An Axiomatic System for a Physical or Digital but Continuous 3-Dimensional Euclidean Geometry, Without Infinite Many Points

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Abstract: **This paper is concerned with finding an axiomatic system, so as to define the 3-dimensional Euclidean space, without utilizing the infinite ,that can imply all the known geometry for practical applied sciences and engineering applications through computers , and for more natural and perfect education of young people in the Euclidean geometric thinking. In other words by utilizing only finite many visible and invisible points and only finite sets, and only real numbers with finite many digits, in the decimal representation. The inspiration comes from the physical matter , rigid, liquid and gaseous, which consists of only finite many particles in the physical reality. Or from the way that continuity is produced in a computer screen from only finite many invisible pixels . We present such a system of axioms and explain why it is chosen in such a way. The result is obviously not equivalent, in all the details, with the classical Euclidean geometry. Our main concern is consistency and adequacy but not independence of the axioms between them. It is obvious that within the space of a single paper, we do not attempt to produce all the main theorems of the Euclidean geometry, but present only the axioms.**

Index Terms— **Axiomatic systems of Euclidean geometry, Digital Mathematics, Digital space, Constructive mathematics, Non-standard mathematics.**

Mathematical Subject Classification: **03F99, 03H99, 93C62, 51M05**

I. INTRODUCTION.

Changing our axiomatic system of the Euclidean geometry so as as to utilize only finite points, numbers and sets, means that we change also our perception our usual mental images and beliefs about the reality. This project is under the next philosophical principles

1) Consciousness is infinite. Conversely the infinite is a function and property of the consciousnesses.

2) But the physical material world is finite.

3) Therefore mathematical models in their ontology should contain only finite entities and should not involve the infinite.

This paper is part of larger project which is creating again the basic of mathematics and its ontology with new axioms that do not involve the infinite at all.

Our perception and experience of the reality, depends on the system of beliefs that we have.

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In mathematics, the system of spiritual beliefs is nothing else than the axioms of the axiomatic systems that we accept. The rest is the work of reasoning and acting.

Quote: "It is not the world we experience but our perception of the world"

The abstraction of the infinite seems sweet at the beginning as it reduces some complexity, in the definitions, but later on it turns out to be bitter, as it traps the mathematical minds in to a vast complexity irrelevant to real life applications. Or to put it a more easy way, we already know the advantages of using the infinite but let us learn more about the advantages of using only the finite, for our perception, modeling and reasoning about empty space. This is not only valuable for the applied sciences, through the computers but is also very valuable in creating a more perfect and realistic education of mathematics for the young people. The new axioms of the Euclidean geometry create a new integrity between what we see with our senses, what we think and write and what we act in scientific applications.

The Euclidean geometry with infinite many points creates an overwhelming complexity which is very often irrelevant to the complexity of physical matter. The emergence of the irrational numbers is an elementary example that all are familiar But there are less known difficult problems like the 3rd Hilbert problem (see [8]). In the 3rd Hilbert problem it has been proved that two solid figures that are of equal volume are not always decomposable in to an in equal finite number of congruent sub-solids! Given that equal material solids consists essentially from the physical point of view from an equal number of sub-solids (atoms) that are congruent, this is highly non-intuitive! There are also more complications with the infinite like the Banach-Tarski paradox (see [1]) which is essentially pure magic or miracles making! In other words it has been proved that starting from a solid sphere S of radius r, we can decompose it to a finite number n of pieces, and then re-arrange some of them with isometric motions create an equal sphere S1 of radius again r and by rearranging the rest with isometric motions create a second solid Sphere S2 again of radius r! In other words like magician and with seemingly elementary operations we may produce from a ball two equal balls without tricks or “cheating”. Thus no conservation of mass or energy!. Obviously such a model of the physical 3-dmensional space of physical matter like the classical Euclidean geometry is far away from the usual physical material reality! I have nothing against miracles, but it is challenging to define a space that behaves as we are used to know. In the model of the 3-dimesional space, as new axiomatic system where such balls have only finite many points such "miracles" are not possible!

The continuous 3-dimensional space, defined axiomatically, is closer to what we know from the continuity of matter and fluids in physical reality, and strictly logically different from the traditional Euclidean space of infinite many points. It is not only the Hilbert's 3rd problem, and the Banach-Tarski paradox which do not hold anymore for the physical or digital 3-dimensional Euclidean space, but also elementary topics like the constructability with ruler and compass.

We know e.g. that the squaring of the circle is not constructible with ruler and compass in the classical Euclidean Geometry with infinite many points. Because it involves the solution of the equation $\pi R^2 = x^2$

and the number π is a transcendental irrational number. But in the digital Euclidean space $E_3(n,m,q)$ the previous equation becomes the next equation of rational numbers

$$[\pi]_m [R]_m{}^2 =_m [x]^2{}_m$$

Where by $[]_m$ we denote the truncation of a real number of infinite decimal points to m only decimal points in the precision level P(m), and by $=_m$ the equlity within the precision level P(m).

And so the constructability with ruler and compass of squaring of the circle must be put together with the next two facts

1) The last Equation is an equation of rational numbers
2) Rational numbers, that is of the form k/l (k, l positive integers) , are constructible with ruler and compass (as linear segments in a line with a unit length)

Still we should not jump in to conclusions. General rational numbers of the form k/l as above may not necessarily belong to the precision level P(m). So it might be necessary to resort to a higher precision level digital geometric space $E_3(n',m',q')$, n'>>n, m'>>m, q'>>q , make the construction with ruler and compass, and then return back to the lower precision level space $E_3(n,m,q)$, to construct a square with equal area with the initial circle.

We shall not only describe a new axiomatic system of the Euclidean geometry but also new axiomatic system of the natural numbers and real numbers , where only finite many numbers with finite many decimal digits are involved. Actually we could start in the meta-mathematics with new axioms and definitions of 1st order and 2nd order formal Logic where only finite many symbols, finite many natural numbers and proofs with finite only steps are involved. But we have not sufficient space for this in this paper, so we shall start only from the natural numbers.

We present such a new system of axioms and explain why it is chosen in such a way. The result is obviously not equivalent, in all the details, with the classical Euclidean geometry. Our main concern is consistency and adequacy but not independence of the axioms between them. In some future research we may analyze finite models of this axiomatic system within the classical Euclidean geometry so as to derive relative consistency (In other words of the classical Euclidean geometry is consistent then so is this digital Euclidean geometry). It is obvious that within the space of a single paper, we do not attempt to produce all the main theorems of the Euclidean geometry, but present only the axioms. The next step is obviously to define a digital differential and Integral calculus over such a digital Euclidean geometry and digital real numbers without convergence of infinite sequences or limits. But again this is not for the space of the current paper but probably of a future such paper. The next presentation of such an axiomatic system is a design of logically organized realistic thinking in the area of numbers and space. It is also a realistic ontology of an operating system for numbers and space, for all practical scientific and engineering applications.

II. THE NEW AXIOMS

As I am a computer programmer too, besides being a mathematician, it became easier for me to, find out the necessary changes of the axioms of traditional mathematics, so as to derive axioms for the digital mathematics.

The axiomatic system adopted here, is that of Hilbert axiomatic system for the Euclidean Geometry, with modifications. (See e.g. [7] or Wikipedia) Surfing among the Euclidean figures of this geometry , is like turning pages in a e-book of a touch-screen mobile. We make here some small modifications of the Hilbert axioms of synthetic visual Euclidean Geometry. Some of the axioms of Hilbert will not hold, (like that which claims that between two points here is always a third) , and some new initial concepts will be added, like that of two types of points visible and invisible, plus some relevant axioms.

I do not claim here that the axioms of the Digital Euclidean Geometry, below, are independent, in other words none of them can be proved from the others. As the elements are finite, there may be such a case. But I am strongly interested a) at first that are non-contradictory, and b) second that are adequate many, so as to describe the intended structure. later simplified and improved in elegance versions of the axioms may be given.

Before we proceed we remind the properties of the axiomatic digital natural numbers and axiomatic decimal digital real numbers, where again no infinite exists.

A. similar to Peano, axioms

We define the natural numbers in two scales (and later precision levels) that are two unequal initial segments of the natural numbers N(ω)< N(Ω) .The number ω is called the Ordinal size ω of the local system of natural numbers N(ω) while the Ω is the cardinal size of the global system of natural numbers. ω<Ω. If we start with integers n_1, n_2, n_3 from N(ω), then their addition and multiplication, have the commutative semiring properties but without closure in N(ω), but with values in N(Ω). We call the N(ω), the *local segment* while the N(Ω) the global segment.

We have here an initial relation among the natural numbers which is called successor or next of a natural number x and it is denoted by S(x).

1) The number 1 is a natural number and belongs both to N(ω), and N(Ω) .

2) There is no natural number whose successor is 1.

3) If x is a natural number of N(ω), its successor S(x) , is also a natural number belonging in N(Ω).

4) If two different numbers of N(Ω) , have the same successor, then they are equal, Formally if S(x)=S(y) then x=y .

*5) (**Peano axiom of induction**) If a property or formal proposition P() holds for 1 (that is P(1)=true) and if when holding for x in N(ω) holds also for P(S(x)) with S(x) in N(Ω) , then it holds for all natural numbers of N(ω).*

*6) **Axiom of sufficient large size**. If we repeat the operations of the commutative semiring starting from elements of the local version N(ω), ω-times, the results are still inside the larger set N(Ω).*

This last Peano axiom of induction is useful only if the natural numbers are formulated within a formal logic (the axiom itself as a formal proposition is in 2nd order formal logic) that its size Ω(l) is less than the size of the objective system of natural numbers Ω. Otherwise for sufficient large Ω(L)>>Ω, we may simply construct a lengthy proof of this axiom starting from P(1) then P(2) ...and finally P(Ω), which then it is a theorem.

Any two models M1 M2 of the digital natural numbers N(Ω),N(ω) of equal size ω, Ω are isomorphic.

B. The axiomatic multi-precision decimal digital real numbers R(n,m,q).

a) The **rational numbers** Q, as we known them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions** k/l of natural numbers k,l exist as numbers and are unique. The cost of course is that when we represent them with decimal representation they may have infinite many but with finite period of repetition decimal digits.

b) The **classical real numbers** R, as we know them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions** of linear segments of Euclidean geometry, exist as numbers and are unique (**Eudoxus** theory of proportions). The cost of course finally is that when we represent them with decimal representation they may have infinite many arbitrary different decimal digits without any repetition.

c) But in the physical or digital mathematical world, such costs are rather not to be accepted. The infinite is not accepted in the ontology of mathematics (only in the subjective experience of the consciousness of the scientist). Therefore in the **multi-precision digital real numbers**, proportions are handled in different way, with priority in the Pythagorean idea of the **creation of all numbers from an integral number of elementary units**, almost exactly as in the physical world matter is made from atoms (here the precision level of numbers in decimal representation) and the definitions are different and more economic in the ontological complexity.

We will choose for all practical applications of the digital real numbers to the digital Euclidean geometry and digital differential and integral calculus, the concept of a system of digital decimal real numbers with three precision levels, lower, low and a high.

Definition II.B.1 The definition of a ***precision level*** P(n,m) where n, m are natural numbers , is that it is the set of all real numbers that in the decimal representation have not more than n decimal digits for the integer part and not more than m digits for the decimal part. Usually we take m=n. In other words as sets of real numbers it is a nested system of lattices each one based on units of power of 10, and as union a lattice of rational numbers with finite many decimal digits. We could utilize other bases than 10 e.g. 2 or 3 etc, but for the sake of familiarity with the base 10 and the 10 fingers of our hands we leave it as it is.

The axioms of the digital real numbers R(n,m,q)

We assume at least three precision levels for an axiomatic decimal system of digital real numbers R(n,m,q) : *the local lower precision level p(n) , the low precision level p(m) , and the high precision level p(q).* Each precision level of order k has $10^{(2k+2\log 2)}$ points or numbers where by log we denote the logarithm with base 10. It has 10^k positive decimal numbers , which are doubled for the negative ones, thus in total at most $2(10^{2k}) \le 10^{(2k+\log 2)}$. And again so many for the integer numbers, thus in total $10^{(4k+2\log 2)}$. Now if for 3-dimensional geometric applications as coordinates of points, this will give $10^{(12k+6\log 2)}$ points inside a big cube.

Whenever we refer to a real number x of a (minimal in precision levels) system of real numbers R(n,m,q) , we will always mean that x belongs to the local lower precision level P(n) and that the system R(n,m,q) has at least three precision levels with the current axioms.

Whenever we write an equality relation $=_m$ we must specify in what precision level it is considered. The default precision level that a equality of numbers is considered to hold, is the low or standard precision level P(n).

The digital real numbers are not anymore linearly ordered field. It has partially defined operations, with values outside the original domain. Still they can be conceived through general algebrae in the context of ***Universal algebra*** (see e.g. [6]).

Some of the Linearly ordered Field operations

The field operations in a precision level are defined in the usual way, from the decimal representation of the numbers. This would be an independent definition, not involving the infinite. Also equality of two numbers with finite decimal digits should be always specified to what precision level. E.g. if we are talking abut equality in P(m) we should symbolize it my $=_m$, while if talking about equality in P(q) we should symbolize it by $=_q$ .If we want to define these operation from those of the real numbers with infinite many **decimal digits, then we** will need the truncation function [a]x of a real number a , in the Precision level P(x).

Symbolism convention: *We denote the multiplication of x and y either as xy or as x*y, and the raising x to power y by* x^y *or x^y.*

Then the operations e.g. in P(n) with values in P(m) n<<m would be

$$[a]_n + [b]_n =_m [a+b]_m$$

$$[a]_n * [b]_n =_m [a*b]_m$$

$$[a]_n^{(-1)} =_m [a^{(-1)}]_m$$

(Although, the latter definition of inverse seems to give a unique number in P(m), there may not be any number in P(m) or not only one number in P(m), so that if multiplied with $[a]_n$ it will give 1. E.g. for n=2 , and m=5 , the inverse of 3, as $([3]_n)$^(-1)$=_m$ $[1/3]_m$ =0.33333 is such that still 0.33333*3$\neq_m$1).

Such a system of double or triple precision digital real numbers, has closure of the linearly ordered field operations only in a specific **local** way. That is if a, b belong to the Local Lower precision, then a+b, a*b , -a, a^(-1) belong to the Low precision level, and the properties of the **linearly ordered commutative field** hold: (here the equality is always in P(m), this it is mean the $=_m$).

1) if a, b, c belong to P(n) then (a+b), (b+c), (a+b)+c, a+(b+c) belong in P(m) and
(a+b)+c=a+(b+c) for all a, b and c in P(m).

2) There is a digital number 0 in P(m) such that

2.1) a+0=a, for all a in P(m).

2.2) For every a in P(n) there is some b in P(m) such that a+b=0. Such a, b is symbolized also by -a , and it is unique in P(m).

3) if a, b, belong to P(n) then (a+b), (b+a), belong in P(m) and
a+b=b+a

4) if a, b, c belong to P(n) then (a*b), (b*c), (a*b)*c, a*(b*c) belong in P(m) and
(a*b)*c=a*(b*c).

5) There is a digital number 1 in P(m) not equal to 0 in P(m), such that

5.1) a*1=a, for all a in P(m).

5.2) For every a in P(n) not equal to 0, there may be one or none or not only one b in P(m) such that a*b=1 . Such b is symbolized also by 1/a, and it may not exist or it may not be unique in P(m).

6) if a, b, belong to P(n) then (a*b), (b*a), belong in P(m) and
a*b=b*a

7) if a, b, c belong to P(n) then (b+c) , (a*b), (a*c), a*(b+c), a*b+a*c, belong in P(m) and
a*(b+c)=a*b+a*c

Which numbers are **positive** and which **negative** and the **linear order** of digital numbers in the precision levels P(n), P(m), P(q) is something known from the definition of precision levels in the theory of classical real numbers in digital representation.

If we denote by PP(n) the positive numbers of P(n) and PP(m) the positive numbers of P(m) then

8) For all a in PP(n), one and only one of the following 3 is true

8.1) a=0

8.2) a is in PP(n)

8.3) -a is in PP(n) (-a is the element such that a+(-a)=0)

9) If a, b are in PP(n), then a+b is in PP(m)

10) If a, b are in PP(n), then a*b is in PP(m)

It holds for the inequality a>b if and only if a-b is in PP(m)
a<b if b>a
a<=b if a<b or a=b
a>= b if a> b or a=b
and similar for PP(m).

11) Similar properties as the ones from P(n) to P(m) hold if we substitute n with m, and m with q.

12) Also, the **Archimedean property** holds only recursively in respect e.g. to the local lower precision level P(n). In other words, if a, b, a<b belong to the Local lower precision level P(n) then there is n integer in the Low precision level P(m) such that a*n>b. And similarly for the precision levels P(m) and P(q).

13) The corresponding to the **Eudoxus-Dedekind completeness** in the digital real numbers also is relative to the three precision levels.

Definition II.B.2 We define that **two visible points A, B, are in contact or of zero distance** distance(A,B)=0, if and only if in their Cartesian coordinates they are at a face , at an edge or at a vertice successive. If this is so then there are invisible points A' belonging to A (see axioms of incidence) and B' belonging to B, so that distance(A'.B')<=1/(10^2q). Two visible points in contact do not have in general the same Cartesian measures distance The distance of the invisible points is defined from the coordinates of the invisible points in the precision level P(q) of R(n,m,q) from the standard formula of Euclidean distance , that is a Cartesian measure as in **Definition II.C.I.2** or with the Archimedean measures but the values are identical in the standard or low precision level P(n).

In other words for every visible point A in the Low precision level , there are exactly two other points B1, B2 again in the Low precision level with B1<A<B2 , such that the distance between A and B1, and A, B2 is zero in the Low precision level, and there is no other visible point C strictly between A and B1 and a and B2. This can be derived also from the requirement that all possible combinations of decimal digits in the local lower, low and high precision levels are being used as numbers of the system of digital real numbers.

Sufficient Mutual inequalities of the precision levels

We impose also axioms for the sufficiently large size of the high precision level relative to the other two, and the sufficient large size of the low precision level relative to the local lower precision level. That is for the mutual relations of the integers m, n, q.

It may seem that these differences of the resolution or the precision levels are very severe and of large in between distance, and not really necessary. It may be so, as the future may show. But for the time being we fell safe to postulate such big differences.

14) If we repeat the operations of addition and multiplication of the linearly ordered commutative field starting from numbers of the local lower precision level P(n), so many times as the numbers of the local lower precision level P(n), then the results are still inside the low precision level P(m). (This in particular gives that

$$(10^n)^{10^n} \le 10^m$$

15) The largest error in the high precision level P(q), which we may also identify as the smallest magnitude in the low precision level P(m) in other words the 10^{-m}, will appear as zero error in the low precision level P(n), even after additive

repetitions that are as large as the cardinal number of points of the lower precision level P(n). This is e.g. is guaranteed if 5n+2log2<m or rounded 6n<m (Where by log we denote the logarithm with base 10) . The points in 1-dimensional geometry are $10^{(4n+2\log 2)}$ *and if an error of order* 10^{-m} *is repeated so many times and still be less than* 10^{-n} *, then* $10^{(4n+2\log 2)}10^{-m} \leq 10^{-n}$*, thus 5n+2log2<m. For the Euclidean geometry cube, this requires that* $10^{(12n-6\log 2)}10^{-m} \leq 10^{-n}$ *thus 13n+6log2<m or rounded 14n<m.*

16) The smallest magnitude in the high precision level P(q) in other words the 10^{-q}*, will appear as zero error in the low precision level P(m), even after additive repetitions as large as the cardinal number of points of the low precision level P(m). This is e.g. guaranteed if 5m+2log2<q or rounded 6m<q, and for Euclidean geometry applications 13m+6log2<q or rounded 14m<q.*

If instead of three precision levels P(n), P(m), P(q), we would introduce four precision levels (still another (P(r)), with the same mechanism of recursive axioms, then we would denote it by R(n,m,q,r) and we would call it a 4-precisions levels system of digital real numbers.

Two digital systems of Real numbers R(n,m,q) , R(n',m',q') with n=n', m=m', q=q' and the above axioms are considered isomorphic.

C. An axiomatic system of the physical or digital but continuous 3-dimensional Euclidean geometry $E_3(n,m,q)$.

We have as initial concepts of objects

a) The High resolution or precision points, or invisible points or atoms

b) The Low resolution or precision points, or visible points or pixels.

c) The Lower or standard precision level of measurements.

Remark II.C.1

We introduce in the digital Euclidean geometry the next two types of points:

1) All visible points (or low precision level points) are finite in number. And of *non zero but minimum possible dimension in the single or Low precision,* but not in the High precision level. Between two visible points there is not always another visible point. The case of non-existence on intermediate points will be used in the concept of completeness up to some density or resolution and continuity of the space. Visible points are called visible in our usual material realizations of geometric figures because if we put our eyes close enough to the paper surface or screen where a line or a circle is drawn, we can see the point, while at a normal distance we cannot see the points but only the linear segment or circle arc. E.g. **pixels** of lines on the computer screen. Nevertheless the smallest magnitude of the standard precision level is by far larger than the visible points.

2) All invisible points or pixels or atoms (or high precision level points) are finite in number and of minimum dimension in the high precision level. Invisible points are called invisible in our usual material realizations of geometric figures because no matter how close we may put our eyes to the paper surface or screen where a line or a circle is drawn, we cannot see these points. E.g. **atoms** of a metallic material surface. The main reason of introducing here the invisible points is so as to have at least two alternative systems of measures (lengths, areas, volumes), that of Archimedes and that of Cartesius. The full significance of the invisible points will become apparent only when introducing digital curved space like digital Riemannian space or manifolds, which is not in the scope of the current paper.

For the *axiomatic digital or physical Euclidean geometry* we do not intent to use the system of Real numbers as it is defined as the minimal complete linearly ordered commutative field (in the order to topology), but instead all measurements of linear geometric segments lengths, areas, volumes etc will be done with a Low Precision level and a high precision level of real numbers. The definition of a *precision level* P(n,m) where n, m are natural numbers , is that it is the set of all real numbers that in the decimal representation have not more than n decimal digits for the integer part and not more than m digits for the decimal part. In other words as sets of real numbers it is a nested system of lattices each one based on units of power of 10, and as union a lattice of rational numbers with finite many decimal digits. We could utilize other bases than 10 e.g. 2 or 3 etc, but for the sake of familiarity with the base 10 and the 10 fingers of our hands we leave it as it is.

Whenever we refer to a real number x of as (minimal in precision levels) system of real numbers r we will always mean that x belongs to the local lower precision level and that the system has at least three precision levels with the current axioms.

Whenever we write an equality relation we must specify in what precision level it is considered. The default precision level that a equality of numbers and geometric elements of geometric figures like length, area and volume, is considered to hold, is the standard or *lower precision level*

We continue with the initial concepts of objects

d) The visible lines

e) The visible planes

f) We may apply finite sets only on the points of the digital Euclidean geometry

g) And of course we may apply digital formal logic to make arguments and proofs.

h) Besides the ***congruence*** *as equivalence relations we have the next initial relations among visible or invisible elements.*

An invisible point A belongs to a visible point B, denoted by A є B

A visible point A belongs to a line L , denoted by A є L

A visible point A belongs to a plane P , denoted by A є P

A line L belongs to a Plane P, denoted by L є P

A visible point A is ***between*** *two visible points B, C.*

We design 7 groups of axioms

1) Of finite decimal coordinates

2) Of lengths, areas and volumes

3) Of Incidence

4) Of Order

5) Of Congruence
6) Of Continuity
7) Of Resolution or density

In the next axioms the term point if we do not specify that it is invisible, refers to visible or low precision point. It has the minimum no-zero size (length) in the Low resolution real numbers that can be constructed on a geometric line, by the Cartesian coordinates as tiny cube, as we shall see. We use the axioms of Hilbert, but we modify them and add more axioms.

I Axioms of finite decimal coordinates of points

1) Every invisible point P has 3 numerical coordinates $P(x_1)$, $P(x_2)$, $P(x_3)$ that are rational numbers that in decimal notation have finite many digits so many as the definition of the High measurement precision.

2) Every visible point P has 3 numerical coordinates $P(x_1)$, $P(x_2)$, $P(x_3)$ that are rational numbers that in decimal notation have finite many digits so many as the definition of the Low measurement precision.

3) The density of the visible is uniform through-out the spherical space. For every triad of decimal rational numbers of the low precision level P(m) , there is an invisible point with these coordinates.

4) The density of the invisible is uniform through-out the spherical space. For every triad of decimal rational numbers of the high precision, there is an invisible point with these coordinates.

Remark II.C.I.1

The visible and invisible points due to their orthogonal and rectangular coordinates may be considered **tiny little cubes**. Then of course we may define their cross sectional length, area and volumes as 10^{-m}, 10^{-2m}, 10^{-3m} for the visible and 10^{-q}, 10^{-2q}, 10^{-3q} for the invisible points.

Definition II.C.I.1 ***of the local lower LLS and low resolution LS finite sphere or space.*** There is a central visible point O of the space with coordinates (0,0,0) such that all the visible and invisible points of the space that have distance at most ω of it, where ω belongs in the P(n,n), and ω=10^n is called the local lower resolution space or in short LLS. If we take the corresponding sphere from the center with coordinates (0,0,0) with all visible and visible points with radius Ω=10^m, is called the Low resolution space or LS.

Definition II.C.I.2 Cartesian measures of length, areas and volumes

From the elementary Cartesian analytic geometry, we may define the distance of two points $A(x_1,y_1,z_1)$ $B(x_2,y_2,z_2)$ through the Pythagorean or Euclidean formula of distance (norm with rule of parallelogram) . We may similarly define the area of three points not lying in a line, through the well known formula that is involving the determinant and their coordinates, and similarly for the 3-dimensional simplex or tetrahedron. Then we may define the area of finite sets of points that are in contact (see Definition xyz below) by triangulation with non-overlapping triangles. Similarly define the volume of finite sets of points that are in contact or connected (see **Definition II.C.VI.1** below) by simplicialization with non-overlapping tetrahedral (simplexes). Such measures of area, and volumes of finite sets of points that are in contact (connected) we call in the next the **Cartesian measures of areas and volumes.**

Remark II.C.I.2

Notice that in the synthetic axioms that we introduce here we do not impose geometric structure to the invisible points, but only to the visible points. In other words we do not define invisible lines and invisible planes. But we could as well do so, from the coordinates of the invisible points and the standard equations of lines and planes in the analytic geometry.

II Axioms of Archimedes measures of length, area, and volumes and compatibility with the coordinates.

1) Every invisible point P, as belonging to a line L, has a non-zero length l(P) which is a rational number that in decimal notation has finite many digits so many as the definition of the High measurement precision while it is zero in the low measurement precision.

2) Every invisible point P, as belonging to a plane E, has a non-zero area a(P) which is a rational number that in decimal notation has finite many digits so many as the definition of the High measurement precision while it is zero in the low measurement precision.

3) Every invisible point P, has a non-zero volume v(P) which is a rational number that in decimal notation has finite many digits so many as the definition of the High measurement precision while it is zero in the low measurement precision.

4) For every visible point P, there are a sets VIn(P) of invisible points of it, so that volume of the visible point is defined as the sum of the lengths, of the volumes of the invisible points of the above sets correspondingly. These sets Vln(P) for the volume are not unique for the point P , but all the alternative such sets give the same values volume of the point, and the same for all visible points.

5) For every visible point P, there are a sets LIn(P) , AIn(P) of invisible points of it, so that the length and are of the visible point is defined as the sum of the lengths, of the lengths and areas of the invisible points of the above sets correspondingly. These sets LIn(P) , AIn(P) and also their values for the lengths and areas are not unique for the point P, but depend and their values depend also, on the linear segment or plane correspondingly that the point P is considered that it belongs.

6) The length of linear segment is defined as the sum of the lengths of its visible points that in their turn define a partition of the invisible points of the segment. The length of the unit segment OA , with coordinates of O, (0,0,0) and (0,0,1) is equal to 1. (similarly by cyclic permutation of the coordinates and the other unit lengths from O).

7) The area of figure (set of visible point) is defined as the sum of the areas of all of its visible points that in their turn define a partition of the invisible points of the figure. The length of the unit square OA-OB , with coordinates of O, (0,0,0) and (0,0,1), (1,1,1), (1,0,0) is equal to 1 (similarly by cyclic permutation of the coordinates and the other unit squares)

8) The volume of a figure is defined as the sum of the volumes of its visible points that in their turn define a partition of the invisible points of the figure. The volume of *the unit cube , with coordinates (0,0,0) and (0,0,1), (1,1,1), (1,0,0) , (0,1,0) , (1,1,0), (0,1,1) ,(1,0,1) is equal to 1*

9) Congruent sets of points (of the LLS) have length, area, and volumes either in the Cartesian measures or the Archimedean measures , correspondingly that differ only by errors that are zero in the standard or low precision level. Furthermore they remain zero error, even if are repeated additively as many times as the cardinal number of elements of the low precision level P(n).

10) For a finite connected set of visible points (of the LLS) the difference of its measure in the Cartesian measure and the Archimedean measure , correspondingly differ only by error that is zero in the standard or low precision level. Furthermore it remains zero error, even if it is repeated additively as many times as the cardinal number of elements of the low precision level P(n).

Remark II.C.II.1

Both types of measures Cartesian measures and Archimedes measures of, areas and volumes have the additive property of disjoint unions of finite sets of points in contact (connected sets of points see **Definition II.C.VI.1**).

After the above axioms and definitions of such measures , it can be shown that the lengths, areas and volumes, are set functions l, a, v of sets of visible points, (but also of invisible points) ,with values in the positive Low precision level of decimal numbers, with the additive property of disjoint unions: (By $\cap$ we denote the intersection and by $\cup$ the union of sets)

$$l(A \cup B) =_n l(A) + l(B) - l(A \cap B)$$

$$a(A \cup B) =_n a(A) + a(B) - a(A \cap B)$$

$$v(A \cup B) =_n v(A) + v(B) - v(A \cap B)$$

AB congruent to A'B' then $d(AB) =_n d(A'B')$

Furthermore more properties for linear segments AB, BC, AD hold like

$l(AB) =_n l(BA)$

$l(AC) <=_n l(AB)+l(BC)$

Also angular measures ang() again with values in positive Low precision level are defined , through areas of circular sectors of unit circular discs or of the length of the corresponding circular segment of unit circler discs.

Axioms of Sufficient many points (visible and invisible) and mutual inequalities of the precision levels for length, areas and volumes.

We impose also axioms for the sufficiently large size of the high precision level relative to the other two, and the sufficient large size of the low precision level relative to the local lower precision level. That is for the mutual relations of the integers m, n, q.

It may seem that these differences of the resolution or the precision levels are very severe and of large in between distance, and not really necessary. It may be so, as the future may show. But for the time being we fell safe to postulate such big differences.

The axioms are essential those of the digital real numbers $R^3(n,m,q)$ with numbers 1) 2) 3) as coordinates of the visible and invisible points of the digital Euclidean space $E_3(n,m,q)$.

In all the next axioms we start from visible points and geometric elements of the Local Lower resolution space LLS (which belongs in the coordinates cube P(n,m)) , and we result in to the Lower resolution space LS (which belongs in the coordinates cube P(m,m)), because of the recursive and not absolute closeness in the digital real numbers R(n,m,q). Angles in LLS are essentially circular sectors of length of radius equal to one unit.

III. Incidence

Terminology convention

1) In all the axioms of incidence, order, and congruence below, when we say and write the term "point" without specifying it to be an invisible point we will mean a visible point.
2) when we say and write the term "angle", we will mean, a circular sector of unit radius.
3) when we say and write the term "line", we will mean, a linear segment starting and ending at points of the Local Lower Sphere (LLS). The ending visible points of the line do not count as (interior) visible points of the line
4) when we say and write the term "plane", we will mean, a circular disc , with boundary circle at surface points of the Local Lower Sphere (LLS). The boundary visible points of a plane do not count as (interior) points of the plane.

1. *For every two points A and B (in LLS) there exists a line a (in LLS) that contains them both. We write AB = a or BA = a. Instead of "contains," we may also employ other forms of expression; for example, we may say "A lies upon a", "A is a point of a", "a goes through A and through B", "a joins A to B", etc. If A lies upon a and at the same time upon another line b, we make use also of the expression: "The lines a and b have the point A in common," etc.*
2. *For every two points (in LLS) there exists no more than one line (in LLS) that contains them both; consequently, if AB = a and AC = a, where B ≠ C, then also BC = a.*
3. *There exist at least two points on a line (in LLS) . There exist at least three points that do not lie on a line.*
4. *For every three points A, B, C (in LLS) not situated on the same line there exists a plane α (in LLS) that contains all of them. For every plane (in LLS) there exists a point which lies on it. We write ABC = α. We employ also the expressions: "A, B, C, lie in α"; "A, B, C are points of α", etc.*
5. *For every three points A, B, C (in LLS) which do not lie in the same line, there exists no more than one plane (in LLS) that contains them all.*
6. *If two points A, B of a line a (in LLS) lie in a plane α (in LLS) , then every point of a lies in α. In this case we say: "The line a lies in the plane α," etc.*
7. *If two planes α, β (in LLS) have an (interior) point A in common, then they have at least a second (interior) point B in common.*

8. *There exist at least four points (in LLS) not lying in a plane.*
9. *For every invisible point A, there is a visible point B, so that A belongs to B.*
10. *Two invisible points A, B belong to the same visible point C is an equivalence relation among the invisible points.*

IV. Order

1. *If a point B (of LLS) lies between points A and C (of LLS) , B is also between C and A, and there exists a line containing the distinct points A,B,C.*
2. *Of any three points situated on a line (of LLS) , there is no more than one which lies between the other two.*
3. ***Pasch's Axiom**: Let A, B, C be three points (of LLS) not lying in the same line and let a be a line (of LLS) lying in the plane ABC and not passing through any of the points A, B, C. Then, if the line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.*

V. Congruence

1) *If A, B are two points on a line a, (of LLS), and if A' is a point upon the same or another line a' (of LLS), then, upon a given side of A' on the straight line a' , we can always find a point B' (of LS) so that the segment AB is congruent to the segment A'B' . We indicate this relation by writing AB ≅ A' B'. Every segment is congruent to itself; that is, we always have AB ≅AB.*

We can state the above axiom briefly by saying that every segment can be laid off upon a given side of a given point of a given straight line in at least one way (Always starting from the sphere LLS and resulting in the larger sphere LS).

2. *If a segment AB (of LLS) is congruent to the segment A'B' and also to the segment A"B", then the segment A'B' is congruent to the segment A"B"; that is, if AB ≅ A'B' and AB ≅ A"B", then A'B' ≅ A"B".*

Remark II.C.V.1 ***Limited transitivity?***

As we noticed in the axioms II.9 -II.10, congruent linear segments, and angles have equal measures with zero error in the standard precision level P(n), but non-zero in the low precision level P(m). Therefore in the transitivity of the congruence in the previous axiom, the error may be added and propagated. Still by the same axiom II.9 , the repletion may me s many times as the elements of the standard precision level P(n) and still be zero. Therefore we know that the transitivity of the congruence will still hold up to as many times as the number of the elements of the standard precision level P(n). Now if in the meta-mathematics of the formal logic we utilize the digital natural numbers N(ω) with ω=n, then certainly even the largest allowable number of formal propositions and therefore repetitions of the transitivity of congruence will not lead to an non-zero error in the standard precision level. Therefore we may accept that the transitivity of the congruence is valid for all practical applications, although theoretically it is limited. Another way to keep the transitivity of the congruence is the next: We may define when modelling this axiomatic system to test the consistency, the congruence with a standard types transformation of the coordinates (e.g. isometric transformations). Then as the composition of two isometries are is an isometry, and the error of an isometry can be uniformly bounded for all isometries so as to be zero in the low precision level, the transitivity of the congruence is valid from the point of view of the standard precision level.

3. *Let AB and BC be two segments of a line a (of LLS) which have no points in common aside from the point B, and, furthermore, let A'B' and B'C' be two segments of the same or of another line a' (of LS) having, likewise, no point other than B' in common. Then, if AB ≅ A'B' and BC ≅ B'C', we have AC ≅ A'C'.*
4. *Let an angle ang (h,k) be given in the plane α (of LLS) and let a line a' be given in a plane α' (of LLS) . Suppose also that, in the plane α', a definite side of the straight line a' be assigned. Denote by h' a ray of the straight line a' emanating from a point O' of this line. Then in the plane α' there is one and only one ray k' (of LS) such that the angle ang (h, k), or ang (k, h), is congruent to the angle ang (h', k') and at the same time all interior points of the angle ang (h', k') lie upon the given side of a'. We express this relation by means of the notation ang (h, k) ≅ ang (h', k').*
5. *If the angle ang (h, k) (of LLS) is congruent to the angle ang (h', k') and to the angle ang (h", k"), then the angle ang (h', k') is congruent to the angle ang (h", k"); that is to say, if ang (h, k) ≅ ang (h', k') and ang (h, k) ≅ ang (h", k"), then ang (h', k') ≅ ang (h", k").*
6. *If, in the two triangles ABC and A'B'C' (of LLS) the congruencies AB ≅ A'B', AC ≅ A'C', ang(BAC) ≅ ang(B'A'C') hold, then the congruence ang(ABC) ≅ ang(A'B'C') holds (and, by a change of notation, it follows that ang(ACB) ≅ ang(A'C'B') also holds).*

VI. Continuity and Completeness up to some density or resolution, relative to the digital real numbers R(n,m,q).

The corresponding to the ***Eudoxus-Cartesius-Dedekind, completeness*** also is relative to the three precision levels of R(n,m,q).

Definition II.C.VI.1

We define that *two visible points A, B, are in contact or of zero distance* distance(A,B)=0 in P(n), if and only if in their Cartesian coordinates they are at a face , at an edge or at a vertice successive. If this is so then there are invisible points A' belonging to A (see axioms of incidence) and B' belonging to B, so that distance(A'.B')<=1/(10^2q). Two visible points in contact do not have in general the same Cartesian measures distance The distance of the invisible points is defined from the coordinates of the invisible points in the precision level P(q) of R(n,m,q) from the standard formula of Euclidean distance , that is a Cartesian measure as in **Definition II.C.I.2** or with the Archimedean measures but the values are identical in the standard or low precision level P(n).

1. ***Axiom of Digital Continuity and Completeness***: *For every non-ending visible point A of LLS , of a linear segment a, there are exactly two other divisible points B1, B2 on a in LS, and with B1<A<B2 , such that the distance between A and B1, and A, B2 is zero, and there is no other visible point C strictly between A and B1 and a and B2. This can be derived also from the requirement that all possible combinations of decimal digits in the low and high precision levels are being used as numbers of the system of digital real numbers and correspond to visible and invisible points.*

Remark II.C.VI.1 : *An alternative way that we could formulate the completeness of points is the next*. An extension of a set of visible points on any line , plane and the space , with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-VII based on the given density of Coordinates in R(n,m,q), is impossible. In short we cannot add more visible points relative to the Low precision level of measurements and coordinates, and the same for the visible points and low precision level of coordinates. This comes also from the axiom of the density-completeness of all possible but finite many coordinates of points in R(n,m,q).

VII. Axioms of Resolution or of Density

These axioms are of the same nature as the corresponding axioms of the digital natural numbers, and multi-precision digital real numbers, and the axioms 1,2,3 of the digital real numbers.

1. ***Axiom of sufficient high resolution or density .*** *Let a line a passing from the center of the space O and the units of measurements OA on it, and let ω(a), Ω(a), demote the finite cardinal number which is the cardinal number of visible and invisible points that belong to a. Estimates of them are*
   $\omega(a) = 10^{(4m\_2\log 2)}$
   and $\Omega(a) = 10^{(4q\_2\log 2)}$ *where by log we denote the logarithm with base 10. And let ω(n) be the size of the model of the natural numbers constructed on the line a through congruence and the above axioms.An estimate of it is* $\omega(n) = 10^{(2n+\log 2)}$ *Then it holds that*
   ω(n) <= ω(a) or $10^{(2n+\log 2)} \leq 10^{(4m+2\log 2)}$
   (Strong version of the axiom $2^{\omega(n)} \leq \omega(a)$ *)*
   ω(a) <=Ω(a) or $10^{(4m+2\log 2)} \leq 10^{(4q+2\log 2)}$
   (Strong version of the axiom $2^{\omega(a)} \leq \Omega(a)$ *)*
2. *Let ω(S), Ω(S), demote the finite natural numbers which are the cardinal numbers of visible and invisible points that can belong to the spherical S 3-dimensional space. And let also ω(P) be the cardinal number of invisible points that any visible point may contain . (From the group II of axioms about lengths, areas and volumes and axioms 8,9,10, we have an estimate* $\omega(S) \leq 10^{(12m+6\log 2)}$ *and* $\Omega(S) \leq 10^{(12q+6\log 2)}$ *where by log we denote the logarithm with base 10, and m and q are the orders of the precision level of the visible and invisible points respectively and* $\omega(P) \leq 10^{(q-m)}$
3. *Then it holds that*
   a) Any number of visible points of the total spherical space is less than any number of invisible points that a visible point may contain . In particular ω(S) <= ω(P) or $10^{(12m+6\log 2)} \leq 10^{(q-m)}$
   (Strong version of the axiom: $2^{\omega(S)} \leq \omega(P)$ *).*

   (**Remark II.C.VII.1** This axiom must guarantees that lengths, areas and volumes that are defined by summing the corresponding values of the invisible points, will have in general total errors zero in the low precision, and that the failure if the transitivity of relations congruence due to their limited character can be avoided to occur, with sufficient high resolution of invisible points relative to the size of the total space and our repetitive construction in it.)

   b) The diameter of the total spherical 3--dimensional space in integer number of units of length denoted by ω(n) ,is less that any maximum number of visible points that the spherical space may contain. An estimate of ω(n) is $\omega(n) = 10^{(2n+\log 2)}$ *. In particular ω(n) <=ω(S) or* $10^{(2n+\log 2)} \leq 10^{(12m+6\log 2)}$
   (Strong version of the axiom $2^{\omega(n)} \leq \omega(S)$)

Remark II.C.VII.2 The axiom guarantees that a lattice of points with say integer coordinates will always be by far less dense than the lattice of visible points.

Remark II.C.VII.3 Notice that the inequalities of the current axioms of resolution are stronger than those of axioms 1,2,3 of the digital real numbers.

Remark II.C.VII.4.

Notice that we did not postulate anything similar to the **axiom of parallel lines** of Euclid! One reason is that the digital lines are eventually linear segments and do not extend to infinite, as this in the current setting would make them have infinite many points. Furthermore for this reason there are more than one linear segment passing from a point outside another linear segment which do not have any point in common. But this of course does not make the digital geometry **a Lobachevskian or hyperbolic** geometry. Still in the low precision level there would be only one linear segment that passing from a point outside a linear segment so that the angles from a third crossing both linear segment have sum exactly equal to 180 degrees. But there is also another source of non-uniqueness of the "parallel" which is not that the geometry is systematically hyperbolic and non-Euclidean, but due to different levels of precisions in measurements and determination of the lines. As it is the lower precision level which is the standard of measuring angles and lengths, there would be more than one linear segments (defined in the low precision level by visible points) passing from a point outside another linear segment so that a third segment crossing them would have sum of angles 180 degrees measured in the lower precision level. The main reason that we did not postulate the axiom of parallels is that in the digital Euclidean geometry the properties of such parallels in the low precision level are deduced from the axioms of the coordinates.

Remark II.C.VII.5

Any digital space $E_3(n,m,q)$ is determined essentially from the integer parameters n,m,q of the corresponding digital system of real numbers R(n,m,q) which is used as coordinate system. To have that any two finite models of $E_3(n,m,q)$ are isomorphic, one has to define and model appropriately the incidence , order and congruence at first before defining the appropriate form of isomorphism of the models of $E_3(n,m,q)$.

III CONCLUSIONS

The axiomatic system of the digital Euclidean geometry $E_3(n,m,q)$ may seem complicated and elaborate compared to the Hilbert's axioms of the classical Euclidean geometry. But as we remarked from the beginning the cost of the simple Hilbert's axioms with infinite many points is paid later with overwhelming complexity, and many non-intuitive paradoxes like Hilbert's 3rd problem, and Banach-Tarski paradox. In addition the theory of the measures of areas and volume that require integration and limits is very complex. In contrast in the digital Euclidean geometry we may start with elaborate axioms, but later the theory of measures of areas and volumes is very simple and intuitive!

E.g. The areas of circular discs are calculated in the Cartesian measure of area, by triangulation with non-overlapping triangles of all the (finite many) points of the circular disc! And in the Archimedean measure of area the calculation is even simpler as the sum of the areas of the (finite many) points of the circular discs that are tiny rectangles. Integration is simple finite sums.

Similarly other elementary or non-elementary theorems can have easier proofs in the digital Euclidean geometry! We even have a new type of proofs not possible in the classical Euclidean Geometry: **Proof by induction on the number of (visible) points!** It is quite interesting to re-formulate classical non-solved so far problems in the context of digital Euclidean geometry e.g. Riemann hypothesis of the roots of the zeta function , and try to prove it by induction on the number of visible points.

APENDIX

A dialogue of the immortals mathematicians on the occasion of the new axioms of the axiomatic digital Euclidean geometry by NEWCLID

This is a fictional dialogue of the immortally famous mathematicians of the past that have significantly contribute to the mathematics of the Euclidean Geometry and comment on the new axiomatic system of the Axiomatic Digital Euclidean Geometry. The list is only indicative, not exhaustive.

NEWCLID after presenting the immortals the new axioms of the Axiomatic Digital Euclidean Geometry, invites them in a free discussion about it.

NEWCLID, is an individual representing the collective intelligence of the digital technology but also of mathematics of the 21st century.

The participants of the discussion are the next 20.

1)Pythagoras
2)Aristarchus from Samos
3)Eudoxus
4)Euclid
5)Democritus
6)Archimedes
7)Apollonius
8)Copernicus
9)Galileo
10) Newton
11)Leibnitz
12)Cartesius
13)Cauchy
14)Dedekind
15)Weierstrasse
16)Hilbert
17)Riemann
18) Cantor
19) von Neumann

and a mortal:

20) Newclid

NEWCLID:

Welcome honourable friends that you have become immortals with your fame and contribution in the creation of the science and discipline of Mathematics among the centuries on the planet earth!

Now that you have watched my presentation of the axiomatic system of the new Axiomatic Digital Euclidean Geometry, I would like to initiate a discussion that will involve your remarks and opinions about it. Who would like to start the conversation?

PYTHAGORAS:

Thank you Newclid for the honour in gathering us together. I must express that I like the new approach of the Axiomatic Digital Euclidean Geometry, that as you say is a resume of what already the beginning of the 21st century in the earthly Computer Science has realized through software in the computer operating systems and computer screens and monitors.

I must say that I like the approach! In fact, I was always teaching my students that the integer natural numbers are adequate for creating a mathematical theory of the geometric

space! One only has e.g. to take as unit of measurement of lengths, the length of a visible points and all metric relations in the low precision level of the figures , including the Pythagorean theorem, become relations of positive integer numbers, or solutions of Diophantine equations! But at that time no such detailed and elaborate axiomatic system, neither a well accepted concept that matters consists from atoms, was available in the mathematicians of the ancient Greece, Egypt or Babylon.
EUCLID: I am impressed Newclid for your elaborate axiomatic system. The axioms that I had gathered in my books with title “Elements” for the Euclidean geometry in my time were much less! I would like to ask you a question that puzzles me since I watched your presentation: How do we know that the more than 20 axioms of Hilbert about my Euclidean geometry, or your axioms of the Digital Euclidean Geometry are enough to prove all that we want to prove?
NEUCLID: This is a very good question, Euclid! Maybe our friend here Hilbert might like to answer it!
HILBERT: Well my friends, this is a question that I posed also to myself when writing my more than 20 axioms of the classical Euclidean Geometry! I have not read any such proof! It is by the rule of the thump as they say! I collected them , through my experience and according to the theorems of Euclidean geometry till my time but also according to the standards of proofs in my time!
NEWCLID: What do you mean Hilbert? That maybe in the future we might discover that we need more axioms?
HILBERT: Exactly! That is what the History teaches us!
CARTESIUS: If I may enter the discussion here, I propose that a proof that the axioms of Hilbert are enough could be proving from the Hilbert axioms, the basic numerical axioms of my Analytic Geometry with coordinates! This, in my opinion, would be a proof!
NEWCLID: Very good idea Cartesius! This in my opinion suggests also that my axioms of the Digital Euclidean Geometry, that involve coordinates too, most probably are enough. But I am almost sure that they are not independent and some of them can be proved from the rest. Still I cannot claim that I have any proof, more than just experience and a rule of the thump, that my axioms are adequate! Maybe in the future I may discover that I need a couple more!
ARISTARCHUS: May I ask Newclid if your concept of digital Euclidean space which is in the shape of a spherical ball is intended to be large enough so as to allow e.g. astronomical calculations like my calculations of the size of earth, moon , sun and their mutual distances?
GALILEO: I have the same question Newclid! Good that ARISTARCHUS asked it!
COPERNICUS: Me too Euclid!
NEWCLID: Certainly ARISTARCHUS! The spherical digital Euclidean space can be so large so as to include all the observable galaxies of the astronomical world as we know it! But it can be also small as a planet to accommodate for planetary calculations only too! The axioms do not specify how large or small it should be!
ARCHIMEDES: I like your axiomatic system and concept of space Newclid! It is as my perceptions! Actually my experimental work with solids that I was filling with sand or water to make volume comparisons is just an experimental realisation of your axioms of volumes through those of the points and finite many points!
DEMOCRITUS: Bravo Newclid! Exactly my ideas of atoms! Actually as in my theory of atoms, the water is made from finite many atoms, the volume experiments of Archimedes with water is rather the exact realisation of your axioms of volume through that of the invisible points! Here the atoms of the water are invisible, while the granulation of the sand may resemble your axioms of the visible points!
NEWCLID: Thank you, my friends! I agree!
EUDOXOS: Well in your digital Geometry Newclid, my definition of the ratio of two linear segments which is the base of the complete continuity of the line is not that critical in your axiomatic system, although I thing that it still holds!
DEDEKIND: As I reformulated the idea and definition of equality of ratio of linear segments of Eudoxus, as my concept of Dedekind cuts about the completeness of continuity of the real numbers, I must say the same thing as Eudoxus!
WEIERSTRASSE: The same with my definitions of convergent sequences though the epsilon-and-delta formulation! They still hold in your approach!
APOLLONIUS: I would like to know Newclid, if my theory of circles in mutual contact would be provable as I know it in the classical Geometry of Euclid. E.g. if tow circles are in contact externally, are they in contact in one only point, as I know it, or in more than one point in your geometry?
NEWCLID: I think APOLLONIUS that in my geometry what you observe in the real world is also more or less what is provable with the visible points. For sure two circles in contact even if they have only one common visible point they will have many common invisible points, all those inside the common visible point! But I am afraid that they may even have more than one common visible point , depending on their size and the definition of circle intersecting circle or line. The reason is that it may happen that two different visible points have an error of distance from the centers of the circles which is zero in the Low precision although not zero in the High precision. Still one may gibe an appropriate definition where one of them has a maximal property thus a unique point of contact.
GALILEO: I would like to ask Newclid if your concept of invisible and visible points could be large enough and both of them visible, so as to account for the real planet earth (which is not a perfect sphere) as if a perfect sphere!
NEWCLID: Well GALILEO, the initial intention is the invisible points are indeed small enough to be invisible. But as you understand what is visible and invisible is not absolute and depends at least on the closeness of our eye. Theoretically one could conceive a model of my axioms where both visible and invisible points are visible and even large!
LEIBNITZ: I want to congratulate you Newclid for your approach! In fact my symbols of infinitesimal dx in my differential calculus suggest what I had in mind: A difference dx=x2-x1 so that it is small enough to be zero in the Low precision but still non-zero in the High precision! Certainly a finite number!
NEWTON: I must say here that the Leibnitz idea of infinitesimal as a finite number based on the concepts of Low and High precision is not what I had in my mind when I was writing about infinitesimals. That is why I was calling them fluxes and symbolized them differently. The theory of null sequences of numbers (converging to zero) of Cauchy and Weierstrasse is I think the correct formulation of my

fluxes. So that such fluxes fit to a Geometry as Euclid and Hilbert was thinking it and not as Newclid formulated here. Still for physical applications I thing that Newclid's concept of space with finite many points only is better and closer to the physical reality! I was believing in my time that matter consist from finite many atoms , but I never dared to make a public scientific claim of it, as no easy proof would convince the scientist of my time!

I want to ask an important question to Newclid: Is your differential and integral calculus based on three levels of precision more difficult or simpler that the classical differential and integral calculus based on limits and infinite many real numbers?

NEWCLID: Well Newton thank you for the good words! Actually I have not yet developed all of a differential and integral calculus based on digital real numbers and digital Euclidean geometry, therefore the question runs ahead of our presentation. But I have thought myself about it, and I can remark the next: A differential and integral calculus based on three levels of precision is certainly less complicated than (and also not equivalent to) the classical calculus with infinite sequences or limits. But a differential and Integral calculus of 3 ,4 or more precision levels is by far more complicated than the classical differential and Integral calculus. Only that this further complication is a complexity that does correspond to the complexity of the physical material reality, while the complexities of the infinite differential and integral calculus (in say Lebesgue integration theory or bounded variation functions etc) is a complexity rather irrelevant to the physical material complexity.

CARTESIUS: I want to congratulate you Newclid for your practical , finite but axiomatic too approach for the physical space, and the introduction of my idea of rectangular coordinates right from the beginning of the axioms! I have a question though! You correspond points to coordinates, but they also have volume. If we think of a cubic lattice with its points and coordinates, which of the 8 cubes that surround the point you assume as voluminous point in your geometry?

NEWCLID: If I understand your question well CARTESIUS, it is the cube that its left upper corner is the point. Thanks for your praise!

CAUCHY: I wish I had thought of such an axiomatic system of space with finite many only points, and the concept of infinitesimals as Leibnitz mentioned with your Local, Low and High precision levels! But there is a reason for this! Your axioms are much more elaborate and complicated that the Hilbert axioms of Euclidean Geometry!

NEWCLID: Indeed CAUCHY! But later the proofs of many other theorems, on areas, volumes and even derivatives, will become much simpler!

HILBERT: I like your brave and perfect approach Newclid! No infinite in your axioms so as to have easy physical applications, as nothing in the physical material reality is infinite. Congratulations! I am glad that my axioms of the classical Euclidean Geometry were of a good use to your work.

Von NEUMANN: I like tooyou axiomatic system Newclid! I believe that I could easily make it myself, except at that time I was busy in designing a whole generation of computers! I believe your works is a direct descendant of my work on computers. As you said your ideas came from software developers in the operating system of a computer!

NEWCLID: Indeed von Neumann! Thank you!

CANTOR: Pretty interesting your axiomatic system Newclid! But what is wrong with the infinite? Why you do not allow it in your axiomatic system? I believe that the infinite is a legitimate creation of the human mind! Your Digital Euclidean Geometry lacks the magic of the infinite!

PYTHAGORAS: Let me, Newclid, answer this question of CANTOR! Indeed CANTOR the human mind may formulate with a consistent axiomatic way what it wants! E.g. an axiomatic theory of the sets where infinite sets exist! And no doubt that the infinite is a valuable and sweet experience of the human consciousness! But as in the physical material reality there is nowhere infinite many atoms, mathematical models that in their ontology do not involve the infinite, will be more successful for physical applications! In addition there will not be any irrelevant to the physical reality complexity as in the mathematical models of e.g. of physical fluids that use infinite many points with zero dimensions in the place of the finite many only physical atoms with finite dimensions. The infinite may have its magic, but the axioms of the Digital Euclidean Geometry have their own and different magic!

RIEMANN: Very impressive Newclid your logical approach to the Euclidean space! But what about my Riemannian geometric spaces? Could they be formulated also with Local, Low and High precision levels and finite many visible and invisible points?

NEWCLID: Thank you Riemann! Well my friend any axiomatic system of your Riemannian Geometric spaces, with finite many points would require at least 3 or 4 precision levels! The reason is that at any A point of a Riemannian Space, the tangent or infinitesimal space at A is Euclidean! And here the interior of the point A will be a whole spherical Euclidean space which already requires two precision levels and both the visible and invisible points of the tangent Euclidean space will have to be invisible, while the point A visible point! But let us have patience! When I will be able to develop fully the digital differential calculus on a digital Euclidean Geometry we will reach and answer your question with clarity!

NEWCLID: If there no more questions or remarks, let us end here our discussion, and let us take a nice and energizing walk under the trees in the park close to our building.

At this point the discussion ends.

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