Some Properties of Semi-continuous, Pre-continuous and α-continuous Mappings

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***Abstract*— This paper investigates some new characteristics of semi-continuous, pre-continuous and α-continuous mappings. We provides two theorems that are equivalent to the definitions of pre-continuous and M-semi-continuous mappings. A condition has been proposed, which makes the injective mapping pre-open. We have proved that the domain of the injective α-continuous mapping with closed graph is Housdorff space. In addition, more other conditions put on the α-continuous mapping, which make its graph closed.**

***Index Terms*—closed graph, semi-continuous mappings, pre-continuous mapping, α-continuous mapping, α-open mapping, M-semi-continuous mapping**

I. INTRODUCTION

A subset A of the space X is called a semi-open [5] (resp. α-set [7], pre-open [1], β-open [2], regular open[8]) $A \subset A^{\circ -}$ (resp. $A \subset A^{\circ - \circ}$, $A \subset A^{- \circ}$, $A \subset A^{- \circ -}$, $A = A^{- \circ}$). The complement of a semi-open (resp. α-set, pre-open, β-set, regular open) set, is called a semi-closed[5] (resp. α-closed [6], pre-closed[1], β-closed[2], regular closed[7]). The family of all semi-open (resp. α-set, pre-open, β-open, regular open) sets of a space X will be denoted by SO(X) (resp. α(X), PO(X), βO(X), RO(X)).

A mapping $f : X \to Y$ is called semi-continuous [5] (resp. α-continuous [6], pre-continuous [1], and β- continuous [4]) if the inverse image of every open set in Y is semi-open (resp. α-set, pre-open, β-open) in X.

Theorem 1.1. [5]. Let $f : X \to Y$ be a mapping, then the following statements are equivalent:

i) f is β-continuous.

ii) For every $x \in X$ and every open set $V \subset Y$ containing f(x), there exists a β-open set $W \subset X$ containing x such that $f(W) \subset V$.

iii) The inverse image of each closed set in Y is β-closed in X.

iv) $(f^{-1}(B))^{o-o} \subset f^{-1}(\bar{B})$, for every $B \subset Y$.

v) $f(A^{o-o}) \subset \overline{(f(A))}$, for every $A \subset X$.

Theorem 1.2. [1]. Let $f: X \to Y$ be a mapping, then the following statements are equivalent:

i) f is β-open.

ii) For every $x \in X$ and every neighborhood U of x, there exists a β-open set $W \subset Y$ containing f(x) such that $W \subset f(U)$.

iii) $f^{-1}(B^{o-o}) \subset \overline{(f^{-1}(B))}$, for every $B \subset Y$.

iv) If f is bijective, then $(f(A))^{o-o} \subset f(\bar{A})$, for every $A \subset X$.

Definition 1.1. [3],[5] A mapping $f : X \to Y$ is said to have a closed graph , if its graph $G(f) = \{(x , y) : y = f(x) , x \in X\}$ in the product space $X \times Y$ is a closed set . Equivalently , G(f) is a closed subset of $X \times Y$, if and only of for each $x \in X$, and $y \neq f(x)$, there exist open sets U and V containing x and y respectively such that $f(U) \cap V = \emptyset$.

II. Semi-continuous, Pre-continuous and a -continuous Mappings.

Theorem 2.1. A mapping $f : X \to Y$ is pre-continuous iff $f(\overline{U}) \subset (f(U))^{-}$, for every open set $U \subset X$.

Proof. Let f be a pre-continuous, then by theorem 1.1., $f(U^{\circ -}) \subset (f(U))^{-}$, for every U , since $U \subset X$, since $U \subset X$ is open, then $f(\overline{U}) \subset (f(U))^{-}$.

Conversely, let $V \subset Y$ be open, $W = Y - V$, and let $U = (f^{-1}(W))^{\circ}$ be an open subset of X, then

$f(f^{-1}(W))^{\circ -}) \subset (f((f^{-1}(W)^{\circ}))^{-} \subset (f(f^{-1}(W)))^{-} \subset \overline{W} = W$

. So, $(f^{-1}(W))^{\circ -} \subset f^{-1}(W)$ and f is pre-continuous .

Theorem 2.2. An injective mapping $f : X \to Y$ is pre-open iff $f^{-1}(\overline{B}) \subset (f^{-1}(B))^{-}$, for every open set $B \subset Y$.

Proof. Suppose f is pre-open then by Theorem 1.2., $f^{-1}(B^{\circ -}) \subset (f^{-1}(B))^{-}$. Since $B \subset Y$ is open,

$$f^{-1}(\overline{B}) \subset (f^{-1}(B))^{-}$$

Conversely, let $V \subset X$ be an open set , $W = X - V$ and let $B = (f(W))^{o}$ be an open subset of Y.

Then

$f^{-1}((f(W))^{\circ -}) \subset (f^{-1}(W))^{\circ})^{-} \subset (f^{-1}(f(W)))^{-} = \overline{W} = W$.

Hence $(f(W))^{\circ -} \subset f(W)$, and so, f is pre-open.

Theorem 2.3. Let $f : X \to Y$ be an injective α-continuous mapping with closed graph. Then X is a T_2-space.

Proof. Let $x_1, x_2 \in X$, $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. Since G(f) is closed , then there exist open sets U and V containing x_1 and $f(x_2)$ respectively , such that $f(U) \cap V = \phi$. Thus $f^{-1}(V) \subset X - U$, and $(f^{-1}(V))^{\circ - \circ} \subset (X - U)^{\circ - \circ} = (X - U^{-\circ})^{\circ}$. Since f is α –continuous, $x_2 \in f^{-1}(V) \subset (X - \overline{U}^{\circ})^{\circ}$. Therefore, X is a T_2-space.

Theorem 2.4. Let $f : X \to Y$ be α-continuous mapping where Y is locally connected T_2-space. If f and f^{-1} map connected sets into connected sets, then the graph G(f) of f is closed.

Proof. Let $x \in X$, and $y \in f(x)$, $y \in Y$. Since Y is T_2 -space, there exist two disjoint open sets U and V containing y and f(x), respectively. Since Y is locally connected , there exist an

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open connected set W such that $f(x) \in W \subset V$. So $W \cap U = \phi$ and $f^{-1}(W) \cap f^{-1}(U) = \phi$. Since f is α-continuous,

$$x \in f^{-1}(W) \subset (f^{-1}(W))^{\circ-\circ}.$$

Now since any point $p \in f^{-1}(U)$ is a limit point of the connected set $f^{-1}(W)$, then $\{p\} \cup f^{-1}(W)$ is connected. But, $f(\{p\} \cup f^{-1}(W))$ has points in each of the two disjoint open sets U and W and so, $f(\{p\} \cup f^{-1}(W)))$ is not connected, which is a contradiction to our assumption that f maps connected sets into connected sets . Hence $(f^{-1}(W))^{\circ-\circ} \cap f^{-1}(U) = \phi$ and $f(f^{-1}(W))^{\circ-\circ} \cap U = \phi$, therefore G(f) is closed.

Theorem 2.5. Let f be α-continuous surjection, then Y is connected if X is connected.

Proof. Assume Y is not connected and X is connected, then there are two disjoint open sets $V_i \subset Y$, $i \in \{1,2\}$ such that $\bigcup_i V_i = Y$ and $\bigcap_i V_i = \phi$. Since f is α-continuous and since α-continuity implies β-continuity, then $f^{-1}(V_i) \subset (f^{-1}(V_i))^{\circ-\circ} \subset f^{-1}(\overline{V_i})$. Since V_i is open and closed for every $i \in \{1,2\}$,

$$\bigcap_i f^{-1}(V_i) \subset \bigcap_i (f^{-1}(V_i))^{\circ-\circ} \subset \bigcap_i f^{-1}(\overline{V_i}) \bigcap_i f^{-1}(V_i) = \phi.$$

Hence X is not connected, and this leads to a contradiction which proves that Y is connected.

Theorem 2.6. For a bijective mapping f , f is α-open iff $(f(U))^{-\circ-} \subset f(\overline{U})$, for every $U \subset X$.

Proof. Suppose f is α-open. Let $U \subset X$, then

$$f(X - \overline{U}) \subset (f(X - \overline{U}))^{\circ-\circ} \subset (f(X - U))^{\circ-\circ}).$$

Since f is bijective, $f(\overline{U}) \supset (f(U))^{-\circ-}$.

Conversely, suppose U is an open set of X. Then

$$f(X - U) = f(X - U) \supset (f(X - U))^{-\circ-}.$$

Since f is bijective , $f(U) \subset (f(U))^{\circ-\circ}$, and so , f is α-open.

Definition 2.1.[1]. A mapping $f : X \to Y$ is called M-semi-continuous if the inverse image of every semi-open set in Y is semi-open in X.

The following theorem gives a new property of a semi-continuous mappings.

Theorem 2.7. Let $f : X \to Y$ be semi-continuous and $f^{-1}(\overline{V}) \subset (f^{-1}(V))^{-}$ for every semi-open set $V \subset Y$, then f is M-semi-continuous.

Proof. Let V be semi-open set in X . Since f is semi-continuous,

Then

$$f^{-1}(V) \subset f^{-1}(V^{\circ-}) \subset (f^{-1}(V^{\circ}))^{-} \subset (f^{-1}(V^{\circ}))^{\circ} = f^{-1}(V^{\circ}))^{\circ-} \subset$$

$(f^{-1}(V))^{\circ-}$. Hence f is M-semi-continuous.

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