r_m} \left\langle y, w_{mj} \right\rangle_{L^2(\Omega)} w_{mj} \tag{5}$$

Corollary .2

The system (1)-(2) is exactly ω-gradient remediable on $[0,T]$ if and only if $\exists \gamma > 0$ such that $\forall \theta \in IR^q$, we have

$$\sum_{m\geq 1} \frac{1}{2\lambda_m}\left(e^{2\lambda_m T} - 1\right) \sum_{j=1}^{r_m} \left\langle \chi_\omega^* C^*\theta, \nabla w_{mj} \right\rangle^2_{(L^2(\Omega))^n}$$

$$\leq \gamma \sum_{i=1}^p \int_0^T \left(\sum_{m\geq 1} e^{\lambda_m (T-s)} \sum_{j=1}^{r_m} \left\langle \chi_\omega^* C^*\theta, \nabla w_{mj} \right\rangle \left\langle g_i, w_{mj} \right\rangle \right)^2 ds$$

Proof

Since the corollary .1, the system (1) - (2) is exactly ω-gradient remediable on $[0,T]$ if and only if there exists $\gamma > 0$ such that for every $\theta \in IR^q$, we have

$$\int_0^T \left\| S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\|^2_{X'} ds$$

$$\leq \gamma \sum_{i=1}^p \int_0^T \left\langle g_i, S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\rangle^2_{L^2(\Omega_i)} ds$$

By using the formula (5) of the operator S^*, we obtain

$$\int_0^T \left\| S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\|^2_{L^2(\Omega)} ds$$

$$= \int_0^T \sum_{m\geq 1} e^{2\lambda_m (T-s)} \sum_{j=1}^{r_m} \left\langle \chi_\omega^* \nabla^* C^*\theta, w_{mj} \right\rangle^2 ds$$

$$= \sum_{m\geq 1} \frac{1}{2\lambda_m}\left(e^{2\lambda_m T} - 1\right) \sum_{j=1}^{r_m} \left\langle \chi_\omega^* C^*\theta, \nabla w_{mj} \right\rangle^2$$

and

$$\sum_{i=1}^p \int_0^T \left\langle g_i, S^*(T-s)\nabla^*\chi_\omega^* C^*\theta \right\rangle^2_{L^2(\Omega_i)} ds =$$

$$\sum_{i=1}^p \int_0^T \left(\sum_{m\geq 1} e^{\lambda_m (T-s)} \sum_{j=1}^{r_m} \left\langle \nabla^*\chi_\omega^* C^*\theta, w_{mj} \right\rangle \left\langle g_i, w_{mj} \right\rangle \right)^2 ds$$

Where the result.

By using (5) the formula of the operator C^*, we have the following corollary

Corollary .3

The system (1) - (2) is exactly ω-gradient remediable on $[0,T]$ if and only if $\exists\gamma>0$ such that $\forall\theta\in IR^q$, we have

$$\sum_{m\geq1}\frac{1}{2\lambda_m}\left(e^{2\lambda_m T}-1\right)\sum_{j=1}^{r_m}\sum_{k=1}^{n}\sum_{l=1}^{q}\left\langle\theta_l h_l,\frac{\partial w_{mj}}{\partial x_k}\right\rangle^2_{L^2(D_l)}$$
$$\leq\gamma\sum_{i=1}^{p}\int_0^T\left(\sum_{m\geq1}e^{\lambda_m(T-s)}\sum_{j=1}^{r_m}\langle g_i,w_{mj}\rangle\sum_{k=1}^{n}\sum_{l=1}^{q}\theta_l\left\langle h_l,\frac{\partial w_{mj}}{\partial x_k}\right\rangle\right)^2 ds$$

Proof

Since the corollary .2, the system (1) - (2) is exactly regionally gradient remediable in ω on $[0,T]$ if and only if there exists $\gamma>0$ such that for every $\theta\in IR^q$, we have

$$\sum_{m\geq1}\frac{1}{2\lambda_m}\left(e^{2\lambda_m T}-1\right)\sum_{j=1}^{r_m}\left\langle\chi_\omega^*C^*\theta,\nabla w_{mj}\right\rangle^2_{(L^2(\Omega))^n}$$
$$\leq\gamma\sum_{i=1}^{p}\int_0^T\left(\sum_{m\geq1}e^{\lambda_m(T-s)}\sum_{j=1}^{r_m}\left\langle\chi_\omega^*C^*\theta,\nabla w_{mj}\right\rangle\langle g_i,w_{mj}\rangle\right)^2 ds$$

in addition, by using (4) the formula of the operator C^*, we obtain

$$\left\langle\chi_\omega^*C^*\theta,\nabla w_{mj}\right\rangle_{(L^2(\Omega))^n}=\left\langle C^*\theta,\chi_\omega\nabla w_{mj}\right\rangle_{(L^2(\omega))^n}$$
$$=\left\langle\begin{pmatrix}\sum_{l=1}^{q}\chi_{D_l}\theta_l h_l\\\sum_{l=1}^{q}\chi_{D_l}\theta_l h_l\\\vdots\\\sum_{l=1}^{q}\chi_{D_l}\theta_l h_l\end{pmatrix},\begin{pmatrix}\tilde{\chi}_\omega\frac{\partial w_{mj}}{\partial x_1}\\\tilde{\chi}_\omega\frac{\partial w_{mj}}{\partial x_2}\\\vdots\\\tilde{\chi}_\omega\frac{\partial w_{mj}}{\partial x_n}\end{pmatrix}\right\rangle_{(L^2(\omega))^n}$$
$$=\sum_{k=1}^{n}\sum_{l=1}^{q}\left\langle\chi_{D_l}\theta_l h_l,\tilde{\chi}_\omega\frac{\partial w_{mj}}{\partial x_k}\right\rangle_{L^2(\omega)}$$
$$=\sum_{k=1}^{n}\sum_{l=1}^{q}\theta_l\left\langle h_l,\frac{\partial w_{mj}}{\partial x_k}\right\rangle_{L^2(D_l)}$$

Where the result.

This characterization shows that the remediability of a system may depend on the structure of the actuators and sensors.

By analogy with the concept of regionally gradient strategic actuator, we introduce the notion of regionally gradient efficient actuator, as follows

Definition .3

The actuators $(\Omega_i,g_i)_{1\leq i\leq p}, g_i\in L^2(\Omega_i)$ are said to be regionally gradient efficient if the system (1) - (2) so excited is weakly regionally gradient remediable.

The actuators regionally gradient efficient define actions with the structure (spatial distribution, location and number) can compensate the effect of disturbance distributed on the system.

We have then the following characterization of the regionally gradient efficient actuators.

For $m\geq1$, let M_m be the matrix of order $(p\times r_m)$ defined by $M_m=\left(\langle g_i,w_{mj}\rangle\right)_{ij}$, $1\leq i\leq p$ and $1\leq j\leq r_m$ and let G_m be the matrix of order $(q\times r_m)$ defined by $G_m=\left(\sum_{k=1}^{n}\left\langle h_i,\frac{\partial w_{mj}}{\partial x_k}\right\rangle_{L^2(D_i)}\right)_{ij}$, $1\leq i\leq q$ and $1\leq j\leq r_m$.

Proposition .5

The actuators $(\Omega_i,g_i)_{1\leq i\leq p}, g_i\in L^2(\Omega_i)$ are ω-gradient efficient if and only if

$$\ker\left(\nabla^*\chi_\omega^*C^*\right)=\bigcap_{m\geq1}\ker\left(M_m f_m^\omega\right)$$

where, for $m\geq1$,

$$f_m^\omega:\theta\in IR^q\to f_m^\omega(\theta)=$$
$$\left(\langle\nabla^*\chi_\omega^*C^*\theta,w_{m1}\rangle,\langle\nabla^*\chi_\omega^*C^*\theta,w_{m2}\rangle,\ldots,\langle\nabla^*\chi_\omega^*C^*\theta,w_{mr_m}\rangle\right)^T$$
$$\in IR^{r_m}$$

Proof

1- We assume that the actuators $(\Omega_i,g_i)_{1\leq i\leq p}, g_i\in L^2(\Omega_i)$ are ω-gradient efficient and we show that

$$\ker\left(\nabla^*\chi_\omega^*C^*\right)=\bigcap_{m\geq1}\ker\left(M_m f_m^\omega\right)$$

Since the proposition .1, the system (1) - (2) is weakly ω-gradient remediable on $[0,T]$ if and only if

$$\ker\left(B^*F^*\nabla^*\chi_\omega^*C^*\right)=\ker\left(F^*\nabla^*\chi_\omega^*C^*\right)$$

Let $\theta\in IR^q$, we have

$$B^*F^*\nabla^*\chi_\omega^*C^*\theta = B^*S^*(T-.)\nabla^*\chi_\omega^*C^*\theta$$

$$= \begin{pmatrix} \langle g_1, S^*(T-.)\nabla^*\chi_\omega^*C^*\theta\rangle_{L^2(\Omega_1)} \\ \langle g_2, S^*(T-.)\nabla^*\chi_\omega^*C^*\theta\rangle_{L^2(\Omega_2)} \\ \vdots \\ \langle g_P, S^*(T-.)\nabla^*\chi_\omega^*C^*\theta\rangle_{L^2(\Omega_P)} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle_{L^2(\Omega)} \langle g_1, w_{mj}\rangle_{L^2(\Omega_1)} \\ \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle_{L^2(\Omega)} \langle g_2, w_{mj}\rangle_{L^2(\Omega_2)} \\ \vdots \\ \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle_{L^2(\Omega)} \langle g_P, w_{mj}\rangle_{L^2(\Omega_P)} \end{pmatrix}$$

and we have $\forall m \geq 1$,

$$M_m f_m^\omega(\theta) = \begin{pmatrix} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta; w_{mj}\rangle_{L^2(\Omega)} \langle g_1, w_{mj}\rangle_{L^2(\Omega_1)} \\ \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta; w_{mj}\rangle_{L^2(\Omega)} \langle g_2, w_{mj}\rangle_{L^2(\Omega_2)} \\ \vdots \\ \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta; w_{mj}\rangle_{L^2(\Omega)} \langle g_P, w_{mj}\rangle_{L^2(\Omega_P)} \end{pmatrix}$$

If we assume that $\theta \in \bigcap_{m\geq 1} \ker(M_m f_m^\omega)$, then

$$\theta \in \ker(M_m f_m^\omega), \quad \forall m \geq 1 \Rightarrow$$

$$\sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta; w_{mj}\rangle\langle g_i, w_{mj}\rangle = 0, \forall i \in \{1,2,\ldots,p\}, \forall m \geq 1$$

$$\Rightarrow \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle\langle g_i, w_{mj}\rangle_{L^2(\Omega_i)} = 0,$$
$$\forall i \in \{1,2,\ldots,p\}, \forall m \geq 1$$

$$\Rightarrow B^*F^*\nabla^*\chi_\omega^*C^*\theta = 0 \Rightarrow \theta \in \ker(B^*F^*\nabla^*\chi_\omega^*C^*)$$

where $\bigcap_{m\geq 1} \ker(M_m f_m^\omega) \subset \ker(B^*F^*\nabla^*\chi_\omega^*C^*)$ that is to say

$$\bigcap_{m\geq 1} \ker(M_m f_m^\omega) = \ker(B^*F^*\nabla^*\chi_\omega^*C^*)$$

On the other hand, we have for every $\theta \in IR^q$,

$$F^*\nabla^*\chi_\omega^*C^*\theta = S^*(T-.)\nabla^*\chi_\omega^*C^*\theta$$

$$= \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle w_{mj}$$

We assume that $\theta \in \ker(F^*\nabla^*\chi_\omega^*C^*)$, then
$F^*\nabla^*\chi_\omega^*C^*\theta = 0$

$$F^*\nabla^*\chi_\omega^*C^*\theta = S^*(T-.)\nabla^*\chi_\omega^*C^*\theta$$

$$= \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle w_{mj} = 0$$

$$\Rightarrow \sum_{m\geq 1} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle w_{mj} = 0$$

$$\Rightarrow \nabla^*\chi_\omega^*C^*\theta = 0 \Leftrightarrow \theta \in \ker(\nabla^*\chi_\omega^*C^*)$$

Then,

$$\ker(F^*\nabla^*\chi_\omega^*C^*) \subset \ker(\nabla^*\chi_\omega^*C^*)$$

If we assume that $\theta \in \ker(\nabla^*\chi_\omega^*C^*)$, then $\nabla^*\chi_\omega^*C^*\theta = 0$ that is to say

$$F^*\nabla^*\chi_\omega^*C^*\theta = \sum_{m\geq 1} e^{\lambda_m(T-.)} \sum_{j=1}^{r_m} \langle\nabla^*\chi_\omega^*C^*\theta, w_{mj}\rangle w_{mj} = 0$$

$$\Rightarrow \theta \in \ker(F^*\nabla^*\chi_\omega^*C^*)$$

then $\ker(\nabla^*\chi_\omega^*C^*) \subset \ker(F^*\nabla^*\chi_\omega^*C^*)$ that is to say

$$\ker(\nabla^*\chi_\omega^*C^*) = \ker(F^*\nabla^*\chi_\omega^*C^*)$$

Where the result.

Corollary .4

If there exists $m_0 \geq 1$ such that

$$rank\, G_{m_0}^T = q \tag{6}$$

then the actuators $(\Omega_i, g_i)_{1\leq i\leq p}, g_i \in L^2(\Omega_i)$ are regionally gradient efficient if and only if

$$\bigcap_{m\geq 1} \ker(M_m G_m^T) = \{0\}$$

Proof

Let $\theta \in IR^q$, then

$$\theta \in \bigcap_{m\geq 1} \ker(M_m G_m^T) \Leftrightarrow (M_m G_m^T)\theta = 0, \quad \forall m \geq 1$$

$$\Leftrightarrow \sum_{l=1}^{q} \sum_{j=1}^{r_m} \langle g_i, w_{mj}\rangle_{L^2(\Omega_i)} \left\langle h_l, \sum_{k=1}^{n} \frac{\partial w_{mj}}{\partial x_k}\right\rangle \theta_l = 0,$$
$$\forall m \geq 1, \forall i = 1,\ldots,p$$

$$\Leftrightarrow \sum_{j=1}^{r_m} \langle g_i, w_{mj}\rangle_{L^2(\Omega_i)} \langle\nabla^*\chi_\omega^*C^*, w_{mj}\rangle_{L^2(\Omega)} = 0,$$
$$\forall m \geq 1, \forall i = 1,\ldots,p$$

$$\Leftrightarrow (M_m f_m^\omega)\theta = 0, \forall m \geq 1$$

$$\Leftrightarrow \theta \in \bigcap_{m\geq 1} \ker(M_m f_m^\omega)$$

this gives

$$\bigcap_{m\geq 1} \ker(M_m G_m^T) = \bigcap_{m\geq 1} \ker(M_m f_m^\omega)$$

On the other hand,

$$\theta \in \ker\left(\nabla^* \chi_\omega^* C^*\right) \Leftrightarrow \nabla^* \chi_\omega^* C^* \theta = 0$$

then for m_0 that appear in the hypothesis and by using (4) the formula of the operator C^*, we obtain

$$\left\langle \nabla^* \chi_\omega^* C^* \theta, w_{m_0 j} \right\rangle_{L^2(\Omega)} = \left\langle C^* \theta, \chi_\omega \nabla w_{m_0 j} \right\rangle_{\left(L^2(\omega)\right)^n}$$

$$= \sum_{l=1}^{q} \theta_l \left\langle h_l, \sum_{k=1}^{n} \frac{\partial w_{m_0 j}}{\partial x_k} \right\rangle_{L^2(D_l)} = 0, \quad \forall j = 1, \ldots, r_{m_0}$$

$$\Rightarrow G_{m_0}^T \theta = 0 \Leftrightarrow \theta \in \ker G_{m_0}^T$$

and since $rank\, G_{m_0}^T = q$ then $\ker G_{m_0}^T = \{0\}$ this gives $\theta = 0$. That is to say

$$\ker\left(\nabla^* \chi_\omega^* C^*\right) = \{0\}$$

Finally, the demonstration follows directly from the Proposition .2.

Corollary .5

If there exists $m_0 \geq 1$ such that

$$rank\left(G_{m_0}^T\right) = q$$

and if

$$rank\left(M_{m_0} G_{m_0}^T\right) = q \tag{7}$$

Or

$$rank\left(M_{m_0}\right) = r_{m_0} \tag{8}$$

then the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$ are regionally gradient efficient.

Proof

Assume that there exists $m_0 \geq 1$ such that $rank\left(M_{n_0} G_{n_0}^T\right) = q$.

The matrix $\left(M_{m_0} G_{m_0}^T\right)$ is of order $p \times q$. From the theorem of rank to matrices [14], we have

$$rank\left(M_{m_0} G_{m_0}^T\right) + \dim\left(\ker\left(M_{m_0} G_{m_0}^T\right)\right) = q$$

then $\dim\left(\ker\left(M_{m_0} G_{m_0}^T\right)\right) = 0$ witch is equivalent to

$$\ker\left(M_{m_0} G_{m_0}^T\right) = \{0\} \Rightarrow \bigcap_{m \geq 1} \ker\left(M_m G_m^T\right) = \{0\}$$

Since the Corollary .4, that is equivalent to the regionally gradient efficient of the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$.

Now, we suppose that $rank\left(G_{m_0}^T\right) = q$ and $rank\left(M_{m_0}\right) = r_{m_0}$.

The matrix $\left(G_{m_0}^T\right)$ is of order $r_{m_0} \times q$. By using the theorem of rank to matrices [14], we have

$$rank\left(G_{m_0}^T\right) + \dim\left(\ker\left(G_{m_0}^T\right)\right) = q$$

then, $\dim\left(\ker\left(G_{m_0}^T\right)\right) = 0$. That is equivalent to

$$\ker\left(G_{m_0}^T\right) = \{0\} \tag{9}$$

The same, the matrix $\left(M_{m_0}\right)$ is of order $p \times r_{m_0}$. By using the theorem of rank for matrices [14], we have

$$rang\left(M_{m_0}\right) + \dim\left(\ker\left(M_{m_0}\right)\right) = r_{m_0}$$

And from (8), we obtain $\dim\left(\ker\left(M_{m_0}\right)\right) = 0$ which is equivalent to

$$\ker\left(M_{m_0}\right) = \{0\} \tag{10}$$

On the other hand, let $\theta \in \ker\left(M_{m_0} G_{m_0}^T\right)$, then $\left(M_{m_0} G_{m_0}^T\right)\theta = 0$ which gives $M_{m_0}\left(G_{m_0}^T \theta\right) = 0$.

From (10), we obtain $G_{m_0}^T \theta = 0$ and from (9), we obtain $\theta = 0$ then $\ker\left(M_{m_0} G_{m_0}^T\right) = \{0\}$ and then,

$$\bigcap_{m \geq 1} \ker\left(M_m G_m^T\right) = \{0\}$$

which is equivalent, from the Corollary .4, to the regionally gradient efficient of the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}, g_i \in L^2(\Omega_i)$.

Remark .2

1) The condition (7) $\Rightarrow q \leq p$
2) The condition $q \leq p$ is not necessary for actuators to be regionally gradient efficient. Indeed, in the case of a single actuator (Ω_1, g_1) and of q sensors $(D_i, h_i)_{1 \leq i \leq q}$, with $q > 1$,

$M_m = \left(\left\langle g_1, w_{mj} \right\rangle\right)_{1 \leq j \leq r_m}$ is of order $(1 \times r_m)$ and

$G_m^T = \left(\sum_{k=1}^{n} \left\langle h_i, \frac{\partial w_{mj}}{\partial x_k} \right\rangle\right)_{\substack{1 \leq j \leq r_m \\ 1 \leq i \leq q}}$ is of order $(r_m \times q)$

consequently

$$M_m G_m^T = \left(\sum_{j=1}^{r_m} \sum_{k=1}^{n} \left\langle g_1, w_{mj} \right\rangle \left\langle h_i, \frac{\partial w_{mj}}{\partial x_k} \right\rangle\right)_{1 \leq i \leq q}$$

is of order $(1 \times q)$.

From the Corollary .4, if there exists $m_0 \geq 1$ such that $rank\, G_{m_0}^T = q$, then (Ω_1, g_1) is regionally gradient efficient if and only if $\bigcap_{m \geq 1} \ker\left(M_m G_m^T\right) = \{0\}$.

Then, if there exists $n_1, n_2, \ldots, n_m$ such that $n_i \neq n_j$ for $i \neq j$ and

$$\bigcap_{i=1,m} \ker\left(M_{n_i} G_{n_i}^T\right) = \{0\} \tag{11}$$

then (Ω_1, g_1) is regionally gradient efficient. In particular if $m = q$, the condition (11) is equivalent to

$$\begin{vmatrix} \sum_{j=1}^{r_{n_1}}\sum_{k=1}^{n}\langle g_1, w_{n_1 j}\rangle\left\langle h_1, \frac{\partial w_{n_1 j}}{\partial x_k}\right\rangle \cdots & \sum_{j=1}^{r_{n_1}}\sum_{k=1}^{n}\langle g_1, w_{n_1 j}\rangle\left\langle h_q, \frac{\partial w_{n_1 j}}{\partial x_k}\right\rangle \\ \vdots & \vdots \\ \sum_{j=1}^{r_{n_q}}\sum_{k=1}^{n}\langle g_1, w_{n_q j}\rangle\left\langle h_1, \frac{\partial w_{n_q j}}{\partial x_k}\right\rangle \cdots & \sum_{j=1}^{r_{n_q}}\sum_{k=1}^{n}\langle g_1, w_{n_q j}\rangle\left\langle h_q, \frac{\partial w_{n_q j}}{\partial x_k}\right\rangle \end{vmatrix} \neq 0$$

V. APPLICATIONS

Let Ω be an open and bounded subset of IR^n, with a sufficiently regular boundary $\Gamma = \partial\Omega$. We consider the following diffusion system with a Dirichlet boundary condition

$$\begin{cases} \frac{\partial y}{\partial t}(x,t) = \Delta\, y(x,t) + \sum_{i=1}^{p} \chi_{\Omega_i}\, g_i(x) u_i(t) & \Omega \times]0,T[\\ y(x,0) = y^0(x), & \Omega \\ y(\xi,t) = 0 & \partial\Omega \times]0,T[\end{cases} \quad (12)$$

For $\omega \subset \Omega$ an open subregion of Ω with positive Lesbegue measure, the system (12) is augmented by the regional output equation

$$z = C\,\chi_\omega \nabla y = \left(\sum_{i=1}^{n} \left\langle h_1, \frac{\partial y}{\partial x_i}\right\rangle_{L^2(D_1)}, \ldots, \sum_{i=1}^{n} \left\langle h_q, \frac{\partial y}{\partial x_i}\right\rangle_{L^2(D_q)} \right)^T \quad (13)$$

If the system (12) is disturbed by $f \in L^2(0,T; L^2(\Omega))$, it is written

$$\begin{cases} \frac{\partial y}{\partial t}(x,t) = \Delta\, y(x,t) + \sum_{i=1}^{p} \chi_{\Omega_i}\, g_i(x) u_i(t) + f(x,t) & \Omega \times]0,T[\\ y(x,0) = y^0(x) & \Omega \\ y(\xi,t) = 0 & \partial\Omega \times]0,T[\end{cases} \quad (14)$$

The operator Δ admits an orthonormal basis of eigenfunctions $(w_{mj})_{\substack{m\geq 1 \\ 1\leq j\leq r_m}}$ associated with the eigenvalues $(\lambda_m)_{m\geq 1}$ of multiplicity r_m and given by

$$\Delta w_{mj} = \lambda_m w_{mj}\ ; \quad \forall m \geq 1 \text{ and } j = 1, \ldots, r_m .$$

Application 1:

Case one dimensional with $\omega \subset \Omega =]0,1[$. The eigenfunctions of Δ are given by

$w_m(x) = \sqrt{2}\sin(m\pi x);\ \forall m \geq 1$ and the associated eigenvalues are simple and given by $\lambda_m = -m^2\pi^2$; $\forall m \geq 1$. The semi-group generated by Δ is given by

$$S(t)y = \sum_{m\geq 1} e^{-m^2\pi^2 t}\langle y, w_m\rangle w_m$$

In the case of a sensor (D,h), with $D = \text{supp}(h) \subset \omega$. Let m_0 such that $\left\langle h, \frac{dw_{m_0}}{dx}\right\rangle_{L^2(D)} \neq 0$. By using the Corollary .5, an actuator (Ω_1, g_1) is regionally gradient efficient if and only if $\langle g_1, w_{m_0}\rangle \neq 0$, let $\int_{\Omega_1} g_1(x)\sin(m_0\pi x)dx \neq 0$. Thus for example, if $g_1 = w_{m_0}$, then the actuator (Ω_1, g_1) is regionally gradient efficient.

Application 2:

Case of a rectangle with $\omega \subset \Omega =]0,\alpha[\times]0,\beta[$. In this case, the eigenvectors of Δ are defined by

$$w_{mn}(x,y) = \frac{2}{\sqrt{\alpha\beta}}\sin\left(\frac{m\pi x}{\alpha}\right)\sin\left(\frac{n\pi y}{\beta}\right); \quad m,n \geq 1$$

and the associated eigenvalues are $\lambda_{mn} = -\left(\frac{m^2}{\alpha^2} + \frac{n^2}{\beta^2}\right)\pi^2$; $m,n \geq 1$.

It is known [15], that in the case of a square domain, with $\alpha = \beta = 1$, we have $\lambda_{mn} = -(m^2+n^2)\pi^2$ and $\sup_{m,n\geq 1} r_{mn} = 1$.

From the Corollary .5, the system (14) augmented by (13) is weakly regionally gradient remediable in $\omega \subset \Omega$ if there exists $m_0, n_0 \geq 1$ such that

$$rank\left(G^T_{m_0 n_0}\right) = rank\left(M_{m_0 n_0} G^T_{m_0 n_0}\right) = q$$

Thus, in the case of a sensor, a single actuator may be regionally gradient efficient, for any α and β. Indeed, (Ω_1, g_1) is regionally gradient efficient if there exists m_0 , $n_0 \geq 1$ such that

$$\sum_{j=1}^{r_{m_0 n_0}} \langle g, w_{m_0 n_0 j}\rangle\left\langle h, \frac{\partial w_{m_0 n_0 j}}{\partial x}\right\rangle + \sum_{j=1}^{r_{m_0 n_0}} \langle g, w_{m_0 n_0 j}\rangle\left\langle h, \frac{\partial w_{m_0 n_0 j}}{\partial y}\right\rangle \neq 0 \quad (15)$$

where $(w_{m_0 n_0 j})_{1\leq j\leq r_{m_0 n_0}}$ are the eigenvectors associated with the eigenvalue $\lambda_{m_0 n_0}$. That is the case for example if $g_1 = w_{m_0 n_0 j}$ where $m_0 n_0\ j_0$ are such that

$$\left\langle h, \frac{\partial w_{m_0 n_0 j}}{\partial x}\right\rangle_{L^2(D)} + \left\langle h, \frac{\partial w_{m_0 n_0 j}}{\partial y}\right\rangle_{L^2(D)} \neq 0$$

what is always possible because h is not identically zero. Then, the actuator (Ω_1, g_1) is regionally gradient efficient.

Remark .3

1) In the case where $\frac{\alpha^2}{\beta^2} \notin IQ$, we have $r_{mn} = 1$; $\forall m,n \geq 1$ [15], the condition (15) becomes

$$\langle g_1, w_{m_0 n_0}\rangle\left\langle h, \frac{\partial w_{m_0 n_0}}{\partial x}\right\rangle + \langle g_1, w_{m_0 n_0}\rangle\left\langle h, \frac{\partial w_{m_0 n_0}}{\partial y}\right\rangle \neq 0$$

that is to say

$$\frac{2\pi}{\sqrt{\alpha\beta}}\int_{\Omega_1} g_1(x,y)\sin\left(\frac{m_0\pi x}{\alpha}\right)\sin\left(\frac{n_0\pi y}{\beta}\right)dx\,dy \times \int_D h(x,y)$$
$$\left(\frac{m_0}{\alpha}\cos\left(\frac{m_0\pi x}{\alpha}\right)\sin\left(\frac{n_0\pi y}{\beta}\right)+\frac{n_0}{\beta}\sin\left(\frac{m_0\pi x}{\alpha}\right)\cos\left(\frac{n_0\pi y}{\beta}\right)\right)$$
$$dx\,dy \neq 0$$

with $D = \mathrm{supp}(h) \subset \omega$.

In the case of a square domain, the condition (15) becomes

$$\sum_{j=1}^{r_{m_0n_0}}\int_{\Omega_1} g_1(x,y)\,w_{m_0n_0j}(x,y)\,dx\,dy$$
$$\int_D h(x,y)\left(\frac{\partial w_{m_0nj}}{\partial x}(x,y)+\frac{\partial w_{m_0n_0j}}{\partial y}(x,y)\right)dx\,dy \neq 0$$

One actuator may be regionally gradient efficient. Thus, for example, h such that

$$\left\langle h,\frac{\partial w_{m_0n_0j_0}}{\partial x}\right\rangle_{L^2(D)}+\left\langle h,\frac{\partial w_{m_0n_0j_0}}{\partial y}\right\rangle_{L^2(D)} \neq 0 \text{ and } g_1 = w_{m_0n_0j_0}$$

the actuator (Ω_1, g_1) is regionally gradient efficient.

VI. Conclusion

The concept developed in this paper is related to the regional gradient remediability in connection with the regionally gradient efficient actuators. Then, new notions of weak and exact regional gradient remediability are introduced and characterized. The relation between the notion of regional gradient remediability and the notion of regional gradient controllability is also study. We have shown that a system parabolic is regionally gradient remediable if it is regionally gradient controllable. Main properties and characterization results are given. Furthermore, we have shown that the exact and weak regional gradient remediability of a system may depend on the structure and the number of the actuators and sensors. Various interesting results concerning the choice of the regionally gradient efficient actuators and applications to a diffusion system are given. Many questions remain open, such as the choice of the optimal control ensuring the regional gradient remediability using an extension of Hilbert Uniqueness Method. This question is still under consideration and the results with numerical simulations will appear in a separate paper.

References

[1] L. Afifi., A. Chafiai and A. El Jai, “Compensation Spatial en Temps Fini dans les Systèmes Distribués,” L.T.S, Université de Perpignant, Rapport Interne, 1998, N. 1498.

[2] L. Afifi., A. Chafiai and A. El Jai, “Sensors and actuators for compensation in hyperbolic systems,” Footeeth International Symposium of Mathematical Theory of Networks and Systems, MTNS’2000, Université de Perpignant, France, June 19-23, 2000.

[3] L. Afifi., A. Chafiai and A. El Jai, “Spatial Compensation of Boundary Disturbances by Boundary Actuators,” Int. J. of Applied Mathematics and Computer Science. Vol. 11, N. 4, 2001, pp. 899-920.

[4] L. Afifi, M. Bahadi and A. Chafiai, “A regional asymptotic analysis of the Compensation Problem in Distributed Systems,” Applied Mathematical Sciences. Vol. 1, N. 54, 2007, pp. 2659-2686.

[5] L. Afifi, E. M. Magri. and A. El jai, “Compensation Problem in Finite Linear Dynamical Systems,” Applied Mathematical Sciences. Vol. 2, N. 49, 2008, pp. 2219-2238.

[6] L. Afifi, M. Bahadi, A. El Jai and A. El mizane, “Asymptotic Analysis Approximations and Simulations of the Compensation Problem in Hyperbolic Systems,” Applied Mathematical Sciences. Vol. 3, N. 15, 2009, pp. 737-765.

[7] L. Afifi, M. Hakam, M.. Bahadi and A. El Jai, “Enlarged Asymptotic Compensation in hyperbolic Systems,” International Mathematical Forum, Vol. 5, N. 40, 2010, 1973-1992.

[8] S. Rekkab and S. Benhadid, “Gradient Remediability in Linear Distributed Parabolic Systems Analysis, Approximations and Simulations,” Journal of Fundamental and Applied Sciences, [S.l.], Vol. 9, n. 3, pp. 1535-1558, sep. 2017. ISSN 1112-9867. Available at: <http://www.jfas.info/index.php/jfas/article/view/2203/1414>. Date accessed: 03 Dec. 2017. doi:http://dx.doi.org/10.4314/jfas.v9i3.18.

[9] J. L. Lions, “Controlabilité Exacte. Perturbations et Stabilisations des Systèmes Distribués,” Dunod, Paris, 1988.

[10] E. Zerrik, A. Kamal and A. Boutoulout, “Regional Gradient Controllability and Actuators,” International Journal of Systems Science, Vol. 33, N. 4, 2002, pp. 239-246.

[11] V. Trenoguine, *Analyse fonctionnelle*. Edition Mir, Moscou. 1985.

[12] R. F. Curtain and H. Zwart, “An introduction to infinite dimensional linear systems theory,” Texts in Applied Mathematics, Springer-Verlag, 21, 1995.

[13] A. El Jai and A. J. Pritchard, “Sensors and actuators in the analysis of distributed systems,” Ellis Horwood, Series in Applied Mathematics, J. wiley, 1988.

[14] I. C. F. Ipsen, “Numerical Matrix Analysis : Linear Systems and Least Squares,” Society for Industrial and Applied Mathematics, Philadelphia, 2009.

[15] A. El Jai and S. Yacoubi, “on the number of actuators in parabolic systems,” International journal of Applied Mathematics and Computer Science, Vol. 3, n. 4, 1993, pp. 47-64.

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