Some Characterized Projective δ-cover

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***Abstract*— In this paper we characterize some properties of projective δ-cover and find some new results with δ-supplemented module M. Let M be a fixed R-module. A δ-cover in M is an δ-small epimorphism from M onto P. These concept introduce by Zhou [14]. A δ-cover is projective δ-cover(M-projective δ-cover) in case M is projective.**

***Index Terms*— Singular, non singular, simple, small, δ-small, cover, δ-cover, supplement, δ-supplement.**

I. Introduction

Throughout this paper R is an associative ring with unity and all modules are unitary R-modules. Let M be a fixed module, a sub module L of module M is denoted by L ≤ M. submodule L of M is called essential (large) in M, abbreviated $K \leq_e M$, if for every submodule N of M, L ∩ N implies N = 0. A sub module N of a module M is called small in M, Denoted by N≪M, if for every sub module L of M, the equality N + L = M implies L = M. For each X ⊂ M, the right Ann(X) in R is $r_R(X) = \{r \in R : xr = 0$ for all x in x}. The sub module Z(M) = {x ∈ M : $r_R(x)$ is an essential in R_R} {x} is singleton, is called singular submodule of M. The module M is called singular module if Z(M) = M.(M is non singular if Z(M) = 0). A right R-module is called simple if M ≠ 0 and M has only proper submodules. A sub module N of M is called minimal in M if N ≠ 0 and for every submodules A of M, A ⊂N implies A = N. An epimorphism $f : M \to P$ is called small if $\ker f \prec\prec M$. A small epimorphism $f : M \to P$ is called projective cover if M is projective with $\ker f \prec\prec M$.[Zhou] introduce the concept of δ-small submodule as generalization of small submodules. Let K ≤ M, K is called δ-small if whenever M = N + K and M/N is a singular, we have M = N.(denoted by $\prec\prec_\delta$). The sum of all δ-small submodules is denoted by δ(M). A δ-cover in M is an δ-small epimorphism from M onto P. A δ-cover is projective δ-cover(M-projective δ-cover) in case M is projective.

Definition: Let M be a fixed R- module. An R-module U is called (small) M-projective module, if for every (small) epimorphism $f : M \to P$ and homomorphism $g : U \to P$, there exists a homomorphism $v : U \to M$ such that $f \circ v = g$, i.e. following diagram is commute.

$$\begin{tikzcd} & U \arrow[dl, "v"'] \arrow[d, "g"] & \\ M \arrow[r, "f"'] & P \arrow[r] & 0 \end{tikzcd} \qquad (Ker f \prec\prec M)$$

Example: Every proper sub module of the Z-modules Z_p^{∞} is small in Z_p^{∞}.

Remarks:

i) Every M-projective module is a small M-projective cover.

ii) Every self projective module M is self small projective module and converse is true for M is hollow.

Lemma: [Zhou] Let N be a sub module of M. The following are equivalent:

i) $N \ll_\delta M$

ii)If M = X + N, then M = X ⊕Y for a projective semisimple sub module Y with Y ⊆N.

Proof: [14]

Lemma: If each $f_i : N_i \to M_i$ are M-projective δ-covers for i = 1,2,3,…n, then $\oplus_{i=1}^{n} f_i : \oplus_{i=1}^{n} N_i \to M_i$ is M-projective δ- cover.

Proof: [12]

Lemma: If N is a direct summand of module M and $A \ll_\delta M$, then $A \cap N \ll_\delta N$.

Lemma: Let K be a sub module of a M-projective module U. If U/K has a M-projective δ-cover, then it has a M-projective δ-cover of the form $f : \frac{U}{L} \to \frac{U}{K}$ with $\ker f = \frac{K}{L}$ where L ⊆ K.

Proof: Let K be a sub module of a M-projective module U. Let $f : M \to \frac{U}{K}$ be a M-projective δ-cover of $\frac{U}{K}$, and $\pi : U \to \frac{U}{K}$ is a canonical epimorphism , U is M-projective module, there exists an homomorphism $v : U \to M \;\; s.t. \; f \circ v = \pi.$

$$\begin{tikzcd} & L \arrow[d] & \\ & K \arrow[d] & \\ & U \arrow[dl, "v"'] \arrow[d, "\pi"] & \\ M \arrow[r, "f"'] & \frac{U}{K} \arrow[r] & 0 \end{tikzcd}$$

Then $M = \ker g \oplus \mathrm{Im}\, v$. By lemma [Zhou] $M = N \oplus \mathrm{Im}\, v$ for semi simple sub module N, with N ⊆ Kerf since $\ker(f|_{\mathrm{Im}\, v}) \prec\prec_\delta \mathrm{Im}\, v$. So $f|_{\mathrm{Im}\, v}$ is also

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M-projective δ-cover of $\frac{U}{K}$. But $\frac{U}{\ker v} \cong \operatorname{Im} v$ by isomorphism theorem. Since $f \circ v = \pi$ and $\ker v \subseteq K$. If we consider the isomorphism $v' : \frac{U}{\ker v} \to \operatorname{Im} v$ defined by $v'(\ker v + u) = u \ \forall u \in U, \operatorname{Im} v \leq^{\oplus} U$. Then we obtain $\ker(f_{\operatorname{Im} v}, v') \prec\prec_{\delta} \frac{U}{\ker v}$.//

Lemma: A pair (M, f) is a M-projective δ-cover of finitely generated module U, The there exists a finitely generated direct summand M' of M such that $f\mathrm{I}_{M'}$ is a M-projective δ-cover of U.

Theorem: An R module M has a M-projective δ-cover, then for every epimorphism $f : M \to P$, the following are equivalent:

i) $f : M \to P$ is a M-projective cover.

ii) M is projective, for every epimorphism $f' : M' \to P$, with $M' \leq^{\oplus} M$, there exists a necessarily split epimorphism $h : M' \to M$ such that $f \circ h = f'$.

iii) For every small epimorphism $g : M \to N$, there exists an epimorphism $h : P \to M$ such that $f \circ h = g$

Corollary: Let $f : M \to P$ and $f' : M' \to P,\ M' \leq^{\oplus} M$, be a M-projective cover. Then there is an isomorphism $h : M \to M'$ such that $f' \circ h = f$. In fact if $h : M \to M'$ is a homomorphism with $f' \circ h = f$, then h is an isomorphism.

Proposition: Let $f : M \to P$ be a M-projective δ-cover. If U is M-projective and $g : U \to P$ is an homomorphism, then there exists decomposition $M = A \oplus B$ and $U = X \oplus Y$ such that

i) $A \cong X$

ii) $f\mathrm{I}_A : A \to P$ is a M-projective δ-cover.

iii) $h\mathrm{I}_X : X \to P$ is a M-projective δ-cover.

iv) B is a Projective semi simple with $B \subseteq \ker f$ and $Y \subseteq \ker h$

Proof: Since U is M- projective ,

$$\begin{tikzcd} & U \arrow[d, "g"] \arrow[dl, "h"'] & \\ M \arrow[r, "f"'] & P \arrow[r] & 0 \end{tikzcd}$$

Then there exists $h : U \to M$ such that $f \circ h = g$ .Thus we have $M = \operatorname{Im} h + \ker f$ and $\ker f \prec\prec_{\delta} M$, we have $M = \operatorname{Im} h + B$ for a semi simple module B with $B \subseteq \ker f$, by lemma 9. $f\mathrm{I}_A : A \to P$ is a M-projective δ-cover.

Since direct summand of projective module is projective, so A is projective and homomorphism $h : U \to A$ splits, then there exists $t : A \to U$ such that $h \circ t = I_A$. Thus $U = X \oplus Y = \operatorname{Im} t + \ker h$ this implies $A \cong t(A) = X$. Since $\ker(h\mathrm{I}_A) \prec\prec_{\delta} A\ (M = A \oplus B)$, we have $\ker(h\mathrm{I}_X) = \ker(h\mathrm{I}_A) \prec\prec_{\delta} t(A) = X$. $g(X) = (f \circ h)(X + Y) = (f \circ h)(U) = P$.

Thus $h\mathrm{I}_X : X \to P$ is a M-projective δ-cover.//

Lemma: Let U be a M-projective module and $N \leq^{\oplus} M$, then the following are equivalent;

i) $\frac{M}{N}$ has a M-projective δ-cover.

ii) $M = M_1 \oplus M_2$ for some M_1 and M_2 ,with

$$M_1 \subset N \ and\ M_2 \cap N \prec\prec_{\delta} M.$$

Proof: i)=> ii) Assume that $\frac{M}{N}$ has a M-projective δ-cover. Let $g : U \to \frac{M}{N}$ be a M-projective δ- cover and $\pi : M \to \frac{M}{N}$ is canonical epimorphism, then there exists an homomorphism $h : U \to M$ such that the diagram

$$\begin{tikzcd} & U \arrow[d, "g"] \arrow[dl, "h"'] & \\ M \arrow[r, "\pi"'] & \frac{M}{N} \arrow[r] & 0 \end{tikzcd}$$

is commute. Therefore $M = \operatorname{Im} h + \ker \pi = \operatorname{Im} h + N$. By lemma [Zhou 3.1] there exists a decomposition $M = M_1 \oplus M_2$ such that $\pi\mathrm{I}_X : M_2 \to M$ is a M-projective δ-cover and $M_1 \subseteq \ker \pi = N.$ Thus $M_2 \cap N = \ker(\pi\mathrm{I}_X) \prec\prec_{\delta} X$. Since $M_2 \leq^{\oplus} M\ then\ M_2 \cap N \prec\prec_{\delta} M.$

ii)=> i) it is clear.

Lemma: If $f : U \to M$ and $g : M \to N$ are δ-covers, then $g \circ f$ is a δ-cover.

Proof: [12]

Lemma: Let M, N, P be R-modules , for some homomorphisms $f : M \to P$, $g : M \to N$ and $h : N \to P$ such that $h \circ g = f$ then,

i) F is a small epimorphism if and only if

$$N = \ker h + \operatorname{Im} g.$$

ii) A pair (M, f) is a projective δ-cover if and only if g(M) is a δ-supplement of kerh in N and $\ker g \prec\prec_{\delta} M$

Proof: i) it is clear by lemma R

(ii) => Suppose a pair (M, f) is a δ-cover, by (i) we have $N = \ker h + \operatorname{Im} g$ i.e. f is small epimorphism , we get $g(\ker f) = \ker h + \operatorname{Im} g$ and $\ker f \prec\prec_{\delta} M$.

By lemma [1,1 K. Al-Thakman] $g(\ker f) \prec\prec_{\delta} \operatorname{Im} g$, hence Img is δ-supplement of kerh in N.

$\Leftarrow$ Assume that the $g(M)$ is a δ-supplement of $\ker h$ in N, then $N = \operatorname{Im} g + \ker h$ and $\operatorname{Im} g \bigcap \ker h \prec\prec_{\delta} \operatorname{Im} g$. Since f is epimorphism, consider $\ker f + S = M$ and $\frac{M}{S}$ is singular. So $g(\ker f) + g(S) = g(M)$ but $g(\ker f) = \ker h \bigcap \operatorname{Im} g$,

Hence $g(M) = g(\ker f) \bigcap \operatorname{Im} g + g(S)$, since $\frac{g(M)}{g(S)}$ is singular, being a homomorphic image of singular module and $\operatorname{Im} g \bigcap \ker h \prec\prec_{\delta} \operatorname{Im} g$. We have $g(M) = g(S)$ and so $M = S + \ker g$, by assumption $\ker g \prec\prec_{\delta} M$ and $\frac{M}{S}$ is singular,

so $M = S$. Hence $\ker f \prec\prec_{\delta} M$. //

Theorem: If $M = M_1 + M_2$ then the following are equivalent:

i) M_2 is a small- M_1-projective.

ii) For any sub module N of M such that M_1 is a δ-supplement of N in M. There exists a sub N_1 of N such that $M = M_1 \oplus N_1$.

Proof: [14].

Proposition: If U is a sub module of R-module M, then following are equivalent:

i) $\frac{M}{M_1}$ has a M-projective δ-cover.

ii) If $M_2 \leq M$ and $M = M_1 + M_2$, M_2 has a δ-supplemented $M'_1 \subseteq M_2$ such that M'_1 has a M-projective δ-cover.

iii) M_2 has a δ- supplemented M'_1, which has a M-projective δ-cover.

Proof: (i) $\Rightarrow$(ii) Assume that $\frac{M}{M_1}$ has a M-projective δ-cover. Therefore $f : U \to \frac{M}{M_1}$ be a M-projective δ-cover.

Since $M = M_1 + M_2, g : M_2 \to \frac{M}{M_1}$ is an epimorphism .Given that U is M- projective module, then there exists an homomorphism $h : U \to M_2$ such that $f = g \circ h$. By lemma Q] $M = M_1 + \operatorname{Im} h = M_1 + h(U)$,where $h(U) \prec\prec_{\delta} M_2$. Since $\ker f \prec\prec_{\delta} U$, we have $M_1 \bigcap h(U) = h(\ker f) \prec\prec_{\delta} h(U)$ and $h(U)$ is δ-supplement of M_1 in M. Since $\ker h \leq \ker f \prec\prec_{\delta} U, h : U \to h(U)$ is M-projective δ-cover.

(ii)$\Rightarrow$(iii) it is clear.

(iii)$\Rightarrow$(i) Let $f : U \to M_1'$ be a M-projective δ-cover. Since M_1' is a δ-supplement of M, the natural epimorphism $g : M_1' \to \frac{M_1'}{M_1 \bigcap M_1'} \cong \frac{M_1 + M_1'}{M_1} = \frac{M}{M_1}$ is M-projective δ-cover. Hence $f : U \to \frac{M}{M_1}$ is a M-projective δ-cover, by lemma [A], where $h : \frac{M_1'}{M_1 \bigcap M_1'} \to \frac{M_1 + M_1'}{M_1}$ is an isomorphism. //

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