

Computable Criteria for an Optimal Control of Fractional Integrodifferential Systems in Banach Spaces with Distributed Delays in The Control

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Abstract— In this work, Fractional Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control of the form:

$$\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) + f\left(t, x(t), \int_0^t g(t, s, x(s)) ds\right)$$

is presented for investigation of existence and form of an optimal control of the system. Use is made of the Unsymmetric Fubini Theorem to establish the exact mild solution of the system. The set functions upon which our results hinged are extracted from the mild solution. The concept of the game of pursuit and that of the Signum function are also used to establish results. The main result is built on the maximization of a set function, a technique drawn from the calculus of variation. Necessary and sufficient conditions for existence and form of control for the system are established.

Index Terms— Signum function, optimal control, mild solution, set functions, unsymmetric Fubini theorem, calculus of variation, complete state..

I. INTRODUCTION

According to Bonilla et al (2007), fractional differential equations emerged as a new branch of mathematics. Fractional differential equations have been used for many mathematical models in Sciences and Engineering. The equations are considered as an alternative model to nonlinear differential equations. The theory of fractional differential equations has been studied extensively by many authors (.Delbosco(1996) and Lakshmilkanthan(2008)). While the problems of stability for fractional differential systems are discussed in Bonnet (2000), Nec and (2007), Balachandran(2009).

Apart from stability, another important qualitative behavior of a dynamical system is controllability. Systematic study of controllability started over years at the beginning of the sixties when the theory of controllability based on the description in the form of state space for both time-varying and time-invariant linear control systems are carried out. Roughly speaking, controllability generally means that, it is possible to steer a dynamical control system from an initial state $x(0)$ of the system to any final state $x(t)$ in some finite

time using the set of admissible controls Oraekie(2013). The concept of controllability plays a major role in both finite and infinite dynamical systems, that is systems represented by ordinary differential equations and partial differential equations respectively. So it is natural to extend this concept to dynamical systems represented by fractional differential equations. Many partialfractional differential equations and Integrodifferential equation can be expressed as fractional differential equations and Integrodifferential equations in Banach spaces Elsayeed (1966).

There exist many works on finite dimensional controllability of linear systems (Klamka 1993) and infinite dimensional systems in abstract spaces (Curtain (1978)). The controllability problems of nonlinear systems and Integrodifferential systems with delays have been carried out by many researchers in both finite and infinite dimensional spaces. Balachandran (1989) and Balachandran(2002)). Controllability fractional differential systems in finite dimensional space has been studied by Chen(2006) and Shamardan(2000). While Balachandran (2009) studied Controllability of fractional Integrodifferential systems in Banach spaces.

In this paper, we shall consider the Fractional Integrodifferential Systems in Banach spaces with Distributed Delays in the control of the form:

$$\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) + f\left(t, x(t), \int_0^t g(t, s, x(s)) ds\right) \quad (1.1)$$

$x(0) = x_0; t \in J = [t_0, t_1]$ with the main objective of investigating the existence and form of an optimal control the system(1.1).

II. PRELIMINARIES

Let n be a positive integer and $E = (-\infty, \infty)$ be the real line. Denote E^n = the space of real n – tuples called the Euclidean space with norm denoted by $|\cdot|$. If $J = [t_0, t_1]$ is any interval of E , L_2 is Lebesgue space of square integrable functions from J to E^n written as $L_2([t_0, t_1], E^n)$. Let $h > 0$ be positive real number and let $C([t_0, t_1], E^n)$ be

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the Banach space of continuous functions with norm of uniform convergence defined by

$$\|\phi\| = \sup \phi(s); \phi \in C([t_0, t_1], E^n).$$

If x is a function from $[-h, \infty)$ to E^n , then x_t is a function defined on the delay interval

$[-h, 0]$ given as :

$$x_t(s) = x(t-s); s \in [-h, 0], t \in [0, \infty).$$

Definition 2.1 (Balachandran(2009))

The Riemann

– Liouville fractional integral operator of order $\beta > 0$ of

function $f \in C_n$, $n \geq -1$ is defined as:

$$I^\beta f(t) = \frac{1}{\rho(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds$$

Definition 2.2 (fractional derivative)

If the function f

$\in C^m$ and m is positive integer, then we can define the fractional derivative of $f(t)$ in the Caputo sense as:

$$\frac{d^n f(t)}{dt^n} = \frac{1}{\rho(m-n)} \int_0^t (t-s)^{m-n-1} f^m(s) ds; m-1 < n \leq m.$$

If $m = 1$, then $m-1 < n \leq m$ becomes $0 < n \leq 1$. Then

$$\begin{aligned} \frac{d^n f(t)}{dt^n} &= \frac{1}{\rho(1-n)} \int_0^t (t-s)^{1-n-1} f^1(s) ds \\ &= \frac{1}{\rho(1-n)} \int_0^t (t-s)^{-n} f^1(s) ds \\ &= \frac{1}{\rho(1-n)} \int_0^t \frac{1}{(t-s)^n} f^1(s) ds \\ &= \frac{1}{\rho(1-n)} \int_0^t \frac{f^1(s)}{(t-s)^n} ds, \end{aligned}$$

where $f^1(s)$

$$= \frac{df(s)}{ds} \text{ and } f \text{ is an abstract function with values in } X.$$

A. VARIATION OF CONSTANT FORMULA

Consider the following system represented by the fractional integrodifferential equations in Banach spaces with distributed delays in the control of the form:

$$\begin{aligned} \frac{d^n f(t)}{dt^n} &= Ax(t) + \int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) \\ &+ f \left(t, x(t), \int_0^t g(t, s, x(s)) ds \right) \end{aligned} \quad (1.1)$$

$$x(0) = x_0; t \in J = [t_0, t_1].$$

where the state $x(\cdot)$ takes values in the Banach space X , $0 < n < 1$, the control

function u

$\in L_2([t_0, t_1], U)$, a Banach space of admissible control functions with

U as a Banach space. $H(t, \theta)$ is an $n \times m$ matrix function continuous at t and of bounded variation

in θ on $[-h, 0]$, $h > 0$ for each $t \in [t_0, t_1]$; $t_1 > t_0$.

The integral is in the Lebesgue

– Stieltjes sense and is denoted by the symbol d_θ . And the nonlinear operators $f: J \times X \times X \rightarrow X$, $g: \Delta \times X \rightarrow X$ are continuous; $\Delta = \{(t, s): 0 \leq s \leq t \leq t_1\}$.

$$\text{If, } Gx(t) = \int_{t_0}^t g(t, s, x(s)) ds,$$

then the equation (1.1) becomes equivalent to the following nonlinear integral equation

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} Ax(s) ds \\ &+ \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} \left[\int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) \right. \\ &\left. + \theta \right] ds \\ &+ \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} f(t, x(t), Gx(s)) ds \end{aligned}$$

And the mild solution of the system (1.1) is given by

$$\begin{aligned} x(t) &= T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s) \\ &\left[\int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) \right. \\ &\left. + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s) f(t, x(t), Gx(s)) ds \right] ds \end{aligned} \quad (1.2)$$

which is similar to the concept defined in the book of Pazy(1983).

For the limiting case, n

$\rightarrow 1$, the above system(1.2) representation becomes

$$\begin{aligned} x(t) &= T(t)x_0 + \int_{t_0}^t T(t-s) \int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) ds \\ &+ \int_{t_0}^t T(t-s) f(t, x(t), Gx(s)) ds \end{aligned} \quad (1.3)$$

Which is the mild solution of

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + \int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) \\ &+ f(t, x(t), Gx(s)) \end{aligned}$$

With initial condition $x(0) = x_0 \in X$.

Analogous to the conventional controllability concept.

A careful observation of the solution

of the system(1.1) given as system(1.2) shows that the values of the control function $u(t)$

for t

$\in [-h, t_1]$ enter the definition of complete state thereby creating the need for an

explicit variation of constant formula. The control in the

2nd term of the formula(1.2),
 therefore, has to be separated in the intervals
 $[-h, 0]$ and $[0, t_1]$.

To achieve this that 2nd term of system
 (1.2) has to be transformed by applying the method
 of Klamka as contained in Chukwu(1992).

Finally, we interchange the order of integration
 using the Unsymmetric Fubini theorem to have

$$x(t) = T(t)x_0 + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s)H(s-\theta)u(s, \theta)ds \right)$$

$$+ \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s)f(t, x(t), Gx(s))ds \quad (2.0)$$

$$\Rightarrow x(t) = T(t)x_0 + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{t_0+\theta}^{t+\theta} (t-s)^{n-1} T(t-s)H(s-\theta, \theta)u(s-\theta, \theta)ds \right)$$

$$+ \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s)f(t, x(t), Gx(s))ds \quad (2.1).$$

Simplifying system(2.1), we have

$$x(t) = T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s)f(t, x(t), Gx(s))ds + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{0+\theta}^t (t-s)^{n-1} T(t-s)H(s-\theta, \theta)u_0(s)ds \right) + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_0^{t+\theta} (t-s)^{n-1} T(t-s)H(s-\theta, \theta)u(s)ds \right) \quad (2.2)$$

Using again the Unsymmetric Fubini Theorem
 on the change of the order of integration and
 incorporating H^* as defined below:

$$H^*(s-\theta, \theta) = \begin{cases} H(s-\theta, \theta), & \text{for } s \leq t \\ 0, & \text{for } s \geq t \end{cases} \quad (2.3)$$

System(.2.2) becomes

$$x(t) = T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s)f(t, x(t), Gx(s))ds$$

$$+ \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{0+\theta}^0 (t-s)^{n-1} T(t-s)H(s-\theta, \theta)u_0(s)ds \right) + \int_{t_0}^t \left[\frac{1}{\rho(n)} \int_{-h}^0 (t-s)^{n-1} T(t-s)d_\theta H^*(s-\theta, \theta)u(s)ds \right] \quad (2.4)$$

Integration is still in the Lebesgue Stieltjes sense in the
 variable θ in H . For brevity, let

$$\alpha(t, s) = T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s)f(t, x(t), Gx(s))ds \quad (2.5)$$

$$\beta(t, s) = \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{0+\theta}^0 (t-s)^{n-1} T(t-s)H(s-\theta, \theta)u_0(s)ds \right) \quad (2.6)$$

$$\mu(t, s) = \frac{1}{\rho(n)} \int_{-h}^0 (t-s)^{n-1} T(t-s)d_\theta H^*(s-\theta, \theta)u(s)ds \quad (2.7)$$

Substituting equations (2.5), (2.6) and (2.7) in equation (2.4),
 we have a precise variation of constant formula for the
 system (1.1) as:

$$x(t, x_0, u) = \alpha(t, s) + \beta(t, s) + \int_{t_0}^t \mu(t, s)ds \quad (2.8).$$

III. MAIN RESULTS

The optimal control problem can be best understood in the
 context of a game of pursuit Hajek(1994), Heyman(1994).
 The emphasis here is the search for a control energy that can
 steer the state of the system of interest to the target set (which
 can be a moving point function or a compact set function) in a
 minimum time. In other words, the optimal control problem
 is best stated as follows:

If $t^* = \text{infimum}\{t: A(t, t_0) \cap G(t, t_0) \neq \emptyset, \text{ for } t \in t_0, t_1\}$.

Is there an admissible control u^*

$\in U$ such that the solution(state) $x(t^*)$ of the
 system with this admissible control u^* be steered into
 the target? Theorem(3.1)and
 Theorem(3.2) that follow answer the question.
 Theorem(3.1).

Consider system(3.1) below as a differential game of
 pursuit with its basic assumptions as of the
 system(1.1).

$$\frac{d^n f(t)}{dt^n} = Ax(t) + \int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) + f \left(t, x(t), \int_0^t g(t, s, x(s)) ds \right) \quad (3.1)$$

$$x(0) = x_0; t \in J = [t_0, t_1].$$

where the state $x(\cdot)$ takes values in the Banach space X , $0 < n < 1$, the control

function u

$\in L_2([t_0, t_1], U)$, a Banach space of admissible control functions with

U as a Banach space. $H(t, \theta)$ is an $n \times m$ matrix function continuous at t and of bounded variation in θ on $[-h, 0]$, $h > 0$ for each $t \in [t_0, t_1]$; $t_1 > t_0$.

The integral is in the Lebesgue

– Stieltjes sense and is denoted by the symbol d_θ .

And the

nonlinear operators $f: J \times X \times X \rightarrow X$, $g: \Delta \times X \rightarrow X$ are continuous; $\Delta = \{(t, s): 0 \leq s \leq t \leq t_1\}$.

$$\text{If, } Gx(t) = \int_{t_0}^t g(t, s, x(s)) ds,$$

then the equation (3.1) becomes equivalent to the following nonlinear integral equation

$$x(t) = x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} Ax(s) ds + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} \left[\int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) + \theta \right] ds + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} f(t, x(t), Gx(s)) ds$$

Suppose that the attainable set $A(t, t_0)$ and the target set $G(t, t_0)$ are compact set functions,

respectively, then there exists an admissible control u

$\in U$ such that the state $z(t)$ of the

weapon for the pursuit of the target satisfies

system(3.1) if and only if

$$A(t, t_0) \cap G(t, t_0) \neq \emptyset, \quad \text{for } t \in [t_0, t_1]$$

Or if and only if the system(3.1) is

relatively controllable.

PROOF.

Suppose that the state $z(t)$ of the weapon for pursuit of target satisfies system(3.1) on the

time interval $[t_0, t_1]$ then $z(t) \in G(t, t_0)$ for t

$\in [t_0, t_1]$. We need to show that there

exists a state $x(t, u)$

$\in A(t, t_0)$ of the system(3.1) such that $z(t)$

$= x(t, u)$ for some

$u \in U$ and $t \in [t_0, t_1]$.

Let $\{u^m\}$ be an infinite sequence of points in U . Since

U lies in the closed interval $[-1, 1]$,

it is a constraint control set and it is also a compact set.

Then the infinite sequence $\{u^m\}$

Has a limit u as $m \rightarrow \infty$.

i. e. $\lim_{m \rightarrow \infty} u^m = u$

Now, $x(t, x_0, u^m) \in A(t, t_0)$ and from this, we have

$$x(t, x_0, u^m) = x(t) = T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s) f(t, x(t), Gx(s)) ds + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{0+\theta}^0 (t-s)^{n-1} T(t-s) H(s - \theta, \theta) u_0^m(s) ds \right) + \int_{t_0}^t \left[\frac{1}{\rho(n)} \int_{-h}^0 (t-s)^{n-1} T(t-s) d_\theta H^*(s - \theta, \theta) u^m(s) \right] ds \quad (3.2)$$

Taking limit on both sides of system(3.2) as $m \rightarrow \infty$, we have

$$\lim_{m \rightarrow \infty} x(t, x_0, u^m) = x(t) = T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s) f(t, x(t), Gx(s)) ds + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{0+\theta}^0 (t-s)^{n-1} T(t-s) H(s - \theta, \theta) \lim_{m \rightarrow \infty} u_0^m(s) ds \right) + \int_{t_0}^t \left[\frac{1}{\rho(n)} \int_{-h}^0 (t-s)^{n-1} T(t-s) d_\theta H^*(s - \theta, \theta) \lim_{m \rightarrow \infty} u^m(s) \right] ds \quad (3.3)$$

Implies that

$$x(t, x_0, u) = x(t) = T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s) f(t, x(t), Gx(s)) ds$$

$$\begin{aligned}
 & + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{0+\theta}^0 (t-s)^{n-1} T(t-s) H(s) \right. \\
 & \quad \left. - \theta, \theta) u_0(s) ds \right) \\
 & + \int_{t_0}^t \left[\frac{1}{\rho(n)} \int_{-h}^0 (t-s)^{n-1} T(t-s) d_\theta H^*(s - \theta, \theta) u(s) \right] ds \\
 & = x(t, x_0, u) \in A(t, t_0) \text{ , since } A(t, t_0) \text{ is compact.}
 \end{aligned}$$

Thus there exists a control energy $u \in U$ such that $x(t, x_0, u) = z(t)$ for $t > t_0$.

Since $z(t)$

$\in G(t, t_0)$ and also belongs to $A(t, t_0)$, it follows that : $A(t, t_0) \cap G(t, t_0) \neq \emptyset$, for $t \in [t_0, t_1]$.

Observation

We observed that in any game of pursuit described by a fractional Integrodifferential system in a Banach space with distributed delays in the control, it is always possible to obtain a control energy function to steer the systems state to the target in finite time. However, the next theorem (Theorem (3.2)) is, therefore, a consequence of this understanding and provides sufficient conditions for the existence of the control energy function that is capable of steering the state of the system (3.1) to the target set in minimum time.

Theorem 3.2. (Sufficient condition)

Consider system (3.1) given as :

$$\begin{aligned}
 \frac{d^n f(t)}{dt^n} &= Ax(t) + \int_{-h}^0 d_\theta H(t, \theta) u(t + \theta) \\
 & + f \left(t, x(t), \int_0^t g(t, s, x(s)) ds \right) \quad (3.1)
 \end{aligned}$$

$$x(0) = x_0 ; t \in J = [t_0, t_1].$$

With its basic assumptions.

Suppose that the system(3.1)is relatively controllable on the finite interval $J = [t_0, t_1]$,

Then there exists an optimal control. Onwuatu (1993) .

PROOF

By the definition of relative controllability, the intersection condition holds

i.e. $A(t, t_0) \cap G(t, t_0) \neq \emptyset$, for $t \in [t_0, t_1]$

then there exists a solution $z(t)$

$\in A(t, t_0)$ such that $z(t)$

$\in G(t, t_0)$, for $z(t)$ arbitrary.

This implies that $z(t) = x(t, x_0, u)$.

Hence , $x(t, x_0, u) \in A(t, t_0)$ and $x(t, x_0, u) \in G(t, t_0)$

Recall that an attainable set $A(t, t_0)$ is a translation of reachable set through the origin η

i.e. $A(t, t_0) = \eta + R(t, t_0)$.

Here, $\eta = T(t)x_0 + \frac{1}{\rho(n)} \int_{t_0}^t (t-s)^{n-1} T(t-s) - s f(t, x, Gx) ds$

$$\begin{aligned}
 & + \int_{-h}^0 d_{H_\theta} \left(\frac{1}{\rho(n)} \int_{0+\theta}^0 (t-s)^{n-1} T(t-s) H(s) \right. \\
 & \quad \left. - \theta, \theta) u_0(s) ds \right) .
 \end{aligned}$$

Thus, it follows that $z(t) \in R(t, t_0)$ for $t \in [t_0, t_1]$, $t > t_0$ can be defined as :

$$Z(t) = \int_{t_0}^t \left[\frac{1}{\rho(n)} \int_{-h}^0 (t-s)^{n-1} T(t-s) d_\theta H^*(s - \theta, \theta) u(s) \right] ds.$$

Now, let $t^* = \text{infimum}\{t : z(t) \in A(t, t_0), t \in [t_0, t_1]\}$ and $0 \leq t_m \leq t_1$

which implies that $\{t_m\}$ is an infinite sequence of times and a corresponding infinite

sequence of controls $\{u_m\}$

$\subset U$, with $\{t_m\}$ converging to t^*

– the minimum time and

$\{u_m\}$ converging to u^* – the optimal control

Let $z(t_m) = y(t_m, u_m) \in R(t, t_0)$, then

$$|z(t^*) - y(t^*, u_m)|$$

$$= |z(t^*) - z(t_m) + z(t_m) - y(t^*, u_m)|$$

$$\leq |z(t^*) - z(t_m)| + |z(t_m) - y(t^*, u_m)|$$

$$\leq |z(t^*) - z(t_m)| + |y(t_m, u_m) - y(t^*, u_m)|$$

$$\leq |z(t^*) - z(t_m)| + \int_{t^*}^{t_m} \|y(s, u_m)\| ds.$$

By the continuity of $z(t)$ which follows the continuity of $R(t, t_0)$ as a continuous set function

and the integrability of $\|y(t)\|$, it follows that

$$y(t^*, u_m) \rightarrow z(t^*)$$

as $m \rightarrow \infty$, where $z(t^*) = y(t^*, u^*) \in R(t, t_0)$.

For some $u^* \in U$ and by definition of t^* ,

u^* is an optimal control. This establishes the existence of an optimal control for the fractional integrodifferential system(3.1)

vis – a – vis system(1.1).

IV. CONCLUSION

The explicit variation of constant formula for the system (1.1) visa-à-vis system (3.1) was established using the Unsymmetric Fubini theorem. The set functions upon which our studies hinged were extracted from the Mild Solution.

According to Hajek (1994), Hayman (1994), the optimal control problem can be best understood in the context of a game of pursuit. Here, the emphasis is the search for a control energy function that can steer the state of the system of interest to the target set in a minimum time.

Employing this context of a game of pursuit, we stated and proved in theorem(3.1), that in any game of pursuit described by a Fractional Integrodifferential Systems in Banach Spaces with Distributed Delays in the Control, it is always possible to obtain a control energy function to steer the systems' state to the target set in finite time.

While the sufficient conditions for the existence of the control energy function (optimal control) which is capable of steering the state of system (3.1) visa-a--via system (1.1) to the target set in minimum time is for the system (3.1) visa-a--via system (1.1) to be relatively controllable. This sufficient condition was stated and proved in theorem (3.2).

REFERENCES

- [1]. B.Bonilla, M.Rodriguez-germa, J.J.Trujillo (2007), Fractional Integrodifferential Equations As Alternative Models to Nonlinear Differential Equations, Applied Mathematics and Computation vol.187 page79-88

- [2]. D.Delbosco, L.Rodino(1996), Existence and Uniqueness for a FractionalDifferential Equations, Journal Mathematical Analysis and Applications.vol.204 pp609-625.
- [3]. V.Lakshmikantham (2008), Theory of FractionalFunctional Differential Equations, Journal of Nonlinear Analysis.vol.69 pp2677-2682
- [4]. C.Bonnet, J.R.Partington(2000), Oprime Factorizations and Stability of Fractional Differential systems, Systems and Control Letters vol.42, pp167-174
- [5]. Y.Nec, A.A.Npomnyashcha(2007), Linear Stability of Fractional Integrodifferential Systems Mathematical Modelling of Natural Phenomena.vol.2 pp77-105
- [6]. K. Balachandran, J.Y.Park(2009), Controllability ofFractional Integrodifferential Systems inBanach Spaces, Journal of Nonlinear Analysis vol.3 pp363-367
- [7]. P.A.Oraekie (2013), Relative Controllability of Neutral Volterra Integrodifferential Systems with Zero in the Interior of Reachable Set, vol.14 No.1 pp3271-3282.
- [8]. M. A.A.El-sayed (1966), Fractional Order Diffusion Wave Equation Integrodifferential, Journal of Theoretical Physics, vol.35.pp311-322.
- [9]. J. Klamka (1993), Controllability of Dynamical Systems, Dowdiest Waver Academic Publisher
- [10]. R.F Curtain, A.J. Pritchard (1978), Infinite Dimensional Linear Systems Theory, Springer-Verlag, New York.
- [11]. K.Balachandran and J.P. Dauer(1989), Relative Controllability of Perturbations of Nonlinear Systems, Journal of Optimization Theory and Applications, vol.16, pp 07-28.
- [12]. K.Balachandran and J.P. Dauer (2002), Controllability of Nonlinear Systems, Journal of Optimization Theory and Applications, vol.115, pp51-56.
- [13]. Y.Q.Chea, H.S.Ahu, D.Xue, Robust (2006), Controllability of Integrodifferential Systems, Signal Processing, vol.86, pp2794-2802.
- [14]. A. Abdel-Chatan, M.K.A.Moubarak, A.B. Shamardan (2000), Integrodifferential Nonlinear Control Systems, Journal of Fractional Calculus, vol. 17.Pp59-69.
- [15]. A. Pazy (1983), Semi-groups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York.
- [16].E.N. Chukwu (1992), Control of Global Economic Growth with Centre Hold, Ordinary and Delay Differential Equation (ed.J. Weiner and J. K. Hale) London Scientific and Technical pp19-25.
- [17]. K. Balachandran (992), Controllability of Neutral Volterra Integrodifferential Systems,Journal of Australian Mathematical Society, vol34, pp18-25.
- [18]. P. Balasubramaniam (1997), Asymptotic Neutral Volterra Integrodifferential Systems, Journal of Mathematical Systems Estimation and Control, vol.7 pp1-4
- [19]. V. A. Iheagwam and J. U. Onwuatu(2005), Relative Controllability and Null Controllability of Generalized Systems with Distributed Delays in the State and Control, Journal of Nigerian Association of Mathematical Physics, vol.9,pp221-238..
- [20]. O. Hajek (1994), Duality for Differential Games and Optimal Control, Journal of Mathematical Systems Theory, vol.8.
- [21]. M. Heyman and J. R.itov (1994), On Linear Pursuit Games with an Unknown Trap, Journal of Optimization Theory and Applications, vol.42, 421-425.