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Global Behavior of A System of Two Nonlinear Difference Equation

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Abstract— **In this paper, we study the global behavior for a system of two nonlinear difference equations**

$$x_{n+1}=\frac{x_n}{A+ay_n},\quad y_{n+1}=\frac{y_n}{B+bx_n},$$

where $n=0,1,\cdots,A,B,a,b\in(0,\infty)$, $x_0,y_0\in(0,\infty)$.
We find that the unique positive equilibrium is global asymptotically stable under certain conditions. Finally, some illustrative examples are given to show the effective of results obtained.

Index Terms—**difference equation, global attractor, locally asymptotic stability**

I. INTRODUCTION

In this paper we investigate the global behavior of solutions of the following system

$$x_{n+1}=\frac{x_n}{A+ay_n},\quad y_{n+1}=\frac{y_n}{B+bx_n},\ n=0,1,\cdots, \tag{1}$$

where

$$A,B,a,b\in(0,\infty)\ , \tag{2}$$

and initial conditions $x_0,y_0\in(0,\infty)$.

In Eq.(1) if $A=B,a=b,x_n=y_n,n=0,1,\cdots,$ then Eq.(1) can be rewritten as the following difference equation.

$$x_{n+1}=\frac{x_n}{A+ax_n},\ n=0,1,\cdots, \tag{3}$$

where

$$A,a\in(0,\infty). \tag{4}$$

The equilibrium of Eq.(3) is the solution of the following Equation

$$x=\frac{x}{A+ax} \tag{5}$$

So $\bar{x}=0$ is always an equilibrium, and when

$$A<1 \tag{6}$$

Eq.(3) has a unique positive equilibrium $\bar{x}=\dfrac{1-A}{a}$.

Eq.(3) has been studied in [3] and whose global behavior of solutions is described by the results as follows

Theorem 1.1. [1] Assume that (4) and (6) hold, then the unique positive equilibrium $\bar{x}$ of Eq.(3) is globally asymptotically stable.

In 1993,Kocic et al.[4] have investigated the stability of all positive solutions of the following difference equation

$$x_{n+1}=\frac{x_n}{A+\sum_{i=0}^{k}a_ix_{n-i}},n=0,1,\cdots, \tag{7}$$

where

$$A,a_i\in(0,\infty),i=0,1,\cdots,k. \tag{8}$$

and obtained the following main results.

Theorem 1.2. [4] Assume that (8) holds. Then the following statements are true.

(i) Assume $A\geq 1$, then every positive solution of Eq.(7) decreases zero.

(ii) Assume that $A<1$ holds. Then Eq.(7) is permanent.

(iii) Assume that $A<1$ holds, and one of the following three conditions is satisfied:

(a) $\sum_{i=0}^{k}a_i\bar{x}k\leq 1$;

(b) $\sum_{i=1}^{k}a_i\bar{x}k\leq 1+a_0\bar{x}$;

(c) $\sum_{i=0}^{k}a_i\bar{x}(k-1)\leq 1$.

Then $\bar{x}=\dfrac{1-A}{\sum_{i=0}^{k}a_i}$ is a global attractor of all positive solutions of Eq. (7).Furthermore, Kulenovic et al. [5] proposed an open problem [4], i.e.,Let $A,a_i\in(0,\infty),i=0,1,\cdots,k$, the global asymptotical stability of the equilibrium points of Eq.(7) with positive initial conditions. Other related results can refer to [2,6,7].

In 2007, Hu and Li [8] investigated this open problem and obtained the following theorem

Theorem 1.3. [8] Assume that (8) and $A<1$ hold. Then the unique positive equilibrium $\bar{x}$ of Eq. (7) is globally asymptotically stable provided that $\sum_{i=1}^{k}a_i>a_0$.

Although difference equations are sometimes very simple in their forms, they are extremely difficult to understand thoroughly the behavior of their solutions. In book [3] Kocic and Ladas have studied global behavior of nonlinear difference equations of higher order. Similar nonlinear systems of rational difference equations were investigated (see [9]). Other related results reader can refer [12-20].

Motivated by above discussion, we study the global behavior all positive solutions of Eq.(1) under certain conditions.

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A pair of sequences of positive real numbers (x_n, y_n) that satisfies Eq.(1) is a positive solution of Eq.(1). The equilibrium $(\overline{x}, \overline{y})$ of Eq.(1) is the solution of the following equation

$$x = \frac{x}{A+ay}, y = \frac{y}{B+bx}. \quad (9)$$

so $\overline{x} = 0, \overline{y} = 0$ is always an equilibrium point.

A positive solution (x_n, y_n) of Eq.(1) is bounded and persists if there exist positive constants M, N such that

$$M \le x_n, y_n \le N, \ n = 0,1,\cdots.$$

II. Main Results

The equilibrium of equation (1) are (0,0) and $(\frac{1-B}{b}, \frac{1-A}{a})$, for $A<1$ and $B<1$. In addition if $A=1, B<1$, then (0,0) and $(\frac{1-B}{b}, 0)$ are equilibrium point of system (1), and if $A<1, B=1$, then (0,0) and $(0, \frac{1-A}{a})$ are the equilibrium point of system (1). Finally, if $A>1$ and $B>1$, (0, 0) is the unique equilibrium point.

We summarize the local stability of the equilibrium of equation (1) as follows.

Theorem 2.1 Considering the system of difference equations (1). Assume that (2) holds and

$$A<1, B<1. \quad (10)$$

Then the following statements are true.

(i) The equilibrium (0, 0) is a repel.

(ii) Eq.(1) has a unique positive equilibrium $(\overline{x}, \overline{y})$ with

$$\overline{x} = \frac{1-B}{b}, \quad \overline{y} = \frac{1-A}{a}.$$

(iii) The unique positive equilibrium $(\overline{x}, \overline{y})$ is locally unstable .

Proof. (i) The linearized equation of system (1) about (0, 0) is

$$\Psi_{n+1} = B\Psi_n \quad (11)$$

where

$$\Psi_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{A} & 0 \\ 0 & \frac{1}{B} \end{pmatrix}.$$

The characteristic equation of (10) is

$$(\lambda - \tfrac{1}{A})(\lambda - \tfrac{1}{B}) = 0 \quad (12)$$

This shows that all the roots of characteristic equation lie outside unit disk. So the unique equilibrium (0,0) is a repel.

(ii) Let $(\overline{x}, \overline{y})$ satisfy (9). By simply calculating and noting (10) we have

$$\overline{x} = \frac{1-B}{b}, \ \overline{y} = \frac{1-A}{a}$$

from which it follows that $(\overline{x}, \overline{y})$ is a unique positive equilibrium of Eq.(1). This completes the proof of (ii).

(iii) We can easily obtain that the linearized system of (1) about the positive equilibrium $(\frac{1-B}{b}, \frac{1-A}{a})$ is

$$X_{n+1} = X_n + \tfrac{a}{b}(B-1)Y_n, \ Y_{n+1} = \tfrac{b}{a}(A-1)X_n + Y_n. \quad (13)$$

from which we can easily obtain there are eigenvalues $\lambda = 1 \pm \sqrt{(A-1)(B-1)}$ of Jacobian matrix of (13).i.e , one lies outside the unit disk and the other lies inside the unit disk. This implies that the positive equilibrium $(\frac{1-B}{b}, \frac{1-A}{a})$ is locally unstable and the equilibrium is called a saddle point. The proof of theorem is completed.

Theorem 2.2. Suppose that

$$A>1, B>1 \quad (14)$$

holds, then the unique equilibrium (0,0) is globally asymptotically stable.

Proof. From (12) and (14) we know that all the roots of characteristic equation lie inside unit disk. So the unique equilibrium (0,0) is locally asymptotically stable.

So, let (x_n, y_n) be a solution of system (1). It suffices to show that

$$\lim_{n\to\infty} x_n = 0, \quad \lim_{n\to\infty} y_n = 0.$$

Since

$$0 \le x_{n+1} = \frac{x_n}{A+ay_n} < \frac{1}{A}x_n < x_n,$$

$$0 \le y_{n+1} = \frac{y_n}{B+bx_n} < \frac{1}{B}y_n < y_n, \ n = 0,1,\cdots,$$

So $\lim_{n\to\infty} x_n = 0, \ \lim_{n\to\infty} y_n = 0$. This completes the proof.

Theorem 2.3. Assume (10) holds, and let (x_n, y_n) be a solution of system (1) such that

(i) $x_0 < \overline{x}, y_0 > \overline{y}$; or (ii) $x_0 > \overline{x}, y_0 < \overline{y}$.

Then (x_n, y_n) non-oscillates about the equilibrium $(\overline{x}, \overline{y})$.

Proof. Assume that the case (i) holds. (The case (ii) is similar and will be omitted). Then

$$x_1 = \frac{x_0}{A+ay_0} < \frac{\overline{x}}{A+a\overline{y}} = \overline{x},$$

$$y_1 = \frac{y_0}{B+bx_0} > \frac{\overline{y}}{B+b\overline{x}} = \overline{y}.$$

We prove it by induction.

For $n=k, \ x_k < \overline{x}, \ y_k > \overline{y}$ hold. It follows that, for $n = k+1$

$$x_{k+1} = \frac{x_k}{A+ay_k} < \frac{\overline{x}}{A+a\overline{y}} = \overline{x},$$

$$y_{k+1} = \frac{y_k}{B+bx_k} > \frac{\overline{y}}{B+b\overline{x}} = \overline{y}.$$

This completes the proof of Theorem 2.3.

III. Some Illustrative Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, some numerical examples are considered in this section. These examples represent different types of qualitative behavior of solutions to the system of nonlinear difference equations.

Example 3.1. If the initial conditions $x_0 = 0.7, y_0 = 0.9$ and $A = 0.8, B = 0.9, a = 1.2, b = 1.6$, we have the following system

$$x_{n+1} = \frac{x_n}{0.8 + 1.2y_n}, \quad y_{n+1} = \frac{y_n}{0.9 + 1.6x_n}$$

It is clear that $A < 1, B < 1$. Then the equilibrium (0,0) is locally unstable.(See Theorem 2.1, Fig 1)

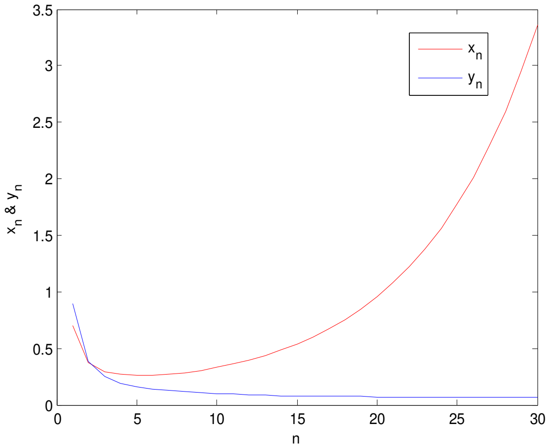

Fig 1: The equilibrium point (0,0) and $(\bar{x}, \bar{y})$ is locally unstable

Example 3.2 If the initial conditions $x_0 = 1.9, y_0 = 2.9$ and $A = 1.8, B = 1.2, a = 1.4, b = 1.6$, we have the following system

$$x_{n+1} = \frac{x_n}{1.8 + 1.4y_n}, \quad y_{n+1} = \frac{y_n}{2.9 + 1.6x_n}$$

It is clear that $A > 1, B > 1$. Then the equilibrium (0,0) is locally unstable.(See Theorem 2.2, Fig 2)

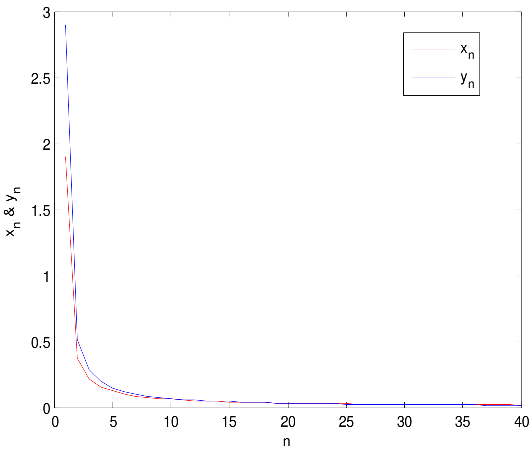

Fig 2:The equilibrium point (0,0) is globally asymptotically stable

IV. CONCLUSION

This paper is concerned with dynamical behavior of second order nonlinear difference equations system. The main results obtained are:

(i) the equilibrium (0,0) and positive equilibrium $(\bar{x}, \bar{y})$ are locally unstable for $A < 1, B < 1$.

(ii) the equilibrium (0,0) is globally asymptotically stable for $A > 1, B > 1$.

(iii) the positive solution of system is non-oscillation under some initial conditions.

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