International Journal of New Technology and Research (IJNTR)
ISSN:2454-4116, Volume-4, Issue-2, February 2018 Pages 52-57

Relative Controllability of Fractional Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control

Dr Sir Paul Anaetodike Oraekie

***Abstract*— In this work, Fractional Integro-differential Systems in Banach Spaces with Distributed Delays in the Control of the form:**

$$\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) + f\left(t, x(t), \int_0^t g(t,s,x(s))ds\right)$$

is presented for controllability analysis. Necessary and Sufficient Conditions for the system to be relatively controllable are established. The Set Functions upon which our results hinged were extracted. Uses were made of: Unsymmetric Fubini theorem, the Controllability Standard and the Concept of Fractional Calculus to establish results.

***Index Terms*— Relative Controllability, Fractional Integro-differential Systems,Banach Spaces,Fractional Calculus, Unsymmetric Fubini Theorem, Positive Definite.**

I. Introduction

According to **Bonilla etal (2007),** fractional differential equations emerged as a new branch of mathematics. Fractional differential equations have been used for many mathematical models in Sciences and Engineering. The equations are considered as an alternative model to nonlinear differential equations. The theory of fractional differential equations has been studied extensively by many authors **Dulbecco (1996) and Lakshimilkanthan(2008) .**While the problems of stability for fractional differential systems are discussed in **Bonnet (2000) , Nec(2007), Balachandran(2009).**

Apart from stability, another important qualitative behavior of a dynamical system is controllability. Systematic study of controllability started over years at the beginning of the sixties when the theory of controllability based on the description in the form of state space for both time-varying and time-invariant linear control systems are carried out. Roughly speaking, controllability generally means that, it is possible to steer a dynamical control system from an initial state x(o) of the system to any final state x(t) in some finite time using the set of admissible controls**Oraekie(2013) .**The concept of controllability plays a major role in both finite and infinite dynamical systems, thatis systems represented by ordinary differential equations and partial differential equations respectively. So it is natural to extend this concept to to dynamical systems represented by fractional differentialequations. Many partialfractional differential equations and Integro-differential equation can be expressed asfractional differential equations and Integro-differential equations in Banach spaces**Elsayeed (1966)**).

There exist many works on finite dimensional controllability of linear systems **Klamka (1993))** and infinite dimensional systems in abstract spaces **Curtain (1978) .** The controllability problems of nonlinear systems and Integro-differential systems with delays have been carried out by many researchers in both finite and infinite dimensional spaces **Balachandran (1989)) and Balachandran (2002).**

Controllabilityfractional differential systems in finite dimensional space have been studied by **Chen (2006)** and **Shamardan(2000).**While **Balachandran (2009)** studied Controllability offractionalIntegro-differential systems in Banach spaces.

In this paper, we study the relative controllability of fractionalIntegro-differential systems in Banach spaces with distributed delays in the control the controllability standard of dynamical control systems and the unsymmetric Fubini theorem to establish results.

II. PRELIMINARIES

Let n be a positive integer and E $= (-\infty, \infty)$ *be the real line . Denote* E^n = *the space of real n − tuples called the Euclidean space with norm denoted by* $|.|$. *If* $J = [t_0, t_1]$ *is any interval of* E, L_2 *is Lebesgue space of square integrable functions from* J *to* E^n *written as* $L_2([t_0,t_1],E^n)$. *Let* $h > 0$ *be positive real number and let* $C([t_0,t_1],E^n)$ *be the Banach space of continuous functions with norm of uniform convergence defined by*

$$\|\phi\| = sup\phi(s);\ \phi\epsilon\ C([t_0,t_1],E^n).$$

If x is a function from $[-h,\infty)$ *to* E^n, *then* x_t *is a function defined on the delay interval* $[-h,0]$ *given as :*

$$x_t(s) = x(t-s);\ s \in [-h,0], t \in [0,\infty).$$

Definition2.1 (Balachandran(2009))

The Riemann − Liouville fractional integral operator of order $\beta > 0$ *of function* $f \in C_n$, $n \geq -1$ *is defined as:*

$$I^\beta f(t) = \frac{1}{\rho(\beta)}\int_0^t (t-s)^{\beta-1} f(s)ds$$

Definition 2.2 (fractional derivative)

Dr Sir Paul Anaetodike Oraekie, Department of Mathematics, Faculty of Physical Sciences Chukwuemeka Odumegwu Ojukwu University, Uli-Campus.

$If the function\ f \in C^m\ and\ m\ is\ positive\ integer, then\ we\ can\ define\ the\ fractional\ derivative\ of\ f(t)\ in\ the\ Caputo\ sense\ as:$

$$\frac{d^n f(t)}{dt^n} = \frac{1}{\rho(m-n)}\int_0^t (t-s)^{m-n-1} f^m(s)ds\ ;\ m-1 < n \le m.$$

$If\ m = 1, then\ m-1 < n \le m\ becomes\ 0 < n \le 1. Then$

$$\frac{d^n f(t)}{dt^n} = \frac{1}{\rho(1-n)}\int_0^t (t-s)^{1-n-1} f^1(s)ds$$

$$= \frac{1}{\rho(1-n)}\int_0^t (t-s)^{-n} f^1(s)ds$$

$$= \frac{1}{\rho(1-n)}\int_0^t \frac{1}{(t-s)^n} f^1(s)ds$$

$$= \frac{1}{\rho(1-n)}\int_0^t \frac{f^1(s)}{(t-s)^n} ds,$$

$where\ f^1(s) = \frac{df(s)}{ds}\ and\ f\ is\ an\ abstract\ function\ with\ values\ in\ X.$

2.1. VARIATION OF CONSTANT FORMULA

$Consider\ the\ following\ system\ represented\ by\ the\ fractional\ Integro-differential\ equations\ in\ Banach\ spaces\ with\ distributed\ delays\ in\ the\ control\ of\ the\ form:$

$$\frac{d^n f(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta) + f\left(t, x(t), \int_0^t g(t,s,x(s))ds\right) \quad (1.1)$$

$$x(0) = x_0\ ; t \in J = [t_0, t_1].$$

$where\ the\ state\ x(.)\ takes\ values\ in\ the\ Banach\ space\ X, 0 < n < 1, the\ control\ function\ u \in L_2([t_0,t_1], U)\ , a\ Banach\ space\ of\ admissible\ control\ functions\ with\ U\ as\ a\ Banach\ space.$ $H(t,\theta)$ is an nxm matrix function continuous at t and of bounded variation in θ on $[-h, 0]$, h > 0 $for\ each\ t \in [t_0, t_1]; t_1 > t_0$. The integral is in the Lebesgue − Stieltjes sense and is denoted by the symbol d_θ. And the nonlinear operators f: JxXxX → X , g: ΔxX → X are continuous; Δ= $\{(t,s): 0 \le s \le t \le t_1\}$.

If, $$Gx(t) = \int_{t_0}^{t} g(t,s,x(s))\, ds,$$

then the equation (1.1)becomes equivalent to the following nonliear integral equation

$$x(t) = x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t} (t-s)^{n-1} Ax(s)ds + \frac{1}{\rho(n)}\int_{t_0}^{t} (t-s)^{n-1}\left[\int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta)\right] ds + \frac{1}{\rho(n)}\int_{t_0}^{t} (t-s)^{n-1} f(t, x(t), Gx(s))ds$$

And the mild solution of the system (1.1) is given by

$$x(t) = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t} (t-s)^{n-1}\, T(t-s)\left[\int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta)\right] ds + \frac{1}{\rho(n)}\int_{t_0}^{t} (t-s)^{n-1}\, T(t-s) f(t, x(t), Gx(s))ds \quad (1.2)$$

which is similar to the concept defined in the book of **Pazy(1983).**

$For\ the\ limiting\ case, n \to 1\ , the\ above\ system (1.2)\ reprsentation\ becomes$

$$x(t) = T(t)x_0 + \int_{t_0}^{t} T(t-s)\int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta)ds + \int_{t_0}^{t} T(t-s)\, f(t, x(t), Gx(s))ds \quad (1.3)$$

.

$Which\ is\ the\ mild\ solution\ of$

$$\frac{dx(t)}{dt} = Ax(t) + \int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta) + f(t, x(t), Gx(s))$$

$With\ initial\ condition\ x(0) = x_0 \in X.$

$Analogus\ to\ the\ conventional\ controllability\ concept.$ $A careful\ observation\ of\ the\ solution\ of\ the\ system (1.1) given\ as\ system (1.2) shows\ that\ the\ values\ of\ the\ control\ function\ u(t)\ for\ t \in [-h, t_1] enter\ the\ definition\ of\ complete\ state\ thereby\ creating\ the\ need\ for\ an\ explicit\ variation\ of\ constant\ formula. The\ control\ in\ the\ 2nd\ term\ of\ the\ formula (1.2), therefore, has\ to\ be\ separated\ in\ the\ intervals\ [-h\ ,0] and\ [0\ , t_1].$

$To\ achieve\ this\ that\ 2nd\ term\ of\ system\ (1.2) has\ to\ be\ transformed\ by\ applying\ the\ method\ of\ Klamka\ as\ contained\ in$ ***Chukwu*(1992)**.

$Finally, we\ interchange\ the\ order\ of\ integration\ using\ the\ Unsymmetric\ Fubuni\ theorem\ to\ have$

$$x(t) = T(t)x_0 + \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}T(t-s)H(s,\theta)u(s+\theta)ds\right)$$

$$+ \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}T(t-s)f(t,x(t),Gx(s))ds \qquad (2.0)$$

$$\Rightarrow x(t) = T(t)x_0 + \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{t_0+\theta}^{t+\theta}(t-s)^{n-1}T(t-s)H(s-\theta,\theta)u(s-\theta+\theta)ds\right)$$

$$+ \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}T(t-s)f(t,x(t),Gx(s))ds \qquad (2.1).$$

Simplifying system (2.1), we have

$$x(t) = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}T(t-s)f(t,x(t),Gx(s))ds + \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}T(t-s)H(s-\theta,\theta)u_0(s)ds\right)$$

$$+ \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0}^{t+\theta}(t-s)^{n-1}T(t-s)H(s-\theta,\theta)u(s)ds\right) \qquad (2.2)$$

Using again the Unsymmetric Fubuni Theorm on the change of the order of integration and incorporating H^* as defined below:

$$H^*(s-\theta,\theta) = \begin{cases} H(s-\theta,\theta), for\ s \le t \\ 0 \qquad , for s \ge t \end{cases} \qquad (2.3)$$

System (.2.2) becomes

$$x(t) = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}T(t-s)f(t,x(t),Gx(s))ds + \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}T(t-s)H(s-\theta,\theta)u_0(s)ds\right)$$

$$+ \int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)u(s)\right]ds \qquad (2.4)$$

Integration is still in the Lebesgue Stieltjes sense in the variable θ in H.

For brevity, let

$$\boldsymbol{\alpha}(\boldsymbol{t},\boldsymbol{s}) = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}T(t-s)f(t,x(t),Gx(s))ds \qquad (2.5)$$

$$\boldsymbol{\beta}(\boldsymbol{t},\boldsymbol{s}) = \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}T(t-s)H(s-\theta,\theta)u_0(s)ds\right) \qquad (2.6)$$

$$\boldsymbol{\mu}(\boldsymbol{t},\boldsymbol{s}) = \frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)u(s)ds \qquad (2.7)$$

Substituting equations (2.5), (2.6) and (2.7) in equation (2.4), we have a precise variation of constant formula for the system (1.1) as:

$$x(t,x_0,u) = \boldsymbol{\alpha}(\boldsymbol{t},\boldsymbol{s}) + \boldsymbol{\beta}(\boldsymbol{t},\boldsymbol{s}) + \int_{t_0}^{t}\boldsymbol{\mu}(\boldsymbol{t},\boldsymbol{s})ds \ (2.8).$$

2.2. BASIC SET FUNCTIONS AND PROPERTIES

Definition 2.2.1 (Reachable set)

The reachable set of the system (1.1) denoted by $R(t,t_0)$ is given as :

$$R(t,t_0) = \left\{\int_{t_0}^{t}\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)u(s)ds : u \in U; |u_j| \le 1 ; j = 1,2,\dots,m\right\}$$

Where $U = \{u \in L_2([t_0,t_1],E^m)\}$

Definition 2.2.2(Attainable set)

The attainable set of the system (1.1) denoted by $A(t,t_0)$ is given as :

$$A(t,t_0) = \{x(t,x_0,u) : u \in U; |u_j| \le 1 ; j = 1,2,\dots,m\}, where\ U = \{u \in L_2([t_0,t_1],E^m)\}$$

Definition 2.2.3 (Target set)

The Target set for the system (1.1) denoted by $G(t,t_0)$ is given by

$$G(t,t_0) = \{x(t,x_0,u) : t \ge \tau > t_0 , for\ some\ fixed\ \tau\ and\ u \in U\}$$

Definition 2.2.4 (Controllability grammian or Map)

The controllability grammian orcontrollability map of the system (1.1) denoted by $W(t,t_0)$

is given as : $W(t,t_0) = \int_{t_0}^{t} \boldsymbol{\mu(t\,,s)\mu(t\,,s)^T}$, *where T denotes matrix transpose*

Definition 2, 2.5 (Positive Definite)

The controllability grammian or map W is said to be positive definite if W varnishes only at the origin and $W(x) > 0$ *for all* $x \neq 0, x \in D$, *where* $D = \{x \in E^n : \|x\| \leq r\,; r > 0\} \subset E^n$

2.3. RELATIONSHIP BETWEEN THE SET FUNCTIONS

We shall first establish the relationship between the attainable set and the reachable set, to enable us see that once a property has been proved for one set function, then it is applicable to the other. From equation (2.4),

$A(t,t) = [\eta(t) + R(t,t_0)]$, *for* $u \in U; t \in [t_0, t_1]$, *where* , $\boldsymbol{\eta(t) = \alpha(t\,,s) + \beta(t\,,s)}$.

This means that the attainable set is the translation of the reachable set through the *origin* $\eta \in E^n$. *Using the attainable set, therefore, it is easy to show that the set functions possess the properties of convexity, closedness, and compactness. Not alone, the set functions are con tenuous on* $[0,\infty)$ *to the metric space of compact subsets of* E^n

CHUKWU **(1988)***and****Gyori***(**1982**)*gave impetus for adaptations of the proofs of these properties for system*(1.1).

Definition 2.3.1 (Relative controllability)

The system(1.1)*is relatively controllable on the interval* $[t_0,t_1]$ *if*

$A(t,t_0) \cap G(t,t_0) \neq \phi, t > t_0 \in [t_0,t_1]$'

Definition 2.3.1 (Properness)

The system(1.1) *is proper in* E^n *on the interval* $[t_0,t_1]$ *if* $spanR(t\,,t_0) = E^n$

$$i.e.if,\quad C^T \int_{t_0}^{t} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta)\right] = 0 \;\; a.e\,, \Rightarrow C = 0\,; C \in E^n.$$

Definition 2.3.1 (Complete state)

We denote the complete state of system(1.1) *at time t by*

$$\boldsymbol{z(t) = \{x(t), u_t\}}.$$

Then , the initial complete state of systm(1.1) *at time* t_0 *is given by*

$$\boldsymbol{z(t_0) = \{x_0\,, u_{t_0}\}}$$

III. MAIN RESULTS

The issue of relative controllability of Neutral Volterra Integro − differential Equations have been settled in ***Balachandran*** **(1992)**, ***Balachandran*** **(1989)**, ***Balachandran***(**1997**).

From the results of these studies the following equivalent statements emerge.

Theorem 3.1.(Necessary conditions)

Consider the system

$$\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t,\theta) u(t+\theta) + f\left(t, x(t), \int_{0}^{t} g\big(t,s,x(s)\big) ds\right) \qquad (3.1)$$

$x(0) = x_0\,; t \in J = [t_0\,,t_1]$.

with the same conditions on the system' sparameters as in the system(1.1), *then the following statements are equivalent* :

(1). *System*(3.1)*is relatively controllable on the interval* $J = [t_0\,,t_1]$.

(2). *The controllability grammian* $\boldsymbol{W(t\,,t_0)}$*of system*(3.1) *is non − singular*.

(3). *System*(3.1) *is proper on the interval* $J = [t_0\,,t_1]$.

PROOF:

$\big((\mathbf{1}) = (\mathbf{2}).\big)$

Recall: *The controllability grammian* $\boldsymbol{W(t\,,t_0)}$ *of the system*(3.1) *is non − singular, is equivalent to saying that* $\boldsymbol{W(t\,,t_0)}$ *is positive definite, which in turn is equivalent to saying that the controllability index of the system*(3.1)*is equal to zero almost everywhere on the interval* $[t_0\,,t_1]$, *implying that* $C = 0$.

$$i.e.\, C^T \int_{t_0}^{t} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta)\right] = 0 \;\; a.e\,, \Rightarrow C = 0\,; C \in E^n,$$

which is properness of the system(3.1) *since the integral is non − negative*.

This , therefore, showsed that (1) *is equivalent to* (2), *or* (1) = (2).

To show that (**2**) ***and***(**3**) ***are equivalent***.

By the definition of properness of the system(3.1), *we have* (2) *given as* :

$$C^T \int_{t_0}^{t} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta)\right] = 0 \;\; a.e\,, \Rightarrow C = 0\,; C \in E^n\,,$$

for each $s \in [t_0\,,t_1]$, *then*

$$\int_{t_0}^{t} C^T \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta)\right] u(s) ds = C^T \int_{t_0}^{t} \left[\frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta)\right] u(s) ds = 0 \;\; , for\; u \in L_2 \;(3.2)$$

It follows from this last equation (3.2) that C is orthogonal to the reachable set

$$R(t,t_0)=\left\{\int_{t_0}^{t}\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)u(s)ds : u\in U; |u_j|\le 1\,;\, j=1,2,\dots,m\right\}$$

If we assume the relative controllability of the system(3.1) *now,* $R(t,t_0)=E^n$ *, sothat* $C=0$*, showing that* (3) *implies* (2)*. Or* (1) *is equivalent to*(2) *and*(2)*is equivalent to* (3) *and vis − a − vis* (3) *to* (2) *to* (1).

Conversely, assume that system(3.1) *is not controllable, sothat the reachable* $R(t,t_0)\neq E^n$ *for* $t>t_0$*. Then, there exists* $C\neq 0$, $C\in E^n$ *, such that*

$$C^T R(t,t_0)=0\,.$$

Itfollows that for all admissible controls $u\in L_2$ *that*

$$0=C^T\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]u(s)ds\ ,for\ u\in L_2$$

$$=\int_{t_0}^{t}C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]u(s)ds$$

Hence,

$$C^T\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]u(s)ds=0\,,a.e\,;\ s\in[t_0,t_1]\ ,C\neq 0.$$

By definition of properness, it implies that the system(3.1) *is not proper, since* $c\neq 0$.
Hence the system(3.1)*is relatively controllable.*

Theorem 3.2. (Sufficientconditions)
Consider the system

$$\frac{d^n x(t)}{dt^n}=Ax(t)+\int_{-h}^{0}d_\theta H(t,\theta)u(t+\theta)+f\left(t,x(t),\int_0^t g(t,s,x(s))ds\right)\qquad(3.2)$$

$x(0)=x_0\ ;t\in J=[t_0\,,t_1]$.
with the same conditions on the systems` parameters as in the system(1.1)*, then the system*(3.2) *is relatively controllable on the interval* $J=[t_0\,,t_1]$ *if and only if zero is in the interior of the reachable set.*

PROOF

The reachable set $R(t\,,t_0)$*is closed and convex subset of* E^n *. Therefore, a point* $y_1\in E^n$ *on the boundary implies that there is a support plane* $\boldsymbol{\pi}$ *of* $R(t\,,t_0)$ *through* y_1.

$$i.e.\qquad C^T(y-y_1)\le 0\,,$$

for each $y\in R(t\,,t_0)$*, where* $C\neq 0$ *is an outward normal to the the support plane* $\boldsymbol{\pi}$.
If u_1 *is the corresponding control to* y_1 *, we have*

$$C^T\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]u(s)ds\le C^T\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]u_1(s)ds\qquad(3.2)$$

Foreach u $\in$ ***U and since U is a unit sphere, the inequality***(3.2) ***becomes***

$$\left|C^T\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]u(s)ds\right|\le\int_{t_0}^{t}\left|C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right].1\right|ds$$

$$=\int_{t_0}^{t}C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]sgn\,C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right](3.3).$$

Comparing (3.2) *with* (3.3)*, we have*

$$u_1(t)=sgn\,C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right](3.4).$$

More so, as y_1 *is on the boundary since we always have* $0\in R(t,t_0)$.
If zero were not in the interior of the reachable set $R(t,t_0)$*, then it is on the boundary.*
Hence, from the preceding argument, it implies that

$$0=\int_{t_0}^{t}C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]$$

Sothat,

$$C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right]$$
$$= 0\ a.e., since\ the\ integral\ is\ not\ zero.$$
$$This, by\ the\ definition\ of\ properness\ implies\ that\ the\ system (3.2) is\ not\ proper$$
$$since\ C \neq 0. However, if\ 0 \in Interior R(t,t_0)\ for\ t > t_0\ ;\ t > 0\ ,$$
$$C^T\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}T(t-s)d_\theta H^*(s-\theta,\theta)\right] = 0\ a.e$$
$$\Rightarrow C = 0$$
$$Which\ is\ the\ properness\ of\ the\ system\ and\ by\ the\ equivalence\ in\ theorem (3.1), the\ relative\ controllability\ of\ the\ system (3.1)\ on\ the\ interval\ J = [t_0\ , t_1]\ is\ established.$$

IV. CONCLUSION

The explicit variation of constant formula for the system (1.1) visa-à-via system (3.1) was established using the Unsymmetric Fubini theorem. The set functions upon which our studies hinged were extracted from the Mild Solution.

We established the necessary condition for the system (1.1) to be relatively controllable. This is stated and proved in theorem (3.1). While the sufficient condition for the system (1.1) to be relatively controllable is stated and proved in theorem (3.2).That is, we established that- a Fractional Integrodifferential Systems in a Banach Space with Distributed Delays in the Control, is relatively controllable on the interval $J = [t_0\ , t_1\ ]$ if and only if **zero** is in the **interior** of the **reachable set** of the system (1.1).

REFERENCES

[1]. B.Bonilla, M.Rodiguez-germa, J.J.Trujillo (2007), Fractional Integrodifferential Equations asAlternative Models to Nonlinear Differential Equations, Applied Mathematics and Computation vol.187 page79-88

[2]. D.Delbosco, L.Rodino(1996), Existence and Uniqueness for a FractionalDifferential Equations, Journal Mathematical Analysis and Applications.vol.204 pp609-625.

[3]. V.Lakshmilkantham (2008), Theory of FractionalFunctional Differential Equations, Journal of Nonlinear Analysis.vol.69 pp2677-2682

[4]. C.Bonnet, J.R.Partington(2000), Oprime Factorizations and Stability of Fractional Differential systems, Systems and Control Letters vol.42, pp167-174

[5]. Y.Nec, A.A.Npomnyashcha(2007), Linear Stability of Fractional Integrodifferential Systems Mathematical Modelling of Natural Phenomena,vol.2 pp77-105

[6]. K. Balachandran, J.Y.Park(2009), Controllability ofFractional Integrodifferential Systems inBanach Spaces, Journal of Nonlinear Analysis vol.3 pp363-367

[7]. P.A.Oraekie (2013), Relative Controllability of Neutral Volterra Integrodifferential Systems with Zero in the Interior of Reachable Set, vol.14 No.1 pp3271-3282.

[8]. M. A.A.El-sayeed (1966), Fractional Order Diffusion Wave Equation Integrodifferential, Journal of Theoretical Physics, vol.35.pp311-322.

[9]. J. Klamka (1993), Controllability of Dynamical Systems, DowdiestWaver Academic Publisher

[10]. R.F Curtain, A.J. Pritchard (1978), Infinite Dimensional Linear Systems Theory, Springer- Verlag, New York.

[11]. K.Balachandran and J.P. Dauer(1989), Relative Controllability of Perturbations of Nonlinear Systems, Journal of Optimization Theoryand Applications vol.16, pp 07-28.

[12]. K.Balachandran and J.P. Dauer (2002), Controllability of Nonlinear Systems, Journal of Optimization Theory and Applications, vol.115, pp51-56.

[13]. Y.Q.Chea, H.S.Ahu, D.Xue, Robust (2006), Controllability of Integrodifferential Systems, Signal Processing, vol.86, pp2794-2802.

[14]. A. Abdel-Chattan, M.K.A.Moubarak, A.B. Shamardan (2000), Integrodifferential Nonlinear Control Systems, Journal of Fractional Calculus, vol. 17.Pp59-69.

[15]. A, Pazy (1983), Semi-groups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York.

[16].Chukwu (1992), Control of Global Economic Growth with Centre Hold, Ordinary and Delay Differential Equation (ed.J. Weiner and J. K. Hale) London Scientific and Technical pp19-25.

[17]. K. Balachandran (992), Controllability of Neutral Volterra Integrodifferential Systems, Journal of Australian Mathematical Society, vol34, pp18-25.

[18]. P. Balasubramaniam (1997), Asymptotic Neutral Volterra Integrodifferential Systems,Journal of Mathematical Systems Estimation and Control, vol.7 pp1-4

[19]. V. A. Iheagwam and J. U. Onwuatu(2005), Relative Controllability and Null Controllability of Generalized Systems with Distributed Delays in the State and Control, Journal of Nigerian Association of Mathematical Physics, vol.9,pp221-238..

[20]. O. Hajek (1994), Duality for Differential Games and Optimal Control, Journal of Mathematical Systems Theory, vol.8.

[21]. M. Heyman and J. R.itov (1994), On Linear Pursuit Games with an Unknown Trap, Journal of Optimization Theory and Applications, vol.42, 421-425.