Complex of Lascoux in Partition (6,6,3)

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Abstract— **In this paper, the complex of Lascoux in the case of partition (6,6,3) has been studied by using diagrams, divided power of the place polarization $\partial_{ij}^{(k)}$,Capelli identities and the idea of mapping cone.**

Index Terms— **Divided power algebra, Resolution of Weyl module, Place polarization, Mapping Cone**

I. Introduction

Let R be the commutative ring with 1, F be a free module and D_sF be the divided power of degree s. Another type of maps are used in Buchsbaum whose images define schur and Weyl modules which send an element $a \otimes b$ of $D_{p+k} \otimes D_{q-k}$ to $\sum a_p \otimes a'_k b$, where $\sum a_p \otimes a'_k$ is the component of the diagonal of a in $D_p \otimes D_k$,the generalization of this map to ones , where there more factors were called in the 'box map'.

The complex of characteristic zero is studied in [3],[4] and [5] in the partition (2,2,2) ,(3,3,3) and (4,4,3), using this modified and the letter place methods [3] , In this paper we study the complex of Lasoux in the case of partition (6,6,3) as a diagram by using the idea of the mapping Cone [6] , and the map $\partial_{ij}^{(k)}$ which means the k^{th} divided power of the place polarization ∂_{ij} where *j* must be less than *I* with it's Caplli identities [1] , specificly in this work we used only the following identities

$$\partial_{32}^{(l)}\partial_{21}^{(k)} = \sum_{\alpha\geq 0}\partial_{21}^{(k-\alpha)}\partial_{32}^{(l-\alpha)}\partial_{31}^{(\alpha)} \qquad (1.1)$$

$$\partial_{21}^{(k)}\partial_{32}^{(l)} = \sum_{\alpha\geq 0}(-1)^{\alpha}\partial_{32}^{(l-k)}\partial_{21}^{(k-\alpha)}\partial_{31}^{(\alpha)}$$

(1.2)

$$\partial_{21}^{(1)}\circ\partial_{31}^{(1)} = \partial_{31}^{(1)}\circ\partial_{21}^{(1)} \qquad \text{and}$$

$$\partial_{32}^{(1)}\circ\partial_{31}^{(1)} = \partial_{31}^{(1)}\circ\partial_{32}^{(1)} \qquad (1.3)$$

Where ∂_{ij} is the place polarization from place *j* to place *i*.

II. The terms of Lascoux complex in the case of partition (6,6,3)

The terms of the lascoux complex are obtained from the determinantal expansion of the Jocobi-trudi matrix of the partition. The positions of the terms of the complex are determined by the length of the permutation to which they correspond [2],[3]. Now in the 6,6,3), we have the following matrix: λ=(case of the partition

$$\begin{bmatrix} D_6 & D_5 & D_1 \\ D_7 & D_6 & D_2 \\ D_8 & D_7 & D_3 \end{bmatrix}$$

Then the Lascoux complex has the correspondence between it's terms as follows:

$D_6F \otimes D_6F \otimes D_3F \leftrightarrow identity$
$D_5F \otimes D_7F \otimes D_3F \leftrightarrow (12)$
$D_6F \otimes D_2F \otimes D_7F \leftrightarrow (23)$
$D_5F \otimes D_2F \otimes D_8F \leftrightarrow (123)$
$D_1F \otimes D_7F \otimes D_7F \leftrightarrow (132)$

So, the complex of Lascoux in the case of the partition $\lambda = (6,6,3)$ has the form:-

$$D_8F\otimes D_6F\otimes D_1F \to \begin{matrix} D_8F\otimes D_5F\otimes D_2F \\ \oplus \\ D_7F\otimes D_7F\otimes D_1F \end{matrix} \to \begin{matrix} D_7F\otimes D_5F\otimes D_3F \\ \oplus \\ D_6F\otimes D_7F\otimes D_2F \end{matrix} \to D_6F\otimes D_6F\otimes D_3F$$

III. The complex of Lascoux as a diagram

Consider the following diagram :

$$\begin{tikzcd} D_8F\otimes D_6F\otimes D_1F \arrow[r, "S_1"] \arrow[d, "b_1"] \arrow[dr, phantom, "M"] & D_8F\otimes D_5F\otimes D_2F \arrow[r, "S_2"] \arrow[d, "b_2"] \arrow[dr, phantom, "N"] & D_7F\otimes D_5F\otimes D_3 \arrow[d, "b_3"] \\ D_7F\otimes D_7F\otimes D_1F \arrow[r, "t_1"] & D_6F\otimes D_7F\otimes D_2F \arrow[r, "t_2"] & D_6F\otimes D_6F\otimes D_3F \end{tikzcd}$$

So, if we define

$S_1 : D_8F \otimes D_6F \otimes D_1F \longrightarrow D_8F \otimes D_5F \otimes D_2F$
as, $S_1(V) = \partial_{32}(V)$ $\quad$ *where* ; $V \in D_8F \otimes D_6F \otimes D_1F$
$b_1 : D_8F \otimes D_6F \otimes D_1F \longrightarrow D_7F \otimes D_7F \otimes D_1F$
as, $b_1(V) = \partial_{21}(V)$ $\quad$ *where* $V \in D_8F \otimes D_6F \otimes D_1F$
$b_2 : D_8F \otimes D_5F \otimes D_2F \longrightarrow D_6F \otimes D_7F \otimes D_2F$
as, $b_2(V) = \partial_{21}^{(2)}(V)$ $\quad$ *where* $V \in D_8F \otimes D_5F \otimes D_2F$

Now , we have to define the following map which makes the diagram M commutative:

$t_1 : D_7F \otimes D_7F \otimes D_1F \longrightarrow D_6F \otimes D_7F \otimes D_2F$

So we have:

$t_1 \circ b_1 = b_2 \circ S_1$

Which implies that

$t_1 \circ \partial_{21} = \partial_{21}^{(2)} \circ \partial_{32}$

Now we use Capelli identities from

$$\partial_{21}^{(2)} \circ \partial_{32} = \partial_{32} \circ \partial_{21}^{(2)} - \partial_{31} \circ \partial_{21}$$
$$= \left(\tfrac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}\right) \circ \partial_{21}$$

Thus , $t_1 = \frac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}$

On the other hand, if we define

$t_2 : D_6F \otimes D_7F \otimes D_2F \longrightarrow D_6F \otimes D_6F \otimes D_3F$
$t_2(v) = \partial_{32}(v)$ $\quad$ where; $v \in D_6F \otimes D_7F \otimes D_2F$
and $b_3 : D_7F \otimes D_5F \otimes D_3F \longrightarrow D_6F \otimes D_6F \otimes D_3F$
$b_3(v) = \partial_{21}(v)$ $\quad$ *where;* $v \in D_7F \otimes D_5F \otimes D_3F$ $\quad$ *as,*

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Now we need to define S_2 to make the diagram N commute:

$S_2 : D_8F \otimes D_5F \otimes D_2F \longrightarrow D_7F \otimes D_5F \otimes D_3F$

Such that $b_3 \circ S_2 = b_3 \circ S_2$ *i.e.* $\partial_{21} \circ S_2 = \partial_{32} \circ \partial_{21}^{(2)}$

Again by using Caplli identities we get

$$\begin{aligned}\partial_{32} \circ \partial_{21}^{(2)} &= \partial_{21}^{(2)} \circ \partial_{32} + \partial_{21} \circ \partial_{31} \\ &= \partial_{21}\left(\tfrac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right)\end{aligned}$$

Then $S_2 = \frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}$

Now consider the following diagram :

$$\begin{tikzcd} D_8F \otimes D_6F \otimes D_1F \arrow[r, "S_1"] \arrow[d, "b_1"'] & D_8F \otimes D_5F \otimes D_2F \arrow[r, "S_2"] & D_7F \otimes D_5F \otimes D_3F \arrow[d, "b_3"] \\ D_7F \otimes D_7F \otimes D_1F \arrow[r, "t_1"] \arrow[urr, "Z"] \arrow[ur, phantom, "H"] & D_6F \otimes D_7F \otimes D_2F \arrow[r, "t_2"] \arrow[ur, phantom, "G"] & D_6F \otimes D_6F \otimes D_3F \end{tikzcd}$$

Define $z : D_7F \otimes D_7F \otimes D_1F \longrightarrow D_7F \otimes D_5F \otimes D_3F$

By $z(v) = \partial_{32}^{(2)}$ where $v \in D_7F \otimes D_7F \otimes D_1F$.

Proposition 3.1:- The diagram H is commutative.

Proof :- To prove H is commutative , we need to prove $S_2 \circ S_1 = z \circ b_1$

$$\begin{aligned}S_2 \circ S_1 &= \left(\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right) \circ \partial_{32} \\ &= \partial_{21} \circ \partial_{32}^{(2)} + \partial_{32} \circ \partial_{31} \\ &= \partial_{32}^{(2)} \circ \partial_{21} - \partial_{32} \circ \partial_{31} + \partial_{32} \circ \partial_{31} \\ &= \partial_{32}^{(2)} \circ \partial_{21} \\ &= z \circ \partial_{21} .\end{aligned}$$

■ **Proposition 3.2:-** The diagram G is commutative

Proof :-

$$\begin{aligned}t_2 \circ t_1 &= \left(\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right) \circ \partial_{32} \\ &= \partial_{21}\partial_{32}^{(2)} + \partial_{32}\partial_{31} \\ &= \partial_{32}^{(2)} \circ \partial_{21} - \partial_{32} \circ \partial_{31} + \partial_{32} \circ \partial_{31} \\ &= \partial_{32}^{(2)} \circ \partial_{21} \\ &= z \circ \partial_{21} .\end{aligned}$$

■

Finally by using the mapping Cone we can define the maps σ_1, σ_2 and σ_3 where:

$$\sigma_3 : D_8F \otimes D_6F \otimes D_1F \longrightarrow \begin{matrix} D_8F \otimes D_5F \otimes D_2F \\ \oplus \\ D_7F \otimes D_7F \otimes D_1F \end{matrix}$$

$$\sigma_2 : \begin{matrix} D_7F \otimes D_5F \otimes D_3F \\ \oplus \\ D_7F \otimes D_7F \otimes D_1F \end{matrix} \longrightarrow \begin{matrix} D_8F \otimes D_5F \otimes D_2F \\ \oplus \\ D_6F \otimes D_7F \otimes D_2F \end{matrix}$$

and

$$\sigma_1 : \begin{matrix} D_7F \otimes D_5F \otimes D_3F \\ \oplus \\ D_6F \otimes D_7F \otimes D_2F \end{matrix} \longrightarrow D_6F \otimes D_6F \otimes D_3F$$

$D_8F \otimes D_5F \otimes D_2F$

$D_7F \otimes D_5F \otimes D_3F$

$0 \longrightarrow D_8F \otimes D_6F \otimes D_1F \longrightarrow$ $\oplus$ $\longrightarrow$

$\oplus$ $\longrightarrow D_6F \otimes D_6F \otimes D_3F$

$D_6F \otimes D_7F \otimes D_2F$

$D_7F \otimes D_7F \otimes D_1F$

by

$\bullet \sigma_3(x) = (s_1(x), b_1(x))$; $\forall x \in D_8F \otimes D_6F \otimes D_1F$

$D_8F \otimes D_5F \otimes D_2F$

$\bullet$ $\sigma_2((x_1,x_2)) = (s_2(x_1) - z(x_2), b_1(x_2) - b_2(x_1))$; $\forall (x_1,x_2) \in$ $\oplus$

$D_7F \otimes D_7F \otimes D_1F$

$D_7F \otimes D_5F \otimes D_3F$

$\bullet \sigma_1((x_1,x_2)) = (b_3(x_1) + t_2(x_2))$; $\forall (x_1,x_2) \in$ $\oplus$

$D_6F \otimes D_7F \otimes D_2F$

Propsition 3.3:

$D_8F \otimes D_5F \otimes D_2$ F

$D_7F \otimes D_5F \otimes D_3F$

0

$\rightarrow$

$D_8F \otimes D_6F \otimes D_1F \xrightarrow{\sigma_3}$ $\oplus$

$\xrightarrow{\sigma_2}$ $\oplus$ $\xrightarrow{\sigma_1} D_6F \otimes D_6F \otimes D_3F$

$D_7F \otimes D_7F \otimes D_1F$ $D_6F \otimes D_7F \otimes D_2F$

is complex.

Proof:-

Since $\partial_{32}^{(1)}$ and $\partial_{21}^{(1)}$ are injective from it is definition (see [1]), then we get σ_3 is injective.

Now

$$\begin{aligned}\sigma_2 \circ \sigma_3 &= \sigma_2(s_1(x), b_1(x)) \\ &= \sigma_2(\partial_{32}(x), \partial_{21}(x))\end{aligned}$$

$$= \left(s_2(\partial_{32}(x)) - z(\partial_{21}(x)), t_1(\partial_{21}(x)) - b_2(\partial_{32}(x))\right).$$

Now

$$s_2(\partial_{32}(x)) - z(\partial_{21}(x)) = \left(\frac{1}{2}\partial_{21} \circ \partial_{32} + \partial_{31}\right) \circ \partial_{32}(x) - \partial_{32}^{(2)} \circ \partial_{21}(x)$$

$$= (\partial_{21} \circ \partial_{32}^{(2)} + \partial_{31} \circ \partial_{32} - \partial_{32}^{(2)} \circ \partial_{21})(x)$$

$$= (\partial_{21} \circ \partial_{32}^{(2)} + \partial_{32} \circ \partial_{31} - \partial_{21} \circ \partial_{32}^{(2)} - \partial_{32} \circ \partial_{31})(x) = 0.$$

$$t_1(\partial_{21}(x)) - b_2(\partial_{32}(x)) = \left(\frac{1}{2}\partial_{32} \circ \partial_{21} - \partial_{31}\right) \circ \partial_{21}(x) - \partial_{21}^{(2)} \circ \partial_{32}(x)$$

$$= (\partial_{32} \circ \partial_{21}^{(2)} - \partial_{21} \circ \partial_{31} - \partial_{32} \circ \partial_{21}^{(2)} + \partial_{21} \circ \partial_{31})(x) = 0.$$

So we get $(\sigma_2 \circ \sigma_3)(x) = 0$.

and

$(\sigma_1 \circ \sigma_2)(x_1,x_2) = \sigma_1(s_2(x_1) - z(x_2), t_1(x_2) - b_2(x_1))$

$= \sigma_1\left(\left(\frac{1}{2}\partial_{21}\circ\partial_{32}+\partial_{31}\right)(x_1)-\partial_{32}^{(2)}(x_2),\left(\frac{1}{2}\partial_{32}\circ\partial_{21}-\right.\right.$
$\left.\left.\partial_{31}\right)(x_2)-\partial_{21}^{(2)}(x_1)\right)$
$= \partial_{21}\left(\frac{1}{2}\partial_{21}\partial_{32}+\partial_{31}\right)(x_1)-\partial_{21}\circ\partial_{32}^{(2)}(x_2)$
$+\partial_{32}\left(\frac{1}{2}\partial_{32}\circ\partial_{21}-\partial_{31}\right)(x_2)-\partial_{32}\circ\partial_{21}^{(2)}(x_1)$
$= \left(\partial_{21}^{(2)}\circ\partial_{32}+\partial_{21}\circ\partial_{31}-\partial_{32}\circ\partial_{21}^{(2)}\right)(x_1)$
$+\left(\partial_{32}^{(2)}\circ\partial_{21}-\partial_{32}\circ\partial_{31}-\partial_{21}\circ\partial_{32}^{(2)}\right)(x_2).$
then

$$(\sigma_1\circ\sigma_2)(x_1,x_2)=(\partial_{32}\circ\partial_{21}^{(2)}-\partial_{21}\circ\partial_{31}+\partial_{21}\circ\partial_{31}-\partial_{32}\circ\partial_{21}^{(2)})(x_1)$$

$$+(\partial_{21}\circ\partial_{32}^{(2)}+\partial_{32}\circ\partial_{31}-\partial_{32}\circ\partial_{31}-\partial_{21}\circ\partial_{32}^{(2)})(x_2)=0.$$

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