New Inequalities Associated with the Euler-Mascheroni Constant

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***Abstract*— In this paper we propose new sequence approximating the Euler-Mascheroni constant which converge faster towards its limit and we establish better bounds in inequalities for the Euler constant.**

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MSC2010: 40A05, 33B15, 11Y60

I. INTRODUCTION

It is well known that the sequence

$$\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n, \quad n \geq 1,$$

is convergent to a limit denoted $\gamma = 0{,}5772 \ldots$ now known as Euler-Mascheroni constant. Many authors have obtained different estimations for $\gamma_n - \gamma$, for exemple the following increasingly better

$$\frac{1}{2(n+1)} \leq \gamma_n - \gamma < \frac{1}{2(n-1)}, \, n \geq 2, \qquad [9]$$

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}, \, n \geq 1, \qquad [12]$$

$$\frac{1-\gamma}{n} < \gamma_n - \gamma < \frac{1}{2n}, \, n \geq 1,$$

[2]

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \, n \geq 1, \qquad [6,7]$$

$$\frac{1}{2n + \frac{2}{5}} < \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3}}, \, n \geq 1, \qquad [10]$$

$$\frac{1}{2n + \frac{2\gamma - 1}{1-\gamma}} \leq \gamma_n - \gamma < \frac{1}{2n + \frac{1}{3}}, \, n \geq 1. \qquad [1,10]$$

The convergence of the sequence γ_n to γ is very slow. In 1993, DeTemple [4] studied a modified sequence which converges faster and he proved:

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \, n \geq 1,$$

where $R_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n + \frac{1}{2})$.

In 2010, Chen [3] proved that for all integers $n \geq 1$,

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2},$$

with the best possible constants

$$a = \frac{1}{\sqrt{24(-\gamma + 1 - \ln \frac{3}{2})}} - 1 = 0{,}55106 \ldots \text{ and } b = \frac{1}{2}.$$

In 1999, A. Vernescu [10] have found a fast convergent sequence to γ, by having the idea to replace the last term of the harmonic sum. He proved that the sequence

$$x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n, \text{ for } n \geq 2, \text{ is}$$

strictly increasing and convergent to γ. Moreover,

$$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}, \quad n \geq 2.$$

Recently, Chen [3] obtained the following bounds for $\gamma - x_n$

$$\frac{1}{12(n+a)^2} \leq \gamma - x_n < \frac{1}{12(n+b)^2}, \quad n \geq 2,$$

with the best possible constants

$$a = \frac{1}{\sqrt{12\gamma - 6}} - 1 = 0{,}038859 \ldots \quad \text{and} \quad b = 0.$$

In 2015 Cringanu [4] obtained the following bounds for $\gamma - x_n$:

For every $a > 0$ there exists $n_a \in N$, $n_a \geq 2$, such that

$$\frac{1}{12(n+a)^2} < \gamma - x_n < \frac{1}{12n^2}, \text{ for all } n_a \geq 2.$$

Inspired by this result we consider the sequence

$y_n = \gamma - x_n - \frac{1}{12n^2}$. The tool for measuring the speed of convergence is a result stated by Mortici [5] according to which a sequence x_n converging to zero is the fastest possible when the difference $y_n - y_{n+1}$ is the fastest possible. More precisely, if there exists the $\lim_{n\to\infty} n^k (y_n - y_{n+1}) = l$, then $\lim_{n\to\infty} n^{k-1} y_n = \frac{l}{k-1}$. Recent results using this lemma were obtained for example in [2, 6-8].

In our case of y_n, we have

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$$y_n - y_{n+1} = \frac{1}{2n} + \frac{1}{2n+2} - \ln(1+\frac{1}{n}) - \frac{1}{2n^2} + $$
$$+ \frac{1}{12(n+1)^2},$$

and using a Mac-Laurin growth serie we get

$$y_n - y_{n+1} = -\frac{1}{30n^5} + O(\frac{1}{n^6}),$$

and so $\lim_{n\to\infty} n^5(y_n - y_{n+1}) = -\frac{1}{30}$.

By the above result we obtain $\lim_{n\to\infty} n^4 y_n = \frac{l}{k-1} = -\frac{1}{120}$, so that

$$\lim_{n\to\infty} n^4(\gamma - x_n - \frac{1}{12n^2}) = -\frac{1}{120}.$$

Let $K_n = x_n + \frac{1}{12n^2}$, and so

$$\lim_{n\to\infty} n^4(K_n - \gamma) = \frac{1}{120}.$$

From this result we prove in this paper that for all $a > 0$ there exists $n_a \in N$ such that for all $n \geq n_a$ we have

$$\frac{1}{120(n+a)^4} < K_n - \gamma < \frac{1}{120n^4} \quad \text{for all } n \geq n_a,$$

using an elementary sequence method.

II. THE MAIN RESULT

Theorem 2.1. (i) For every integer $n \geq 1$ we have

$$K_n - \gamma < \frac{1}{120n^4};$$

(ii) For every $a > 0$ there exists $n_a \in N$ such that

$$\frac{1}{120(n+a)^4} < K_n - \gamma \quad \text{for all } n \geq n_a.$$

Proof. We define the sequence

$$a_n = K_n - \gamma - \frac{1}{120(n+a)^4} = 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} +$$
$$+ \frac{1}{2n} - \ln n + \frac{1}{12n^4} - \gamma - \frac{1}{120(n+a)^4},$$

for $a \geq 0$, and so $a_{n+1} - a_n = f(n)$, where

$$f(n) = \frac{1}{2n} + \frac{1}{2n+2} - \ln(n+1) + \ln n + \frac{1}{12(n+1)^2} -$$
$$-\frac{1}{12n^2} - \frac{1}{120(n+a+1)^4} + \frac{1}{120(n+a)^4}.$$

The derivative of function f is equal to

$$f'(n) = -\frac{1}{2n^2} - \frac{1}{2(n+1)^2} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{6(n+1)^3} +$$
$$+ \frac{1}{6n^3} + \frac{1}{30(n+a+1)^5} - \frac{1}{30(n+a)^5} =$$
$$= \frac{P(n)}{30n^3(n+1)^3(n+a)^5(n+a+1)^5}, \quad \text{where}$$

$$\begin{aligned}
P(n) &= 30an^9 + 5(39a^2 + 27a - 1)n^8 + \\
&+ 10(58a^3 + 78a^2 + 23a - 2)n^7 + \\
&+ (1045a^4 + 2030a^3 + 1210a^2 + 175a - 31)n^6 + \\
&+ (1260a^5 + 3135a^4 + 2710a^3 + 900a^2 + 45a - 23)n^5 + \\
&+ (1050a^6 + 3150a^5 + 3485a^4 + 1700a^3 + \\
&+ 315a^2 - 10a - 8)n^4 + \\
&+ (600a^7 + 2100a^6 + 2800a^5 + 1745a^4 + \\
&+ 490a^3 + 40a^2 - 5a - 1)n^3 + \\
&+ 25a^3(9a^5 + 36a^4 + 56a^3 + 42a^2 + 15a + 2)n^2 + \\
&+ 25a^4(2a^5 + 9a^4 + 16a^3 + 14a^2 + 6a + 1)n + \\
&+ 5a^5(a^5 + 5a^4 + 10a^3 + 10a^2 + 5a + 1).
\end{aligned}$$

(i) If $a = 0$ then

$$P(n) = -5n^8 - 20n^7 - 31n^6 - 23n^5 - 8n^4 - n^3 < 0,$$

for all $n \geq 1$, and then f is strictly decreasing.

We have $f(\infty) = 0$ and then $f(n) > 0$ for all $n \geq 1$, so that $(a_n)_{n\geq 1}$ is strictly increasing. Since (a_n) converges to zero it results that $a_n < 0$ for all $n \geq 1$, so that

$$K_n - \gamma < \frac{1}{120n^4}, \quad \text{for all } n \geq 1.$$

(ii) If $a > 0$ then there exists $n_a \in N$ such that $P(n) > 0$ for all $n \geq n_a$ and then f is strictly increasing on $[n_a, \infty)$. Since $f(\infty) = 0$ it results that $f(n) < 0$ for all $n \geq n_a$, so that $(a_n)_{n\geq n_a}$ is strictly decreasing.

The sequence (a_n) converges to zero and then it results that $a_n > 0$ for all $n \geq n_a$, so that

$$\frac{1}{120(n+a)^4} < K_n - \gamma \quad \text{for all } n \geq n_a.$$

Now we find the constant n_a in some particular cases.

For exemple, if $a = 0{,}03 = \frac{3}{100}$, then

$$P(n) = \frac{9}{10}n^9 - \frac{1549}{2000}n^8 - \frac{619117}{50000}n^7 -$$
$$-\frac{492106871}{20000000}n^6 - \frac{324441563}{15625000}n^5 -$$

$$-\frac{159353996791}{20000000000}n^4-\frac{549643482989}{500000000000}n^3+$$
$$+\frac{672122172249}{4000000000000000}n^2+\frac{483180932133}{20000000000000000}n+$$
$$+\frac{2817036000549}{2000000000000000000000}>0,$$

for all $n\geq 5$, and so

$$\frac{1}{120(n+\frac{3}{100})^4}<K_n-\gamma<\frac{1}{120n^4} \quad \text{for all } n\geq 5.$$

Let us remark that a direct calculus show that these inequalities hold and for $n=4$, and then

$$\frac{1}{120(n+\frac{3}{100})^4}<K_n-\gamma<\frac{1}{120n^4} \quad \text{for all } n\geq 4.$$

Now, if $a=0{,}01=\frac{1}{100}$, then

$$P(n)=\frac{3}{10}n^9-\frac{7261}{2000}n^8-\frac{881071}{50000}n^7-$$
$$-\frac{582539191}{20000000}n^6-\frac{5614314631}{250000000}n^5-$$

$$-\frac{161335296679}{20000000000}n^4-\frac{522746133947}{500000000000}n^3+$$
$$+\frac{21542563609}{4000000000000000}n^2+\frac{5307080451}{20000000000000000}n+$$
$$+\frac{10510100501}{2000000000000000000000}>0,$$

for all $n\geq 17$, and so

$$\frac{1}{120(n+\frac{1}{100})^4}<K_n-\gamma<\frac{1}{120n^4} \quad \text{for all } n\geq 17.$$

Let us remark that a direct calculus show that these inequalities hold and for $n\in\{12,13,14,15,16\}$, and then

$$\frac{1}{120(n+\frac{1}{100})^4}<K_n-\gamma<\frac{1}{120n^4} \quad \text{for all } n\geq 12.$$

REFERENCES

[1] H. Alzer, *Inequalities for the gamma and polygamma functions*, Abh. Math. Sem. Univ. Hamburg 68 (1998) 363-372.
[2] G.D. Anderson, R.W. Barnard, K.C. Richards, M.K. Vamanamurthy, M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. 345 (1995) 1713-1723.
[3] C.P. Chen, *Inequalities for the Euler-Mascheroni constant*, Applied Mathematics Letters 23 (2010) 161-164;
[4] J. Cringanu, *Better bounds in Chen's inequalities for the Euler constant*, Bull. Aust. Math. Soc. Vol 92, No 1 (2015) pp 94-97.
[5] D.W. DeTemple, *A quicker convergence to Euler's constant*, Amer. Math. Monthly 100 (5) (1993) 468-470;
[6] C. Mortici, *On new sequences converging towards the Euler-Mascheroni constant*, Computers and Mathematics with Applications 59 (2010) 2610-2614.
[7] C. Mortici, A. Vernescu, *An improvement of the convergence speed of the sequence $(\gamma_n)_{n\geq1}$ converging to Euler's constant*, An. Stiint. Univ. "Ovidius" Constanta 13 (1) (2005) 97-100.
[8] C. Mortici, A. Vernescu, *Some new facts in discrete asymptotic analysis*, Math. Balkanica (NS) 21 (Fasc. 3-4) (2007) 301-308.
[9] T. Negoi, *A faster convergence to the constant of Euler*, Gazeta Matematica Seria A 15 (1997) 111-113 (in Romanian).
[10] S. R. Tims, J. A. Tyrrel, *Approximate evaluation of Euler's constant*, Math. Gaz., 55 (1971) 65-67.
[11] L. Toth, *Probem E3432*, Amer. Math. Monthly, 98 (3) (1991) 264.
[12] L. Toth and J. Bukor, *On the alternating series* $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$, J. Math. Anal. Appl. 282 (2003) 21-25.
[13] R. M. YOUNG, Euler's constant, Math. Gaz. 75(472) (1991) 187-190.

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Works list:
http://www.math.ugal.ro/siteFacStiinte/cadre_did/Lucrari/cringanu-lucrari.pdf