

On CR-Structure And F-Structure Satisfying

$$F^{p_1 p_2} + F = 0$$

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Abstract— In this paper, we have studied a relationship between CR-structure and F-structure satisfying $F^{p_1 p_2} + F = 0$, where p_1 and p_2 are twin primes. Nijenhuis tensor and integrability conditions have also been discussed.

Index Terms— Projection operators, distributions, Nijenhuis tensor, integrability conditions and CR-structure.

I. INTRODUCTION

Let M be an n -dimensional differentiable manifold of class C^∞ . Let F be a non-zero tensor of type $(1, 1)$ and class C^∞ defined on M , such that

$$1.1 \quad F^{p_1 p_2} + F = 0$$

where p_1 and p_2 are twin primes.

Let $\text{rank} \left(\left(F \right) \right) = r$, which is constant everywhere. We define the operators on M as

$$1.2 \quad l = -F^{p_1 p_2 - 1}, m = F^{p_1 p_2 - 1} + I$$

where I is the identity operator on M .

Theorem (1.1) Let M be an F -structure satisfying (1.1) Then

$$(1.3) \quad (a) \quad l + m = I$$

$$(b) \quad l^2 = l$$

$$(c) \quad m^2 = m$$

$$(d) \quad lm = ml = 0$$

Proof: From (1.1) and (1.2), we get the results.

Let D_l and D_m be the complementary distributions corresponding to the operators l and m respectively. then

$$\dim \left(\left(D_l \right) \right) = r, \quad \dim \left(\left(D_m \right) \right) = n - r$$

Theorem (1.2) Let M be an F -structure satisfying (1.1). Then

$$(1.4) \quad (a) \quad lF = Fl = F, \quad mF = Fm = 0$$

$$(b) \quad F^{p_1 p_2 - 1} l = -l, \quad F^{p_1 p_2 - 1} m = 0$$

Proof: From (1.1), (1.2), (1.3)(a), (b), we get the results.

From (1.4) (b), it is clear that $F^{(p_1 p_2 - 1)/2}$ acts on D_l as an almost complex structure and on D_m as a null operator.

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II. NIJENHUIS TENSOR:

Definition (2.1) Let X and Y be any two vector fields on M , then their Lie bracket $[X, Y]$ is defined by

$$(2.1) \quad [X, Y] = XY - YX,$$

and Nijenhuis tensor

$N(X, Y)$ of F is defined as

$$(2.2)$$

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]$$

Theorem (2.1) A necessary and sufficient condition for the F -structure to be integrable is

$N(X, Y) = 0$, for any two vector fields X & Y on M .

Theorem (2.2) Let the F -structure satisfying (1.1) be integrable, then

$$(2.3)$$

$$\left(-F^{p_1 p_2 - 2} \right) ([FX, FY] + F^2[X, Y]) = l([FX, Y] + [X, FY]).$$

Proof: using theorem (2.1) in (2.2), we get

$$(2.4)$$

$$[FX, FY] + F^2[X, Y] = F([FX, Y] + [X, FY])$$

Operating by $\left(-F^{p_1 p_2 - 2} \right)$ on both the sides of (2.4) and using (1.2), we get the result.

Theorem (2.3) On the F -structure satisfying (1.1)

$$(2.5) \quad (a) \quad m N(X, Y) = m [FX, FY]$$

(b)

$$m N \left(F^{p_1 p_2 - 2} X, Y \right) = m \left[F^{p_1 p_2 - 1} X, FY \right]$$

Proof: Operating m on both the sides of (2.2) and using (1.4) (a) we get (2.5) (a). Replacing X by $F^{p_1 p_2 - 2} X$ in (2.5) (a), we get (2.5) (b).

Theorem (2.4): On the F -structure satisfying (1.1), the following conditions are all equivalent

$$(2.6) \quad (a) \quad m N(X, Y) = 0$$

$$(b) \quad m [FX, FY] = 0$$

$$(c) \quad m N \left(F^{p_1 p_2 - 2} X, Y \right) = 0$$

$$(d) \quad m \left[F^{p_1 p_2 - 1} X, FY \right] = 0$$

$$(e) \quad m \left[F^{p_1 p_2 - 1} lX, FY \right] = 0$$

Proof: Using (1.4) (a), (b) in (2.5) (a), (b), we get the results.

III. CR-STRUCTURE:

Definiton (3.1) Let $T_c(M)$ denotes the complexified tangent bundle of the differentiable manifold M . A CR-structure on M is a complex sub-bundle H of $T_c(M)$ such that

$$(3.1) \quad (a) \quad H_p \cap \tilde{H}_p = \{0\}$$

$$(b) \quad H \text{ is involutive that is } X, Y \in H \Rightarrow [X, Y] \in H \text{ for complex vector fields } X \text{ and } Y.$$

For the integrable F-structure satisfying (1.1) rank $((F)) = r = 2m$ on M .

we define

$$(3.2) \quad H_p = \{X - \sqrt{-1}FX : X \in X(D_l)\}$$

where $X(D_l)$ is the $F(D_m)$ module of all differentiable sections of D_l .

Theorem (3.1) If P and Q are two elements of H , then

$$(3.3) \quad [P, Q] = [X, Y] - [FX, FY] - \sqrt{-1}(-1)([FX, Y] + [X, FY])$$

Proof:

Defining

$$P = X - \sqrt{-1}(-1)FX, Q = Y - \sqrt{-1}(-1)FY$$

and simplifying, we get (3.3)

Theorem (3.2) for $X, Y, \in X(D_l)$

$$(3.4) \quad l([FX, Y] + [X, FY]) = [FX, Y] + [X, FY]$$

Proof: Using (1.4) (a) and (2.1), we get the result as

$$(3.5) \quad l([FX, Y] + [X, FY]) = l(FXY - YFX + XFY - FYZ)$$

$$= FXY - YFX + XFY - FYZ$$

$$= [FX, Y] + [X, FY]$$

Theorem (3.3) The integrable F-structure satisfying (1.1) on M defines a CR-structure H on it such that

$$(3.6) \quad R_e(H) = D_l$$

Proof: since $[X, FY], [FX, Y] \in X(D_l)$

then from (3.3), (3.4), we get

$$(3.7) \quad l[P, Q] = [P, Q]$$

$$\Rightarrow [P, Q] \in X(D_l)$$

Thus F structure satisfying (1.1), defines a CR-structure on M .

Definition (3.2) Let \tilde{K} be the complementary distribution of $R_e(H)$ to TM . We define a morphism $F : TM \longrightarrow TM$, given by

$$F(X) = 0, \forall X \in X(\tilde{K}) \text{ such that}$$

$$(3.8) \quad F(X) = \frac{1}{2}\sqrt{-1}(-1)(P - \tilde{P})$$

where $P = X + \sqrt{-1}(-1)Y \in X(H_p)$

and \tilde{P} is complex conjugate of P .

Corollary (3.1): From (3.8) we get

$$(3.9) \quad F^2 X = -X$$

Theorem (3.4): If M has CR-structure then $F^{p_1 p_2} + F = 0$ and consequently F-structure satisfying (1.1) is defined on M s.t. D_l and D_m coincide with $R_e(H)$ and \tilde{K} respectively.

Proof: Since p_1 and p_2 are twin primes $\therefore p_1 p_2$ when divided by 4 leaves 3 as a remainder \therefore Repeated application of (3.9) gives,

$$F^{p_1 p_2} = F^3(X)$$

$$= F(F^2 X)$$

$$= F(-X)$$

Thus, $F^{p_1 p_2} + F = 0$

REFERENCES:

- [1] Bejancu, A., CR-submanifolds of a Kaehler manifold 1, *Proc. Amer. Math. Soc.*, 69, 135-143 (1978).
- [2] Demetropoulou-Psomopoulou, D. and Andreou, F. Gouli, On necessary and sufficient conditions for an n-dimensional manifold to admit a tensor field $f (\neq 0)$ of type (1, 1) satisfying $f^{2v+3} + f = 0$ *Tensor (N.S.)*, 42,252-257 (1985).
- [3] Blair, D.E. and Chen. B.Y.. On CR-submanifolds of Hermitian manifolds. *Israel Journal of Mathematics*, 34(4), 353-363 (1979).
- [4] Goldberg, S.I. and Yano, K., On normal globally framed f-manifolds. *Tohoku Math.*, 1. 22, 362-370 (1970).
- [5] Goldberg, S.I., On the existence of manifold with an F-structure. *Tensor (N.S.)*, 26, 323-329 (1972).
- [6] Upadhyay, M.D. and Gupta, V.C.. *Integrability conditions of a structure f_λ satisfying $f^3 + \lambda^2 f = 0$* , *Publications Mathematics*, 24 (3-4), 249-255 (1977).
- [7] Yano, K , On structure defined by a tensor Reid f of type (1, 1) satisfying $f^3 + f = 0$, *Tensor (N.S.)*, 14,99-109(1963).
- [8] Yano, K. and Kon, M., Differential geometry of CR-submanifolds, *Geometriae Ddicala*. 10, 369-391(1981).
- [9] Das, L.S., Nivas, R. and Singh, A., On CR-structures and F-structures satisfying $F^{4n} + F^{4n-1} + \dots + F^2 + F = 0$, *Tesor N.S.* 70. 255-260(2008).
- [10] Das L.S., On CR-structures and $F(2K+1.1)$ -structure satisfying $F^{2K+1} + F = 0$, *JTSI*, India. 22, 1-7(2004).
- [11] Das. L.S., Nikic, J. and Nivas. R., Distributions parallelism and geodesic in $F(a_1, a_2, \dots, a_n)$ structure Lagrangian manifolds. *Differential Geometry Dynamitical Systems*, Romania, 7, 82-89 (2005).
- [12] Ram Nivas and Surendra Yadav: On CR-structures and $F_\lambda(2v + 3, 2) - HSU -$ structure satisfying $F^{2v+3} + \lambda^r F^2 = 0$, *Acta Ciencia Indica*, Vol. XXXVIII M, NO. 4, 645 (2012).